Summary of Differentiable Manifolds

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CHAPTER 1

Manifolds as gluing of models

1. Topological manifolds

Let M be a set. A **chart** on M is an injective map $\alpha : U \to \mathbb{R}^m$ from a subset $U \subset M$ onto an open subset of $\mathbb{R}^m$. An **atlas** $\mathfrak{A}$ is a collection of charts (U_i, α_i) for which U_i cover M: $M = \cup U_i$. An atlas $\mathfrak{A}$ is **topological** if $\alpha_i(U_i \cap U_j) \subset \alpha_i(U_i)$ are open, and the **transition maps**

$$\alpha_{ji} := \alpha_j \alpha_i^{-1} : \alpha_i(U_i \cap U_j) \longrightarrow \alpha_j(U_i \cap U_j) \tag{1}$$

are homeomorphisms. Two topological atlases $\mathfrak{A}$ and $\mathfrak{A}'$ are **compatible** if their union is again a topological atlas.

Exercise 1. *Show that compatibility is an equivalence relation $\sim$ among all topological atlases of M, and that every equivalence class is represented by a unique* maximal *topological atlas $\mathfrak{A}$ — that is, with the property that $\mathfrak{A}' \sim \mathfrak{A}$ implies $\mathfrak{A}' \subset \mathfrak{A}$.*

A **topological manifold** $(M, \mathfrak{A})$ is a set M, endowed with a maximal topological atlas $\mathfrak{A}$. It has a canonical topology — namely, the smallest topology in which the domains U_i of charts in the maximal topological atlas are all open. A map $f : (M, \mathfrak{A}_M) \to (N, \mathfrak{A}_N)$ between topological manifolds is **continuous** if it is continuous for the induced topologies.

Remark 1. A classical theorem from Algebraic Topology asserts that:

Theorem (Invariance of Dimension). *If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are nonempty open subsets, there is no homeomorphism $f : U \to V$ unless $m = n$;*

as a consequence, the **dimension** m of (each connected component of) a topological manifold $(M, \mathfrak{A})$ is well-defined.

Exercise 2. *A map $f : (M, \mathfrak{A}_M) \to (N, \mathfrak{A}_N)$ between topological manifolds is continuous iff for topological atlases $\mathscr{V} = \{(V_j, \beta_j)\}$ of N and $\mathscr{U} = \{(U_i, \alpha_i)\} \prec f^{-1}\mathscr{V}$ of M the* **local representations** *of f, that is, the maps between open sets of Euclidean spaces*

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle \alpha_i}\big\uparrow & & \big\uparrow{\scriptstyle \beta_{\lambda(i)}} \\ U_i & \underset{\beta_{\lambda(i)}^{-1} f \alpha_i}{\dashrightarrow} & V_{\lambda(i)} \end{array}$$

are continuous.

Exercise 3. *A topological space is a topological manifold iff it is modeled on some $\mathbb{R}^m$ — that is, if every point has an open neighborhood homomorphic to an open set of $\mathbb{R}^m$. An open subset of a topological manifold is itself a topological manifold, of the same dimension. If M and N denote the following subspaces of $\mathbb{R}^2$,*

$$M = \{(x, y) \mid y = |x|\}, \qquad\qquad N = \{(x, y) \mid xy = 0\},$$

then M is a topological manifold while N is not.

EXERCISE 4 (Gluing of topological manifolds). *The result of gluing topological manifolds $(M_i)_{i\in I}$ by homeomorphisms $\alpha_{ji}: M_{ij} \to M_{ji}$ satisfying the cocycle conditions is again a topological manifold. Give examples in which each M_i is either Hausdorff or second-countable, but the ensuing M is not.*

Let $\mathscr{U} = (U_i)_{i\in I}$ be an open cover, and let $M_{\mathscr{U}} = \coprod_i U_i$. If U_i is nonempty for uncountably many $i \in I$, $M_{\mathscr{U}}$ is not second-countable. The line with doubled origin — that is, the manifold M which arises from the gluing data

$$M = 0 = \mathbb{R} = M_1,\ M_{01} = M_{10} = \mathbb{R}\setminus\{0\}, \qquad \alpha_{01} = \alpha_{10} = \mathrm{id}.$$

as in Exercise 4 is not Hausdorff.

REMARK 2. *We shall henceforth assume that* manifolds *are also:*

a) Hausdorff *b) second countable;* *c) locally compact.*

So in particular, all manifolds will be henceforth paracompact as topological spaces. For example, we exclude from consideration "manifolds" as the line with doubled origin.

Although not strictly necessary for many purposes, these conditions will ensure many features with which we are not willing to part ways just yet — for example, that vector fields have local flows, admit partitions of unity etc.

2. Differentiable manifolds

A **differential** atlas $\mathfrak{A} = \{(U_i, \alpha_i)\}$ a topological manifold M is a topological atlas for which all transition maps

$$\alpha_{ji} := \alpha_j\alpha_i^{-1} : \alpha_i(U_i \cap U_j) \longrightarrow \alpha_j(U_i \cap U_j) \tag{2}$$

are diffeomorphisms — that is, differentiable homeomorphisms with differentiable inverses. Two such differential atlases $\mathfrak{A}$ and $\mathfrak{A}'$ are **compatible** if $\mathfrak{A} \cup \mathfrak{A}'$ is again a differential atlas.

EXERCISE 5. *Compatibility is an equivalence relation $\sim$ among all differential atlases of M, and every equivalence class is represented by a unique* maximal *differential atlas $\mathfrak{A}$.*

A **differential structure** on a topological manifold M is a choice of maximal differential atlas for it. A **smooth manifold** is a topological manifold M equipped with a differential structure.

EXERCISE 6. *Check that the following differential atlases on $\mathbb{R}$:*

$$\mathfrak{A}_0 = (\mathbb{R}, t \mapsto t), \qquad \mathfrak{A}_1 = (\mathbb{R}, t \mapsto t^3).$$

lead to different differential structures on $\mathbb{R}$.

A continuous map $f : (M, \mathfrak{A}_M) \to (N, \mathfrak{A}_N)$ is **smooth** if its local representations

$$f_{\beta_{\lambda(i)}\alpha_i} := \beta^{-1}_{\lambda(i)} f \alpha_i : U_i \longrightarrow V_{\lambda(i)}$$

are differentiable maps between open subsets of Euclidean space. A **diffeomorphism** is an invertible smooth map, whose inverse is also smooth. More generally, it is a C^r**-map** if its local representatives are r times continuously differentiable, and we set

$$C^r(M, N) = \{f \in C^0(M, N) \mid f \text{ is a } C^r \text{ map}\}, \qquad 0 \leqslant r \leqslant \infty.$$

EXERCISE 7. *If $\mathfrak{A}_i$ are the differential structures of Exercise 6, then the map*

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \qquad f(t) = t^3$$

a) is a smooth homeomorphism $f : (\mathbb{R}, \mathfrak{A}_0) \to (\mathbb{R}, \mathfrak{A}_0)$ which is not a diffeomorphism;
b) is a diffeomorphism $f : (\mathbb{R}, \mathfrak{A}_0) \to (\mathbb{R}, \mathfrak{A}_1)$.

EXERCISE 8. *Find a differential atlas of the m-sphere $\mathbb{S}^m = \{x \in \mathbb{R}^{m+1} \mid ||x|| = 1\}$ with two charts.*

Consider the stereographic projections

$$\alpha_\pm : \mathbb{S}^m\setminus\{\pm e_{m+1}\} \xrightarrow{\sim} \mathbb{R}^m, \qquad \alpha_\pm(x_1, ..., x_{m+1}) = \left(\tfrac{x_1}{1\mp x_{m+1}}, ..., \tfrac{x_m}{1\mp x_{m+1}}\right).$$

Then

$$\alpha_+^{-1}(y_1, ..., y_n) = (\tfrac{2y_1}{|y|^2+1}, ..., \tfrac{2y_m}{|y|^2+1})$$

and so

$$\alpha_-\alpha_+^{-1} : \mathbb{R}^m\setminus\{0\} \longrightarrow \mathbb{R}^m\setminus\{0\}, \qquad \alpha_-\alpha_+^{-1}(y_1, ..., y_m) = (\tfrac{y_1}{|y|^2}, ..., \tfrac{y_m}{|y|^2}).$$

EXERCISE 9. *The set $\mathbb{R}P^m$ of all (real) lines in $\mathbb{R}^{m+1}$ is a smooth manifold of dimension m. More generally, if V is a vector space of dimension m, the set*

$$\mathrm{Gr}_d(V) = \{W \subset V \mid W \textit{ subspace of dimension } d\}$$

is a smooth manifold of dimension $d(m-d)$.

Equip V with a metric $\langle\cdot,\cdot\rangle$, and let $W \in \mathrm{Gr}_d(V)$. Then $V = W \oplus W^\perp$. Let $U = \{Z \in \mathrm{Gr}_d(V) \mid Z \cap W^\perp = 0\}$. Then each $Z \in U$ is of the form $Z = \{w + \alpha(Z)w \mid w \in W\}$ for a unique $\alpha(Z) \in \mathrm{Hom}(W, W^\perp)$; explicitly, $\alpha(Z)\mathrm{pr}_W(z) = \mathrm{pr}_{W^\perp}(z)$. This defines a chart $\alpha : U \to \mathrm{Hom}(W, W^\perp)$ (note that α is onto).

EXERCISE 10 (Gluing of smooth manifolds). *If the result M of gluing smooth manifolds $(M_i)_{i\in I}$ by diffeomorphisms $\alpha_{ji} : M_{ij} \to M_{ji}$ satisfying the cocycle conditions is again Hausdorff and second-countable, then M has the structure of smooth manifold.*

EXERCISE 11. *A set $S \subset M$ has* **measure zero** *if for all charts (V, ϕ) of M, $\phi^{-1}S \subset \mathbb{R}^m$ has measure zero.*

A set of functions $\mathcal{P} \subset C^\infty(M)$ is **locally finite** if every $x \in M$ has an open neighborhood U which meets $\mathrm{supp}(\varrho)$ for finitely many $\varrho \in \mathcal{P}$. In that case,

$$f_{\mathcal{P}}(x) := \sum_{\varrho\in\mathcal{P}} \varrho$$

is a well-defined smooth function, and we call a locally finite set of nonnegative functions $\mathcal{P}$ a **partition of unity** if $f_{\mathcal{P}}$ is identically one. A partition of unity $\mathcal{P}$ is **subordinated** to an open cover $\mathscr{U}$ if the support $\mathrm{supp}(\varrho)$ of an $\varrho \in \mathcal{P}$ lies in some $U \in \mathscr{U}$.

EXERCISE 12. *If an open cover $\mathscr{U} = (U_i)_{i\in I}$ is refined by a cover $\mathscr{V} = (V_j)_{j\in J}$ to which a partition of unity is subordinated has itself a partition of unity subordinated to it.*

Let $\lambda : J \to I$ be the refinement map, and (ϱ'_j) the partition of unity subordinated to $\mathscr{V}$. Because every $x \in X$ has an open neighborhood on which only finitely many ϱ'_j's do not vanish identically, the sum

$$\varrho_i = \begin{cases} 0 & i \notin \lambda J; \\ \sum_{j\in(\lambda)^{-1}(i)} \varrho'_j & i \in \lambda J. \end{cases}$$

defined a smooth function, whose support lies in U_i, and $(\varrho_i)_{i\in I}$ is a partition of unity subordinated to $\mathscr{U}$.

EXERCISE 13. *If $U \subset M$ is an open set around $x \in M$, there is an open neighborhood $x \in V \subset U$ and a smooth function $f : M \to [0, 1]$, which is identically one on V and whose support lies in U.*

The function $a : \mathbb{R} \longrightarrow \mathbb{R}$, $a(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0; \\ 0, & x \leqslant 0 \end{cases}$ is smooth, and $b(x) := a(x)a(1-x)$ is positive for $0 < x < 1$ and zero elsewhere. The function $c(x) := \frac{\int_0^x b(y)\mathrm{d}y}{\int_0^1 b(y)\mathrm{d}y}$ is smooth, and

$$h(x) = 0,\ x \leqslant 0, \qquad 0 < h(x) < 1,\ 0 < x < 1, \qquad h(x) = 1,\ 1 \leqslant x.$$

Using this, one constructs for $x \in \mathbb{R}^m$ and $0 < \delta < \epsilon$ a smooth function $g : \mathbb{R}^m \to [0, 1]$ satisfying

$$g|_{B_\delta(x)} = 1, \qquad g|_{\mathbb{R}^m\setminus B_\epsilon(x)} = 0.$$

Now use a chart $\alpha : U \to \mathbb{R}^m$ containing $\overline{B_\epsilon(x)}$ to construct the smooth function

$$f : M \longrightarrow \mathbb{R}, \qquad f(x) = \begin{cases} g\alpha(x) & \text{if } x \in U; \\ 0 & \text{if } x \notin U. \end{cases}$$

EXERCISE 14. *Every open cover $\mathscr{U}$ of a smooth manifold has a partition of unity subordinated to it.*

First, one can refine $\mathscr{U}$ by a locally finite, precompact open cover. For each $U \in \mathscr{U}$ and each $x \in U$, there is $f_x^U : M \to [0, 1]$ such that $f_x^U(x) = 1$ and $\mathrm{supp}(f_x^U) \subset U$. Let $V_x^U = \{y \in U \mid f_x^U(y) > 0\}$. The collection $\mathscr{V}'$ of such open sets V_x^U is an open cover of M which refines $\mathscr{U}$. Because M is paracompact, $\mathscr{V}'$ has a locally finite refinement $\mathscr{V}$. For each $V \in \mathscr{V}$, there is a smooth *function* $f^V : M \to [0,1]$ *and* $U \in \mathscr{U}$*, such that* $f^V|_V > 0$ *and* $\operatorname{supp}(f^V) \subset U$*. Because* $\mathscr{U}$ *is precompact and* $\mathscr{V}$ *is locally finite, each* $U \in \mathscr{U}$ *meets finitely many* $V \in \mathscr{V}$*; that is,*

$$\mathscr{V}^U := \{V \in \mathscr{V} \mid U \cap V \neq \varnothing\}$$

is a finite set for each U*. Therefore the family of functions* $(f^U)_{U\in\mathscr{U}}$

$$f^U : M \longrightarrow [0,\infty), \qquad f^U(x) := \sum_{V\in\mathscr{V}^U} f^V(x)$$

is smooth, subordinated to $\mathcal{U}$ *(hence locally finite), and*

$$f : M \longrightarrow [0,\infty), \qquad f := \sum_{U\in\mathscr{U}} f^U > 0.$$

Hence $\mathcal{P} \subset C^\infty(M)$, $\mathcal{P} := \{\frac{f^U}{f} \mid U \in \mathscr{U}\}$ *is a partition of unity subordinated to* $\mathscr{U}$*.*

Exercise 15. *In a smooth manifold* M*, every two disjoint closed sets* $C_0, C_1 \subset M$ *can be separated by a smooth function — that is, there exists* $f \in C^\infty(M)$*, such that* $f|_{C_0} = 0$ *and* $f|_{C_1} = 1$*.*

Let $U_0 = M\smallsetminus C_1$*,* $U_1 = M\smallsetminus C_0$*, and consider the open cover* $\mathscr{U} = \{U_0, U_1\}$*. Let* $\{\varrho_0, \varrho_1\}$ *be an open cover subordinated to* $\mathscr{U}$*, and set* $f = \varrho_0$*.*

3. Tangent functor and maps of locally constant rank

A smooth curve c in a smooth manifold M is a smooth map $c : [0,1] \to M$. We say that it starts at $c(0)$ and ends at $c(1)$. Among all smooth curves in M which start at x, we may consider the equivalence relation in which $c_0 \sim c_1$ iff αc_0 and αc_1 have the same velocity at $t = 0$ for any chart (U,α) around x. The set of equivalence classes $v = \frac{d}{dt}c(t)|_{t=0}$ under this relation defines the **tangent space** T_xM to M at x.

Exercise 16. *The disjoint union* $TM = \coprod_x T_xM$ *of all tangent spaces to a smooth manifold of dimension* m *has a natural structure of smooth manifold of dimension* $2m$*, equipped with a smooth map* $\mathrm{pr} : TM \to M$*, assigning* $x \in M$ *to* $v \in T_xM$*, is smooth.*

If $\mathfrak{A} = \{(U_i, \alpha_i)\}$ *is a differential atlas for* M*, define a differential atlas* $T\mathfrak{A} = \{(TU_i, \gamma_i)\}$ *by*

$$\gamma_i : TU_i \longrightarrow \alpha_i U_i \times \mathbb{R}^m, \qquad \gamma_i(\tfrac{d}{dt}c(t)|_{t=0}) = (\alpha_i c(0), \tfrac{d}{dt}\alpha_i c(t)|_{t=0}).$$

Then

$$\gamma_{ji} := \gamma_j\gamma_i^{-1} : \gamma_i T(U_i \cap U_j) \longrightarrow \gamma_j T(U_i \cap U_j), \qquad \gamma_{ji}(x,v) = (\alpha_{ji}(x), D\alpha_{ji}(x,v))$$

Exercise 17 (Tangent functor). *Every smooth map* $f : M \to N$ *induces a smooth map* $f_* : TM \to TN$ *which restricts on tangent spaces to linear maps* $f_* : T_xM \to T_{f(x)}N$*, in such a way that if* $\mathrm{id}_* = \mathrm{id}$ *and* $(gf)_* = g_*f_*$ *for a further smooth map* $g : N \to P$*.*

Exercise 18 (Jet manifolds). *Let* $r \geqslant 0$ *and* $x \in M$*. Consider the equivalence relation on the set* $C^\infty(M,N)$ *of smooth maps from* M *to* N *in which two maps* f, f' *are equivalent,* $f \sim_x^r f'$*, iff they have local representatives around* x *with the same Taylor series of order* r *at* x*. Let* $J_r(M,N)^x$ *denote the set of equivalence classes, and denote by* $j_rf(x)$ *the equivalence class of* f*. Then*

$$J_r(M,N)^x = \bigcup_{x\in M} J_r(M,N)^x$$

has a canonical structure of smooth manifold, in which:

- *A smooth map* $f : M \to N$ *gives rise to a smooth map* $j^rf : M \to J_r(M,N)$*;*
- *The map* $\mathrm{pr} : J_r(M,N) \to M$*,* $j_rf(x) \mapsto x$ *is smooth;*
- *The maps* $J_r(M,N) \to J_{r-1}(M,N)$ *are smooth, and* $J_0(M,N) = M \times N$*.*

The **rank** of a smooth map $f : M \to N$ at $x \in M$ is the rank of its differential $f_* : T_xM \to T_{f(x)}N$. The function $x \mapsto \mathrm{rk}_x f$ is lower-semicontinuous, in the sense that every $x \in M$ has an open neighborhood in which the rank of f is bound below by $\mathrm{rk}_x f$. The map f has **locally constant rank** if every point lies in an open set in which the rank is constant.

EXERCISE 19. *A smooth map has locally constant rank iff it is locally linear — that is, around every point it has local representatives which are linear maps.*

It suffices to show that a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$, which maps zero to zero and has rank r around zero is equivalent to the linear map

$$\mathbb{R}^m = \mathbb{R}^r \times \mathbb{R}^{m-r} \longrightarrow \mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^{n-r}, \qquad (x, y) \mapsto (x, 0).$$

Up to a reordering of coordinates, we may assume that $(\frac{\partial f_i}{\partial x_j})_{1\leqslant i,j\leqslant r}$ is non-singular around zero. Hence

$$\alpha : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \qquad \alpha(x_1, ..., x_m) = (f_1(x), ..., f_r(x), x_{r+1}, ..., x_m).$$

is a diffeomorphism around the origin, by the IFT. Hence

$$f\alpha^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^n, \qquad (f\alpha^{-1})(x_1, ..., x_m) = (x_1, ..., x_r, g_{r+1}(x), ..., g_n(x)), \quad g_i := f_i\alpha^{-1}.$$

Because

$$(f\alpha^{-1})_* = \begin{pmatrix} I_r & 0 \\ * & \frac{\partial g_i}{\partial x_j} \end{pmatrix}$$

and the rank of $(f\alpha^{-1})_*$ is exactly r, it follows that $\frac{\partial g_i}{\partial x_j} = 0$ — that is, the functions g_i do not depend on the variables $x_{r+1}, ..., x_m$: $g_i(x) = g_i(x_1, ..., x_r)$ if $i > r$. Define now

$$\beta(y_1, ..., y_n) = (y_1, ..., y_r, y_{r+1} - g_{r+1}(y_1, ..., y_r), ..., y_n - g_n(y_1, ..., y_r))$$

Then β is a diffeomorphism around zero, and

$$\beta f\alpha^{-1}(x_1, ..., x_m) = (x_1, ..., x_r, 0, ...0).$$

The map $f : M \to N$ is an **immersion** if f_* is injective at all points, and a **submersion** if it is surjective at all points. By Exercise 19, for every $x \in M$, there are local charts (U, α) around $x \in M$ and (V, β) around $f(x) \in N$, such that respectively

$$\beta f\alpha^{-1}(x_1, ..., x_m) = (x_1, ..., x_m, 0, ..., 0), \qquad \beta f\alpha^{-1}(x_1, ..., x_m) = (x_1, ..., x_n).$$

The image of an injective immersion is called an **immersed submanifold**. An injective immersion $f : M \to N$ is called an **embedding** if the topology on M is that induced by f.

EXERCISE 20. *The map*

$$f : (-\pi, \pi) \longrightarrow \mathbb{R}^2, \qquad f(t) = (\sin(2t), \sin(t)).$$

is a smooth immersion which is not an embedding.

The image is the image of $[-\pi, \pi]$ and looks like an eight figure, and is compact, so it cannot be homeomorphic to $(-\pi, \pi)$.

EXERCISE 21. *A proper, injective immersion is an embedding.*

By Exercise 78, such a map is a continuous, closed bijection onto its image, and hence a homeomorphism.

EXERCISE 22. *Let $f : M \to N$ be an injective immersion, and $g : P \to N$ a smooth map. Then there is a unique set-theoretic map $\widetilde{g} : P \to M$ lifting g, $f\widetilde{g} = g$, and a sufficient condition for $\widetilde{g}$ to be smooth is that it be continuous.*

It suffices to show that $\widetilde{g}$ is smooth around an arbitrary point $p \in P$. Because $f : M \to N$ is an immersion, it follows from its local normal form that there are open neighborhoods $V \subset N$ of $f\widetilde{g}(p)$, and $U \subset M$ of $\widetilde{g}(p)$, and a smooth map $h : V \to U$, such that $h \circ f|_U = \mathrm{id}$. Because $\widetilde{g}$ is continuous, $W := \widetilde{g}^{-1}U \subset P$ is open, and $\widetilde{g}|_W = h \circ g|_W$. Hence $\widetilde{g}$ is smooth.

An injective immersion $f : M \to N$ is **initial** if the lift $\widetilde{g} : P \to M$ is smooth for every smooth $g : P \to N$. A subset $X \subset N$ is called **initial submanifold** if it is the image of an initial immersion, and a **submanifold** if it is the image of an embedding.

EXERCISE 23. *If $f : M \to N$ is any smooth map, and $\mathrm{gr}(f) : M \to M \times N$ is its graph $\mathrm{gr}(f)(x) = (x, fx)$, then $\mathrm{gr}(f)M \subset M \times N$ is a submanifold.*

EXERCISE 24. *Submanifolds are initial submanifolds, but not conversely.*

Let $i : X \to M$ be an embedding, and let $f : N \to M$ be a smooth map, such that $fN \subset iX$. Let $U \subset X$ be open. Because i is an embedding, $iU \subset iX \cap V$ for some open set $V \subset M$. Hence if $\widetilde{f} : N \to X$ is the unique set-theoretic lift, then $\widetilde{f}^{-1}U = f^{-1}(V)$ is open, because f is continuous. Hence $\widetilde{f}$ is continuous – hence smooth by Exercise 22.

EXERCISE 25. *For a parameter $a \in \mathbb{R}$, consider*

$$f_a : \mathbb{R} \longrightarrow \mathbb{T}^2, \qquad f(t) = (e^{2\pi i t}, e^{2\pi i a t}).$$

The image of f_a is initial submanifolds. They are dense if $a \notin \mathbb{Q}$, and are embedded iff $a \in \mathbb{Q}$.

If $a = \frac{q}{p}$, then $f_a\mathbb{R} = f_a[0,p]$, so if $r : \mathbb{R} \to \mathbb{S}^1$ is the map $r(t) = e^{\frac{2\pi i t}{p}}$, then $f_a = \widetilde{f_a} r$, where $\widetilde{f_a} : \mathbb{S}^1 \to \mathbb{T}^2$ is an injective *immersion — and hence an embedding since the circle is compact.*

If $a \notin \mathbb{Q}$, consider the submersion $r : \mathbb{R}^2 \to \mathbb{T}^2$, $r(x,y) = (e^{2\pi i x}, e^{2\pi i y})$. Then r is étale, and $f_a\mathbb{R}$ is the image under r of a leaf of the linear foliation $\mathrm{d}y - a\mathrm{d}x$. Let $g : N \to \mathbb{T}^2$ be a smooth map whose image lies in $f_a\mathbb{R}$. Let $g(x_0) = f_a(t_0)$, and consider a chart $\alpha : \mathbb{T}^2 \supset U \to \alpha U \subset \mathbb{R}^2$ with $r\alpha = \mathrm{id}$. It suffices to show that $\widetilde{g}$ is smooth around x_0, and note that

$$\widetilde{g}|_{g^{-1}U} = \mathrm{pr}_1 \alpha g|_{g^{-1}U}.$$

Hence f_a is initial. We show that it is not embedded by proving that $\overline{f_a\mathbb{R}} = \mathbb{T}^2$. Let $(e^{2\pi i r}, e^{2\pi i s}). \in \mathbb{T}^2$. We claim that $e^{2\pi i s}$ lies in the closure of the set $\Lambda = \{e^{2\pi i c r} e^{2\pi i c n} \in \mid n \in \mathbb{Z}\}$ or, equivalently, that $\Lambda_0 = \{e^{2\pi i c n} \in \mid n \in \mathbb{Z}\}$ is dense in the circle. Suppose that were not the case, and let $\gamma \notin \overline{\Lambda_0}$. Then there is a maximal open interval $I_\gamma = (\gamma - \delta, \gamma + \epsilon)$ around γ which does not meet Λ_0. But note that $I_{\gamma+c} = (\gamma + c - \delta, \gamma + c + \epsilon)$. No two of these intervals overlap, since they were assumed to be maximal, and no two can coincide since c is irrational. This leads to infinitely many disjoint open sets of the same positive length in the circle, which cannot be. Hence $f_a\mathbb{R}$ is dnse in the torus, and in particular it is not embedded.

EXERCISE 26. *If $f_i : M_i \to N$ are injective immersions with the same image, and $f_1^{-1} f_0 : M_0 \to M_1$ is continuous, then it is a diffeomorphism.*

By Exercise 22, $\phi = f_1^{-1} f_0$ is smooth, hence a diffeomorphism.

EXERCISE 27. *An initial submanifold has a unique structure of smooth manifold, for which the inclusion map is an immersion.*

Suppose $X \subset N$ be the image of an initial immersion, and $f : M \to N$ is an injective immersion with image $f(M) = X$. Then $f : M \to X$ is continuous; hence by Exercise 26, $f : M \to X$ is a diffeomorphism.

EXERCISE 28. *For a subset X of a smooth manifold M, the following conditions are equivalent:*

i) X is a submanifold of codimension q;

ii) around every $x \in X$, there is a smooth chart (V, β) of M, such that

$$\beta(X \cap V) = (\mathbb{R}^{m-q} \times 0) \times \beta V;$$

iii) around every $x \in X$, there is a submersion $s : U \to \mathbb{R}^q$, such that $X \cap U = s^{-1}(0)$.

i) ⇔ ii) *Assume that the inclusion $i : X \to M$ is an embedding. Let (U, α) and (V', β) be charts around $x \in X$ and $i(x) \in M$, such that $\beta i \alpha^{-1} : U \to V'$ coincides with the restriction of the inclusion of $\mathbb{R}^n$ as $\mathbb{R}^n \times 0 \subset \mathbb{R}^m$. Because i is an embedding, $iU = V \cap iX$ for some open $V \subset V'$. Then the chart (V, β) satisfies ii). Conversely, condition ii) implies that X has an induced smooth structure (by considering only charts as in ii), and extracting $(W \cap \mathbb{R}^{m-q} \times 0, \psi)$ from them), for which the inclusion map $i : X \to M$ is an embedding.*

ii) ⇔ iii) *Assuming ii), we construct a such submersion by $s := \mathrm{pr}_2 \circ \psi^{-1} : U \to \mathbb{R}^q$. For the converse, use the local normal form of submersions to find a chart (W, ψ) of M around $x \in X \cap U$, such that $\psi^{-1}(X) = W \times (\mathbb{R}^{m-q} \times 0)$.*

EXERCISE 29. *Every compact manifold embeds into some $\mathbb{R}^N$.*

Define for a finite differential atlas $\{(U_1, \alpha_1), ..., (U_k, \alpha_k)\}$, and $\varrho_1, ..., \varrho_k$ a partition of unity subordinated to $\{U_i\}$,

$$f : M \longrightarrow \mathbb{R}^{k(m+1)}, \qquad f(x) = (f_0(x), ..., f_k(x)), \qquad f_0(x) = (\varrho_1, ..., \varrho_k), \qquad f_i(x) = \begin{cases} 0, & x \notin U_i, \\ \varrho_i(x)\alpha_i(x), & x \in U_i. \end{cases}$$

Then: a) f is injective: if $f(x) = f(y)$, then $\varrho_i(x) = \varrho_i(y) > 0$ and $f_i(x) = f_i(y)$ for some i, and hence $\alpha_i(x) = \frac{f_i(x)}{\varrho_i(x)} = \frac{f_i(y)}{\varrho_i(y)} = \alpha_i(y)$. So $x = y$. Also: b) f is an immersion: if $f_*(x)v = 0$, then $\mathscr{L}_v \varrho_i(x) = 0$ and $\mathscr{L}_v f_i(x) = 0$. If $\varrho_i(x) > 0$, then

$$\mathscr{L}_v f_i(x) = \varrho_i(x) \mathscr{L}_v \alpha_i(x) + \mathscr{L}_v \varrho_i(x) \alpha_i(x)$$

implies $\mathscr{L}_v \alpha_i(x) = 0$; but α_i is a diffeomorphism, so $v = 0$. The rest follows from M being compact.

4. Differential forms

A **covector** ξ at $x \in M$ is a linear function $\xi : T_xM \to \mathbb{R}$ — that is, an element of $T_x^*M := (T_xM)^*$. The **cotangent bundle** of M is the disjoint union

$$T^*M = \coprod_{x \in M} T_x^*M.$$

Exercise 30. *The cotangent bundle of a smooth manifold of dimension m has a natural structure of smooth manifold of dimension $2m$, equipped with a smooth map* $\mathrm{pr}: T^*M \to M$, *assigning* $x \in M$ *to* $\xi \in T^*_xM$, *which turns* T^*M *into a vector bundle over* M.

A **one-form** is a smooth map $\xi : M \to T^*M$ for which $\mathrm{pr} d f = \mathrm{id}$. The space of all one-forms on M is denoted by $\Omega^1(M)$.

Exercise 31. *For a smooth, real-valued function* $f \in C^\infty(M)$, *determines a one-form* $\mathrm{d}f \in \Omega^1(M)$ *by*

$$\langle \mathrm{d}f, v\rangle = \mathscr{L}_v f, \qquad v \in \mathfrak{X}(M).$$

Exercise 32. *Every* $\xi \in T^*_xM$ *is of the form* $\xi = \mathrm{d}f(x)$ *for some* $f \in C^\infty(M)$.

A p**-form**, is a section of $\wedge^p T^*M \to M$. The space of p-forms is denoted $\Omega^p(M)$. The space of differential forms is $\Omega(M) = \oplus_p \Omega^p(M)$.

Exercise 33 (Wedge product). $\Omega(M)$ *is a commutative graded algebra under the product*

$$(\omega \wedge \eta)(v_1, ..., v_{p+q}) = \sum_{\sigma \in S_{p,q}} (-1)^\sigma \omega(v_{\sigma(1)}, ..., v_{\sigma(p)}) \eta(v_{\sigma(p+1)}, ..., v_{\sigma(p+q)})$$

where $S_{p,q}$ *denotes the set of all* (p,q)*-shuffles — that is, permutations in* $p+q$ *letters, satisfying*

$$\sigma(1) < \cdots < \sigma(p), \qquad \sigma(p+1) < \cdots < \sigma(p+q).$$

Exercise 34 (Pullback by a smooth map). *If* $\phi : M \to N$ *is a smooth map, and* $\omega \in \Omega^p(N)$ *is a differential* p*-form, then*

$$\phi^*(\omega)_x = \phi^*(\omega_{\phi x})$$

defines a p*-form* $\phi^*(\omega) \in \Omega^p(M)$, *and the* **pullback** *map*

$$\phi^* : (\Omega(N), \wedge) \longrightarrow (\Omega(M), \wedge)$$

is a homomorphism of graded commutative algebras.

Exercise 35. *The linear map* $\iota : \mathfrak{X}(M) \longrightarrow \mathrm{End}^{-1}\Omega(M)$ *which to a vector field* $v \in \mathfrak{X}(M)$ *assigns the degree* -1 *endomorphism*

$$(\iota_v\omega)(v_1, ..., v_{p-1}) = \omega(v, v_1, ..., v_{p-1}), \qquad v_i \in \mathfrak{X}(M),\ \omega \in \Omega^p(M):$$

a) is a graded derivation of $(\Omega, \wedge)$;
b) squares to zero: $\iota_v(\iota_v\omega) = 0$ *for all differential form* ω.

5. Vector fields and their local flows

A **vector field** v on a smooth manifold M is an assignment of a vector $v_x \in T_xM$ for each x, varying smoothly in x — that is, a smooth map $v : M \to TM$ such that $\mathrm{pr} v = \mathrm{id}$. The space of all vector fields on M will be denoted by $\mathfrak{X}(M)$.

A **trajectory** of a vector field $v \in \mathfrak{X}(M)$ is a smooth curve $c : (a,b) \to M$ such that

$$\tfrac{d}{dt} c = v \circ c.$$

By the fundamental theorem of ODEs, there is a smooth map

$$\phi : \mathbb{R} \times M \supset \mathrm{dom}(v) \longrightarrow M, \qquad \phi(t,x) = \phi_t(x),$$

where $\mathrm{dom}(v)$ is an open neighborhood of $\{0\} \times M$, such that $\phi_t(x)$ is the maximal trajectory of v with $\phi_0(x) = x$. We call ϕ_t the **local flow** of v, and note that $\phi_t\phi_s(x) = \phi_{t+s}(x)$ whenever either side is defined. A vector field v is **complete** if $\mathrm{dom}(v) = \mathbb{R} \times M$.

Exercise 36. *A vector field whose support is compact is complete.*

If $\mathrm{supp}(v) \subset M$ *a compact set, there is* $\epsilon > 0$ *such that* $(-\epsilon, \epsilon) \times M \subset \mathrm{dom}(v)$. *Then if* $t \in \mathbb{R}$, *let* $n \in \mathbb{N}$ *be such that* $\frac{t}{n} \in (-\epsilon, \epsilon)$. *Then*

$$\phi_t(x) = \underbrace{\phi_{\frac{t}{n}} \phi_{\frac{t}{n}} \cdots \phi_{\frac{t}{n}}}_{n}(x)$$

EXERCISE 37. *A* **derivation** δ *of the space* $C^\infty(M)$ *of smooth, real-valued function* $f: M \to \mathbb{R}$ *is a linear map* $\delta : C^\infty(M) \to C^\infty(M)$ *such that* $\delta(fg) = \delta(f)g + f\delta(g)$*. Denote by* $\mathrm{Der}C^\infty(M)$ *the space of such derivations. Then vector field* $v \in \mathfrak{X}(M)$ *defines a derivation* $\mathscr{L}_v \in \mathrm{Der}C^\infty(M)$ *of* **Lie derivative** *by* v,

$$(\mathscr{L}_v f)(x) := \tfrac{d}{dt}(fc)|_{t=0}, \qquad v_x = [c],$$

and $\mathscr{L} : \mathfrak{X}(M) \longrightarrow \mathrm{Der}C^\infty(M)$ *is a linear isomorphism.*

In local coordinates $(x_1, ..., x_m)$*, a vector field* v *has an expression* $v = \sum v_i \frac{\partial}{\partial x_i}$*, where* v_i *are smooth functions. It acts of a function* f *via* $\mathscr{L}_v f = \sum v_i \frac{\partial f}{\partial x_i}$*. That* $\mathscr{L}_v$ *is a derivation follows from the product rule. If* D *is a derivation, set* $v_i = Dx_i$*. Write an arbitrary function* f *as*

$$f(x) = f(0) + \sum_{i=1}^m x_i f_i(x), \qquad f_i(x) = \int_0^1 \tfrac{\partial f}{\partial x_i}(tx)\mathrm{d}t$$

to deduce that $Df(0) = \sum_{i=1}^m v_i f_i(x) = \mathscr{L}_v f$*. Hence* $D = \mathscr{L}_v$ *for* $v = \sum v_i \frac{\partial}{\partial x_i}$*, and this defines* v *uniquely.*

EXERCISE 38 (Lie bracket of vector fields). *For vector fields* $v, w \in \mathfrak{X}(M)$, $\mathscr{L}_{[v,w]} := [\mathscr{L}_v, \mathscr{L}_w]_c$ *defines a unique vector field* $[v, w] \in \mathfrak{X}(M)$*, and the assignment*

$$[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M), \qquad (v, w) \mapsto [v, w]$$

turns $\mathfrak{X}(M)$ *into a Lie algebra.*

Follows from the fact that graded endomorphisms of the graded vector space $\Omega(M)$ *form a graded Lie algebra.*

EXERCISE 39 (Related vector fields). *We say that vector fields* $v_M \in \mathfrak{X}(M)$ *and* $v_N \in \mathfrak{X}(N)$ *are* f**-related** *by a smooth map* $f : M \to N$ *if* $v_N(f(x)) = f_*(v_M(x))$ *for all* $x \in M$*. If* $v_M \sim_f v_N$ *and* $w_M \sim_f w_N$*, then* $[v_M, w_M] \sim_f [v_N, w_N]$*. If* ϕ_t *and* ψ_t *denote their respective flows, then*

$$f \circ \phi_t = \psi_t \circ f$$

whenever either side is defined.

EXERCISE 40 (Push-forward of a vector field by a diffeomorphism). *If* $f : M \to N$ *is a diffeomorphism, and* $v \in \mathfrak{X}(M)$ *is a vector field, then*

$$f_*(v) \in \mathfrak{X}(N), \qquad f_*(v)(x) = f_*^{-1} v(fx)$$

is a vector field on N*,* f*-related to* v*. If* ϕ_t *is the local flow of* v*, then* $\psi_t = f \circ \phi_t \circ f^{-1}$ *is the local flow of* $f_*(v)$*.*

We say that a section $v \in \Gamma(I \times TM)$ is a **time-dependent** vector field, and a curve c is a trajectory if $\frac{d}{dt}c = v_t \circ c(t)$.

EXERCISE 41 (Time-dependent of vector fields). *Let* v *be a time-dependent vector field on* M*, and let the (usual) vector field* $\widetilde{v} = \frac{\partial}{\partial t} + v \in \mathfrak{X}(I \times M)$ *have local flow* Φ_t*. Then the map* $\phi^{t,s}$ *defined by*

$$\Phi_t(s, x) = (t + s, \phi_{t+s,s}(x)),$$

satisfies

$$\tfrac{d}{dt}\phi_{t,s}(x) = v_t \circ \phi_{t,s}(x), \qquad \phi_{t,s}\phi_{s,r}(x) = \phi_{t,r}(x), \qquad \phi_{t,t}(x) = x.$$

and is called the **local flow** *of* v*. If* $v \in \mathfrak{X}(M)$ *has local flow* ϕ_t *and is regarded as depending (trivially) on time, then* $\phi_{t,s} = \phi_t \phi_s^{-1}$*.*

EXERCISE 42. *If* ϕ_t *denotes the local flow of* $v \in \mathfrak{X}(M)$*, then for all* $f \in C^\infty(M)$ *and* $w \in \mathfrak{X}(M)$ *we have*

$$\tfrac{d}{dt}(\phi_t)^* f = (\phi_t)^*(\mathscr{L}_v f), \qquad \tfrac{d}{dt}(\phi_t)^* w = (\phi_t)^*([v, w]).$$

If more generally v *is a time-dependent vector field, then for all* $f \in C^\infty(I \times M)$ *and* $w \in \Gamma(I \times TM)$*, we have that*

$$\tfrac{d}{dt}(\phi_{t,s})^* f_t = (\phi_{t,s})^*(\mathscr{L}_{v_t} f_t + \tfrac{d}{dt} f_t), \qquad \tfrac{d}{dt}(\phi_{t,s})^* w_t = (\phi_{t,s})^*([v_t, w_t] + \tfrac{d}{dt} w_t).$$

The formula for time-dependent objects comes from the time-independent one via the correspondence

$$\Gamma(I \times TM) \longrightarrow \mathfrak{X}(I \times M), \qquad v \mapsto \tfrac{\partial}{\partial t} + v.$$

For functions, this is checked directly:

$$\tfrac{d}{dt}(\phi^{t,s})^*f = f_*\tfrac{d}{dt}(\phi^{t,s}) = f_* v\phi^{t,s} = \mathscr{L}_v\phi^{t,s} = (\phi^{t,s})^*(\mathscr{L}_v).$$

For vector fields, differentiate both sides of $\mathscr{L}_{(\phi^{t,s})^*w}(\phi^{t,s})^*f = (\phi^{t,s})^*(\mathscr{L}_w f)$ *to get*

$$\mathscr{L}_{\frac{d}{dt}(\phi^{t,s})^*w}(\phi^{t,s})^*f + (\phi^{t,s})^*(\mathscr{L}_w\mathscr{L}_v f) = (\phi^{t,s})^*(\mathscr{L}_v\mathscr{L}_w f),$$

which is to say that

$$\mathscr{L}_{\frac{d}{dt}(\phi^{t,s})^*w}(\phi^{t,s})^*f = (\phi^{t,s})^*(\mathscr{L}_{[v,w]}f) = \mathscr{L}_{(\phi^{t,s})^*[v,w]}(\phi^{t,s})^*(f),$$

and therefore $\frac{d}{dt}(\phi^{t,s})^*w = (\phi^{t,s})^*[v,w]$.

6. Cartan calculus

EXERCISE 43. *The degree 1 linear map* $\mathrm{d} : \Omega(M) \to \Omega(M)$ *defined by*

$$\mathrm{d}\omega(v_0, v_1, ..., v_p) = \sum(-1)^i \mathscr{L}_{v_i}\omega(v_0, ..., \widehat{v_i}..., v_p) + \sum_{i<j}(-1)^{i+j}\omega([v_i, v_j], v_0, ..., \widehat{v_i}, ..., \widehat{v_j}..., v_p)$$

for $\omega \in \Omega^p(M)$ *and* $v_i \in \mathfrak{X}(M)$*:*

a) is a graded derivation of $(\Omega, \wedge)$*;*
b) extends $\mathrm{d}f$ *of Exercise 31 for functions* $f \in C^\infty(M) = \Omega^0(M)$*;*
c) commutes with pullbacks: if $\phi : N \to M$ *is a smooth map, and* ω *a differential form on* M*, then* $\mathrm{d}\phi^*(\omega) = \phi^*(\mathrm{d}\omega)$*;*
d) squares to zero: $\mathrm{d}(\mathrm{d}\omega) = 0$ *for all differential form* ω*.*

EXERCISE 44. *The linear map* $\mathscr{L} : \mathfrak{X}(M) \longrightarrow \mathrm{End}^0\Omega(M)$ *which to a vector field* $v \in \mathfrak{X}(M)$ *assigns the degree zero endomorphism*

$$\mathscr{L}_v := [\iota_v, \mathrm{d}]_c$$

(see Exercise 82) is a graded derivation of $(\Omega, \wedge)$*.*

EXERCISE 45 (Cartan calculus). *For vector fields* $v, w \in \mathfrak{X}(M)$*:*

a) $[\iota_v, \iota_w]_c = 0$*;*
b) $[\iota_v, \mathrm{d}]_c = \mathscr{L}_v$*;*
c) $[\mathrm{d}, \mathrm{d}]_c = 0$*;*
d) $[\mathscr{L}_v, \mathrm{d}]_c = 0$*;*
e) $[\mathscr{L}_v, \iota_w]_c = \iota_{[v,w]}$*;*
f) $[\mathscr{L}_v, \mathscr{L}_w]_c = \mathscr{L}_{[v,w]}$*.*

7. Integration

A **volume element** ρ on a vector space V of dimension m is a function

$$\rho : V \times \cdots \times V \longrightarrow \mathbb{R}, \qquad \text{such that} \qquad \rho(Av_1, ..., Av_m) = |\det A|\rho(v_1, ..., v_m)$$

for all $v_1, ..., v_m \in V$ and linear map $A : V \to V$. Note that volume elements on V form a vector space $\mathscr{D}(V)$ of dimension one.

EXERCISE 46. *If* M *is a smooth manifold of dimension* m*, the set* $\mathscr{D}(M) = \coprod_{x\in M}\mathscr{D}(T_xM)$ *has a canonical structure of smooth manifold of dimension* $m+1$*, for which the canonical map* $\mathrm{pr} : \mathscr{D}(M) \to M$ *is a surjective submersion.*

Let (U, α) *be a chart of* M*, and let* $\rho \in \mathscr{D}(T_xM)$*. Then* $\rho = s\alpha^*|\mathrm{d}x|$ *for a unique* $s \in \mathbb{R}$*. Define*

$$\widetilde{\alpha} : \mathscr{D}(U) \longrightarrow \alpha U \times \mathbb{R}, \qquad \widetilde{\alpha}(\rho) := (\alpha(x), s).$$

Then

$$\widetilde{\beta}\widetilde{\alpha}^{-1} : \alpha(U\cap V) \times \mathbb{R} \longrightarrow \beta(U \cap V), \qquad \widetilde{\beta}\widetilde{\alpha}^{-1}(x, s) = (\beta\alpha^{-1}(x), \tfrac{s}{|\det D(\beta\alpha^{-1})(\alpha(x))|}).$$

The charts $(\mathscr{D}(U), \widetilde{\alpha})$ *turn* $\mathscr{D}(M)$ *into a smooth manifold, and the map* $\mathrm{pr} : \mathscr{D}(M) \to M$ *is locally represented by the canonical projections* $\alpha U \times \mathbb{R} \to \alpha U$*.*

A **density** ρ on a smooth manifold M is a smooth map $\rho : M \to \mathscr{D}(M)$ such that $\mathrm{pr}\rho = \mathrm{id}$, and we write $\mathrm{Dens}(M)$ for the vector space of all densities on M, and $\mathrm{Dens}_c(M)$ for the vector subspace of densities of compact support, where

$$\mathrm{supp}(\rho) := \overline{\{x \in M \mid \rho_x \neq 0\}}.$$

Note that they are modules over $C^\infty(M)$ of dimension one.

EXERCISE 47. *A density ρ on M can be equivalently defined as a rule which to every chart (U, α) of M, assigns a smooth function $\rho_\alpha \in C^\infty(\alpha U)$, in such a way that, for a second chart (V, β),*

$$\rho_\alpha = \rho_\beta \circ (\beta\alpha^{-1})|\det D(\beta\alpha^{-1})|. \tag{3}$$

Note that if (U, α) is a chart of M, then $\rho|_U = \alpha^*(\rho_\alpha|\mathrm{d}x|)$ for a unique $\rho_\alpha \in C^\infty(\alpha U)$, hence for a second chart (V, β), we have that

$$\rho_\alpha|\mathrm{d}x| = (\beta\alpha^{-1})^*(\rho_\beta|\mathrm{d}x|) = (\beta\alpha^{-1})^*(\rho_\beta)(\beta\alpha^{-1})^*(|\mathrm{d}x|) = (\beta\alpha^{-1})^*(\rho_\beta)|\det D(\beta\alpha^{-1})|(|\mathrm{d}x|)$$

which is to say that the smooth functions ρ_α satisfy the required transition rule.

EXERCISE 48. *There is a unique linear map $\int_M : \mathrm{Dens}_c(M) \to \mathbb{R}$ with the property that, for a density ρ whose support lies in a chart (U, α), we have*

$$\int_M \rho = \int_{\alpha U} \rho_\alpha$$

Fix a partition of unity ϱ_i subordinated to a locally finite atlas $\mathfrak{A} = (U_i, \alpha_i)$, and let ρ be a density on M of compact support. Then $\varrho_i\rho$ is a density supported in U_i, and we set

$$\int_M \rho := \sum_i \int_{\alpha_i U_i} (\varrho_i\rho)_{\alpha_i}.$$

Because of the hypotheses, this sum is finite. Clearly $\int_M$ thus defined is a linear map, which does not depend on the choices made: if $\varrho'_j, V_j, \beta_j)$ were another set of choices, then

$$\sum_j \int_{\beta_j(V_j)} (\varrho'_j\rho)_{\beta_j} = \sum_{i,j} \int_{\beta_j(U_i \cap V_j)} (\varrho_i\varrho'_j\rho)_{\beta_j} = \sum_{i,j} \int_{\alpha_i(U_i \cap V_j)} (\varrho_i\varrho'_j\rho)_{\beta_j} \circ (\beta_j\alpha_i^{-1})|\det D(\beta_j\alpha_i^{-1})| = \sum_{i,j} \int_{\alpha_i(U_i \cap V_j)} (\varrho_i\varrho'_j\rho)_{\alpha_i}$$
$$= \sum_i \int_{\alpha_i(U_i)} (\varrho_i\rho)_{\alpha_i}$$

EXERCISE 49. *If $\rho \in \mathrm{Dens}(N)$ and $\phi : M \to N$ is a diffeomorphism, then*

$$\phi^*(\rho)_x(v_1, ..., v_m) := \rho_{\phi(x)}(\phi_* v_1, ..., \phi_* v_m)$$

defines a density $\phi^*(\rho) \in \mathrm{Dens}(M)$. If ρ has compact support, so does $\phi^*(\rho)$, and $\int_M \phi^*(\rho) = \int_N \rho$.

If $\rho = (\rho_\beta)_{(V,\beta)}$ is such a density, there is a unique density $\phi^*(\rho)$ on M for which $\phi^*(\rho)_{\beta\phi} = \rho_\beta$, in which case we see that

$$\int_{\beta\phi\phi^{-1}V_j} \phi^*(\rho)_{\beta\phi} = \int_{\beta V_j} \rho_\beta,$$

which implies the equality $\int_M \phi^*(\rho) = \int_N \rho$ for ρ of compact support.

A density ρ is called **positive** if $\rho_\alpha > 0$ for all charts (U, α). We denote by $\mathscr{D}_+(M)$ the subspace of such densities.

EXERCISE 50. *Positive densities exist.*

For every nowhere-vanishing form $\omega \in \Omega^m(M)$, $|\mu|_x(v_1, ..., v_m) := |\mu_x(v_1, ..., v_m)|$ is a positive density. Hence if (U_i, α_i) are charts of M, ϱ_i is a partition of unity subordinated to it, and $\mu_i \in \Omega^m(\alpha_i U_i)$ are nowhere-vanishing forms, then

$$\rho := \sum_i \varrho_i \alpha_i^*(|\mu_i|)$$

is a positive density on M.

EXERCISE 51. *There is a canonical bilinear map* $\mathrm{div} : \mathrm{Dens}_c(M) \times \mathfrak{X}(M) \to \mathrm{Dens}_c(M)$, *uniquely determined by the property that, for all density ρ and vector field v,*

$$\tfrac{d}{dt}(\phi^{t,s})^*(\rho)|_{t=s} = \mathrm{div}(v, \rho)$$

where $\phi^{t,s}$ denotes the local flow of v. First observe that the LHS of the formula above is a priori a density on M. Let $\rho = (\rho_\alpha)$ and $v = (v_\alpha)$. Observe that the local flow of v_α if given by $\phi_\alpha^{t,s} := \alpha\phi^{t,s}\alpha^{-1}$, and that $(\phi^{t,s})^*(\rho)_\alpha = \rho_\alpha \circ \phi_\alpha^{t,s}|\det D\phi_\alpha^{t,s}|$. Because $\phi_\alpha^{t,t} = \mathrm{id}$, it follows that for s and t sufficiently close, $(\phi^{t,s})^*(\rho)_\alpha = \rho_\alpha \circ \phi_\alpha^{t,s} \det D\phi_\alpha^{t,s}$. Hence

$$\tfrac{d}{dt}(\phi^{t,s})^*(\rho)_\alpha|_{t=s} = (\rho_\alpha)_*(v_\alpha) + \rho_\alpha \mathrm{Tr} D v_\alpha = (\rho_\alpha)_*(\sum_1^m (v_\alpha)_i \tfrac{\partial}{\partial x_i}) + \rho_\alpha \sum_1^m \tfrac{\partial (v_\alpha)_i}{\partial x_i} = \sum_1^m \tfrac{\partial}{\partial x_i}(\rho_\alpha (v_\alpha)_i),$$

and so $\mathrm{div}(v,\rho)$ is the density given by $\mathrm{div}(v,\rho)_\alpha = \sum_1^m \frac{\partial}{\partial x_i}(\rho_\alpha(v_\alpha)_i)$.

EXERCISE 52. *A* **Riemannian metric** *g on M is a section $g \in \Gamma(S^2 T^M)$ with the property that $g(v,v) \geqslant 0$ for all v, and $g(v,v) = 0$ iff $v = 0$. If g is a Riemannian metric, then $g \in \Gamma(S^2(\wedge^r T^M))$ given by*

$$g(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r) := \det(g(v_i, w_j))$$

define metrics on $\wedge^r T^M$. If M is oriented and $\omega \in \Omega^r(M)$, there is a unique $\star\omega \in \Omega^{m-r}(M)$ such that

$$\eta \wedge \star\omega = g(\eta, \omega)\mu_g,$$

where μ_g is the volume form of g and $\eta \in \Omega^r(M)$, and $\star : \Omega^r(M) \to \Omega^{m-r}(M)$ comes from a linear endomorphism of $\wedge T^*M$. Conclude that g induces a degree -1 endomorphism δ_g of $\Omega(M)$ which squares to zero.

EXERCISE 53. *Let g be a Riemannian metric on an oriented manifold with boundary M. The map*

$$D_M : C^\infty(M) \times C^\infty(M) \longrightarrow \mathbb{R}, \qquad D_M(f,g) := \int_M \mathrm{d}f \wedge \star \mathrm{d}g$$

is symmetric, and if Δ denotes

$$\Delta : C^\infty(M) \longrightarrow \Omega^m(M), \qquad \Delta f := \mathrm{d} \star \mathrm{d}f,$$

then

$$\int_{\partial M} (f \star \mathrm{d}g - g \star \mathrm{d}f) = \int_M (f\Delta g - g\Delta f)$$

EXERCISE 54. *Let g be a Riemannian metric on an oriented manifold with boundary M. A function is* **harmonic** *if $\Delta f = 0$. If $M = \mathbb{R}^m$ with its Euclidean metric, then*

$$g(x) = \begin{cases} \log r, & m = 2, \\ r^{m-2}, & m > 2 \end{cases}$$

is harmonic. If $U \subset \mathbb{R}^m$ is open, and $f : U \to \mathbb{R}$ is harmonic, then for all spheres $S_r(0)$ contained in U,

$$f(0) = \frac{\int_{S_r(0)} f\mathrm{d}S}{\int_{S_r(0)} f\mathrm{d}S},$$

where $\mathrm{d}S$ denotes the induced volume form. Conclude that if U is connected and f is harmonic and attains its maximum on U, then f is constant.

8. Critical points and transversality

Let $f : M \to N$ be a smooth map. A point $x \in M$ is a **regular point** if $f_* : T_x M \to T_{f(x)} N$ is onto, and **critical point** otherwise. A point $y \in N$ is **regular value** if $f^{-1}(y)$ has no critical points; otherwise, it is a **critical value**. Note that $y \notin fM$ is automatically a regular value. We denote by

$$\mathrm{Crit}(f) = \{x \in M \mid \mathrm{rk}_x f < \dim N\}$$

the set of critical points, and by $f\mathrm{Crit}(f) \subset N$ the set of critical values.

EXERCISE 55. *If M is compact of positive dimension, and $\partial M = \varnothing$, then there every smooth function on M has at least two points. Give counterexamples if any of the hypotheses is omitted.*

A continuous function f on a compact space M attains its maximum value C and minimum value c; if M is a smooth manifold and f is a smooth function, the points where C and c are attained must be critical points. Counterexamples: $f(t) = t$ where M is either $\mathbb{R}$ or $[0, 1]$.

EXERCISE 56 (Sard theorem). *If $f : M \to N$ is a smooth map, $f\mathrm{Crit}(f) \subset N$ has measure zero.*

It suffices to prove the theorem for a smooth map $f : \mathbb{R}^m \supset U \to \mathbb{R}^n$, and we do so by induction on m. Consider

$$\mathrm{Crit}(f) \supset \mathrm{Crit}^1(f) \supset \cdots \supset \mathrm{Crit}^k(f) \supset \cdots, \qquad \mathrm{Crit}^i(f) = \{x \in U \mid D^\alpha f(x) = 0 \text{ for all } |\alpha| \leqslant i\}.$$

Step 1. $f(\mathrm{Crit}(f)\smallsetminus\mathrm{Crit}^1(f))$ has measure zero. *If $x \in \mathrm{Crit}(f)$ does not lie in $\mathrm{Crit}^1(f)$, then wlog $\frac{\partial f_1}{\partial x_1}(x) \neq 0$. Then $\psi = (f_1, x_2, ..., x_m) : V \to V'$ is a diffeomorphism around $x \in V$, and the smooth map $g = f\psi^{-1} : V \to \mathbb{R}^m$ is of the form $g(x_1, x_2, ..., x_m) = (x_1, g_2(x), ..., g_n(x))$. Hence $g(V \cap \{x_1\} \times \mathbb{R}^{m-1}) \subset \{x_1\} \times \mathbb{R}^{n-1}$ for each x_1, and we let $g_{x_1} : V \cap \{x_1\} \times \mathbb{R}^{m-1} \to \mathbb{R}^{n-1}$ denote the restriction. Note that*

$$\mathrm{Crit}(g) = \psi(\mathrm{Crit}(f) \cap V), \qquad g\mathrm{Crit}(g) = f(\mathrm{Crit}(f) \cap V).$$

and that $\frac{\partial g_1}{\partial x_1} = 1$ implies also that

$$\mathrm{Crit}(g) \cap (\{x_1\} \times \mathbb{R}^{m-1}) = \mathrm{Crit}(g_{x_1}).$$

By the inductive hypothesis, $g_1\mathrm{Crit}(g_{x_1})$ has measure zero in $\{x_1\} \times \mathbb{R}^{n-1}$, and so

$$g\mathrm{Crit}(g) = \cup_{x_1 \in \mathrm{pr}_1 V}\, g_1\mathrm{Crit}(g_{x_1})$$

has measure zero.

Step 2. $f(\mathrm{Crit}^{k+1}(f)\smallsetminus\mathrm{Crit}^k(f))$ has measure zero. *WLOG, $\frac{\partial^{k+1} f_1}{\partial x_1 \partial x_\alpha}(x) \neq 0$, so we consider the diffeomorphism $\psi(x_1, x_2, ..., x_m) = (\frac{\partial^k f_1}{\partial x_\alpha}(x)), x_2, ..., x_m) : V \to V'$ as argue as in Step 1 to show that $f(\mathrm{Crit}^{k+1}(f) \cap V)$ has measure zero.*

Step 3. $f(\mathrm{Crit}^k(f))$ has measure zero for k sufficiently large.

Consider the cube $\square(1) \subset \mathbb{R}^m$ of edges 1. We show that $f(\mathrm{Crit}^k(f) \cap \square(1))$ has measure zero if $k > m/n - 1$. By Taylor's theorem, if $x \in \mathrm{Crit}^k(f)$, then for all y such that $x + [0, 1]y \in \square(1) \cap U$, there is a constant $C > 0$, such that

$$|f(x + y) - f(x)| \leqslant C|y|^{k+1}.$$

Subdivide $\square(1)$ into r^m subcubes of length $\frac{1}{r}$, and let $\square(\frac{1}{r})$ be the subcube which contains x. Then $x + y \in \square(\frac{1}{r})$ implies $|y| \leqslant \frac{\sqrt{m}}{r}$, and by the inequality above, $f(\square(\frac{1}{r}))$ lies in a cube of $\mathbb{R}^n$ of edges $C(\frac{\sqrt{m}}{r})^{k+1}$. Hence the volume of $f(\square(\frac{1}{r}))$ is at most $C^n(\frac{\sqrt{m}}{r})^{n(k+1)}$, and so the volume of $f(\mathrm{Crit}^k(f) \cap \square(1))$ is at most

$$C^n(\sqrt{m})^{n(k+1)} r^{m-n(k+1)}$$

If $n(k + 1) > m$, this quantity goes to zero as r grows.

Two smooth maps $f : M \to N$ and $g : P \to N$ are **transverse** if, for all $(x, y) \in M \times_{(f,g)} P = (f, g)(\Delta_N)$, we have

$$T_z N = f_* T_x M + g_* T_y P, z = f(x) = g(y).$$

We write this as $f \pitchfork g$. If $Y \subset N$ is a submanifold, we say that f is **transverse** to Y, and write $f \pitchfork Y$ to denote $f \pitchfork \mathrm{i}_Y$.

EXERCISE 57. *Two smooth maps $f : M \to N$ and $g : P \to N$ are transverse iff $(f, g) : M \times P \to N \times N$ is transverse to Δ_N.*

EXERCISE 58. *If $f : M \to N$ is transverse to a submanifold $Y \subset N$, then $X := f^{-1}Y$ is a submanifold of M of the same codimension in M as that of Y in N, and $TX = f_*^{-1}TY$.*

Let $s : N \supset U \to \mathbb{R}^q$ be a local submersion with $Y \cap U = s^{-1}(0)$. Then $T_y Y = \ker(s_*)_y$, and we claim that f transverse to Y ensures that $sf : f^{-1}U \to \mathbb{R}^q$ is a submersion; indeed,

$$f_*(T_x M) + T_{f(x)} s^{-1}(0) = f_*(T_x M) + \ker(s_*)_{f(x)} = T_{f(x)} N$$

implies that

$$s_* f_*(T_x M) = s_* T_{f(x)} N = T_{sf(x)} \mathbb{R}^q.$$

Hence the collection $\{(sf)^{-1}(0) = f^{-1}(Y \cap U)\}$, as (U, s) range over such local submersions, defines a submanifold $X = f^{-1}(Y)$, and $TX = \ker(sf)_* = f_*^{-1} \ker(s)_* = f_*^{-1}TY$.

Exercise 59. *If $f : M \to N$ and $g : P \to N$ are transverse smooth maps, then*

$$M \times_{(f,g)} P = \{(x, y) \in M \times P \mid f(x) = g(y)\}$$

is a submanifold of $M \times P$ of codimension $\dim N$*. In particular, if $X, X' \subset M$ are transverse submanifolds, then $X \cap X'$ is again a submanifold, and*

$$\dim M + \dim(X \cap X') = \dim X + \dim X', \qquad T(X \cap X') = TX \cap TX'.$$

CHAPTER 2

Appendix: Recollection

1. Topology

For a set X, we denote by $\mathscr{P}(X)$ the set of all subsets of X. A **topology** on X is a collection $\mathscr{T} \subset \mathscr{P}(X)$ of subsets of X, with the following properties:

a) The union of members of $\mathscr{T}$ is again a member of $\mathscr{T}$;
b) The intersection of finitely many members of $\mathscr{T}$ is again a member of $\mathscr{T}$;

Note that the definition implies that $\varnothing$ and X are members of $\mathscr{T}$, since $\varnothing$ is the union of an empty family of subsets, and X is the intersection of an empty family of subsets.

A subset $U \subset X$ is called **open** in the topology $\mathscr{T}$ if $U \in \mathscr{T}$, and **closed** if its complement is open. A **neighborhood** $N \subset X$ of a subset $Y \subset X$ is a subset which contains an open set $U \subset X$ which contains Y: $Y \subset U \subset N$. A map $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ between topological spaces is **continuous** if $\mathscr{T}_X \supset f^{-1}(\mathscr{T}_Y)$. It is **open** if it maps open sets to open sets, and **closed** if it maps closed sets to closed sets.

Exercise 60. *Describe all topologies on* $X = \{0, 1, 2, 3\}$.

Exercise 61. *For any subset* Y *of a topological space* $(X, \mathscr{T}_X)$, *there is a smallest closed set* $\overline{Y} \subset X$ *containing* Y, *the* **closure** *of* Y *in* X. *Similarly, there is a largest open set* $\mathrm{int}(Y) \subset X$ *contained in* Y, *the* **interior** *of* Y. *A set is open iff* $Y = \mathrm{int}(Y)$, *and it is closed iff* $Y = \overline{Y}$.

Exercise 62. *Let* X *be a set, equipped with a* **closure operator**, *i.e. a map* $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ *with the following properties:*

a) $\mathrm{cl}(\varnothing) = \varnothing$;
b) $A \subset \mathrm{cl}(A)$ *for all* A;
c) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ *for all* A, B;
d) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

Then topologies on X *are in bijective correspondence with such closure operators.*

Let cl *be as in the statement. Note that* $A \subset B$ *implies by the third axiom that* $\mathrm{cl}(A) \subset \mathrm{cl}(B) = \mathrm{cl}(A) \cup \mathrm{cl}(B \setminus A)$. *Hence* $\mathrm{cl}\mathscr{P}(X)$ *is closed under finite intersections. On the other hand, let* $(A_i)_{i \in I}$ *be any family of subsets. Then* $\cap_i \mathrm{cl}(A_i) \subset \mathrm{cl}(A_j)$ *for all* $j \in I$; *hence* $\mathrm{cl}(\cap_i \mathrm{cl}(A_i)) \subset \mathrm{cl}^2(A_j) = \mathrm{cl}(A_j)$ *by the fourth axiom, and so* $\mathrm{cl}(\cap_i \mathrm{cl}(A_i)) \subset \cap_i \mathrm{cl}(A_i)$. *But then the second axiom imlies that* $\cap_i \mathrm{cl}(A_i) = \mathrm{cl}(\cap_i \mathrm{cl}(A_i))$, *and this together with the first axiom shows that* $\mathscr{T}$ *is closed under arbitrary unions.*

Exercise 63. *Let* (X, d) *be a metric space, and let* $\mathscr{T}$ *be the set of all subsets* $U \subset X$ *with the property that* $x \in U$ *implies that* U *contains an* open ball

$$B_r(x) := \{y \in X \mid d(x, y) < r\}$$

of radius r *around* x, *for some* $r > 0$. *Then* $\mathscr{T}$ *is a topology on* X.

Let $\mathrm{Top}(X)$ denote the set of all topologies on the set X. It has an induced partial order: we write $\mathscr{T} \leqslant \mathscr{T}'$, and say that $\mathscr{T}$ is **coarser** than $\mathscr{T}'$, or that $\mathscr{T}'$ is **finer** than $\mathscr{T}$, if $\mathscr{T} \subset \mathscr{T}'$. Note that $\mathscr{T} \leqslant \mathscr{T}'$ exactly when $\mathrm{id} : (X, \mathscr{T}') \to (X, \mathscr{T})$ is continuous.

Exercise 64. *For any subset* $\Omega \subset \mathrm{Top}(X)$ *and topological space* $(Y, \mathscr{T}_Y)$:

a) The **infimum** $\inf \Omega := \bigcap_{\mathscr{T} \in \Omega} \mathscr{T} \in \mathrm{Top}(X)$ *is the finest topology coarser than any topology in* Ω;
b) The **supremum** $\sup \Omega := \inf \Omega'$, *where* $\Omega' := \{\mathscr{T}' \in \mathrm{Top}(X) \mid \mathscr{T} \leqslant \mathscr{T}',\ \mathscr{T} \in \Omega\}$, *is the coarsest topology finer than any topology in* Ω;

c) *For every subset $\mathscr{S} \subset \mathscr{P}(X)$, there is coarsest topology $\mathscr{T} := \langle \mathscr{S} \rangle$ which contains $\mathscr{S}$ (the topology* **generated** *by $\mathscr{S}$);*
d) $\inf \mathrm{Top}(X)$ *is the* **indiscrete topology** $\mathscr{T}_{\text{ind}} := \{\varnothing, X\}$, *and all maps $g : (Y, \mathscr{T}_Y) \to (X, \mathscr{T}_{\text{ind}})$ are continuous;*
e) $\sup \mathrm{Top}(X)$ *is the* **discrete topology** $\mathscr{T}_{\text{disc}} := \mathscr{P}(X)$, *and all maps $f : (X, \mathscr{T}_{\text{disc}}) \to (Y, \mathscr{T}_Y)$ are continuous;*
f) *For any collection of maps $f_i : X \to Y$ from a set X, there is a coarsest topology $\mathscr{T}_X$ on X for which all maps f_i are continuous (the* **initial** *topology);*
g) *For any collection of maps $g_i : Y \to X$ into a set X, there is a finest topology $\mathscr{T}_X$ on X for which all maps f_i are continuous (the* **final** *topology).*

A **homeomorphism** is an invertible, continuous function whose inverse is also continuous.

EXERCISE 65. *Find an example of a set X and topologies $\mathscr{T}_0, \mathscr{T}_1 \in \mathrm{Top}(X)$ for which $\mathrm{id} : (X, \mathscr{T}_0) \to (X, \mathscr{T}_1)$ is a continuous, invertible map which is not a homeomorphism.*

A topological space X is **modeled** on another topological space Y if, for all $x \in X$, there is a homeomorphism $\phi : V \to X$ of an open set $V \subset Y$ onto an open neighborhood $U = \phi(V)$ of x.

EXERCISE 66. *Show that $\mathbb{R}^m$ is modeled on $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m \mid x_m \geqslant 0\}$, but not conversely.*

EXERCISE 67 (Quotient topology). *Let $(X, \mathscr{T}_X)$ be a topological space, and $f : X \to Y$ a set-theoretic surjective map. Then*

$$\mathscr{T}_Y = \{U \subset Y \mid f^{-1}U \in \mathscr{T}_X\}$$

defines a **quotient** *topology on Y, which is such that:*

(1) *$f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous;*
(2) *for all topological spaces $(Z, \mathscr{T}_Z)$ and set-theoretic maps $g : Y \to Z$, $g \circ f : (X, \mathscr{T}_X) \to (Z, \mathscr{T}_Z)$ is continuous iff $g : (Y, \mathscr{T}_Y) \to (Z, \mathscr{T}_Z)$ is continuous.*

In particular, every equivalence relation $\sim$ on a topological space induces a topology on the set of equivalence classes.

EXERCISE 68. *Let I be an index set. For each $i \in I$, let X_i be a topological manifold, and for each $i, j \in I$ let $X_{ij} \subset X_i$ be an open subset, and $\phi_{ji} : X_{ij} \to X_{ji}$ be a homeomorphism, satisfying the cocycle conditions:*

$$\phi_{ii} = \mathrm{id}, \qquad \phi_{kj}\phi_{ji} = \phi_{ki}.$$

Then the disjoint union $\coprod_{i \in I} X_i$ has an equivalence relation in which $x \in X_{ij}$ is identified with $\phi_{ji}(x)$, and the set of equivalence classes X has a canonical topology, in which each X_i is identified with an open subset $U_i \subset X$, in such a way that X_{ij} and X_{ji} correspond to $U_i \cap U_j$.

A topology $\mathscr{T}$ on X is **Tychonoff** **(T_1)** if $\{x\}$ is closed for each $x \in X$; equivalently, if for any two distinct points $x, x' \in X$, there is an open set which contains one but not the other. A topology $\mathscr{T}$ on X is **Hausdorff** **(T_2)** if any two distinct points $x, x' \in X$ have disjoint open neighborhoods. It is **regular** **(T_3)** if a closed set $C \subset X$ and a point $x \notin C$ have disjoint open neighborhoods, and **normal** **(T_4)** if any two disjoint closed sets $C, C' \subset X$ have disjoint open neighborhoods.

EXERCISE 69. *Find examples of T_i-topological spaces which are not T_{i+1}.*

T1 but not T2 *$X = \mathbb{R}$ and $\mathscr{T}$ is the collection with the empty set and the complement of any finite set.*

T2 but not T3 *$X = \mathbb{R}^2$ and $\mathscr{T}$ be the topology with basis*

$$\mathscr{B} = \{B_{\frac{|x_1|}{n}}(x) \mid x_1 \neq 0,\ n \in \mathbb{N}\} \cup \{(B_{\frac{1}{n}}(x) \setminus \{x_1 = 0\}) \cup \{x\} \mid x_1 = 0,\ n \in \mathbb{N}\}.$$

This is clearly T2. Both $\{0\}$ and $L = \{x_1 = 0,\ x_2 \neq 0\}$ are closed, but cannot be separated by open sets.

T3 but not T4 *Let $X = \mathbb{R}$ with the topology generated by sets of the form $[a, b)$. Then $X \times X$ is T3: IF $C \subset X \times X$ is a closed set and $(x_1, x_2) \in (X \times X) \setminus C$, there are $b_1, b_2 \in \mathbb{R}$ such that the open set $U := [x_1, b_1) \times [x_2, b_2)$ lies in $(X \times X) \setminus C$. But*

$$(X \times X) \setminus [x_1, b_1) \times [x_2, b_2) = (((-\infty, x_1] \cup [b_1, \infty)) \times \mathbb{R}) \cup (\mathbb{R} \times ((-\infty, x_2] \cup [b_2, \infty)))$$

is also open in this topology, so $U, (X \times X)\setminus U$ *separate* $\{x\}$ *and* C. *Now consider*

$$C' = \{(t, -t) \mid t \in \mathbb{Q}\}, \qquad C'' = \{(t, -t) \mid t \notin \mathbb{Q}\}.$$

The topology induced on $C = C' \cup C''$ *is discrete, so both* C' *and* C'' *are closed in* C. *Hence there are closed sets* F, F'' *in* $X \times X$ *such that* $C' = F' \cap C$ *and* $C'' = F'' \cap C$. *But no two such* F', F'' *can be separated by open sets.*

EXERCISE 70. *If* $f : X \to Z$ *and* $g : Y \to Z$ *are continuous, and* Z *is Hausdorff, then*

$$X \times_{(f,g)} Y = \{(x, y) \mid f(x) = g(y)\}$$

is closed in $X \times Y$.

The diagonal $\Delta_Z \subset Z \times Z$ *is closed if* Z *is Hausdorff.*

EXERCISE 71 (Urysohn's lemma). *A topological space is normal iff any two disjoint closed sets* $A, B \subset X$ *can be separated by a continuous function: there is* $f : X \to [0, 1]$ *continuous, such that* $f|_A = 0$ *and* $f|_B = 1$.

A space is normal iff for all C *closed and* U *open,* $C \subset U$ *implies that there is an open* V, *such that* $C \subset V \subset \overline{V} \subset U$. *Let* $C_0 = A$ *and* $U_1 = X \setminus B$. *Then* $C_0 \subset U_1$; *hence there is an open* $U_{\frac{1}{2}} \subset X$ *and a closed subset* $C_{\frac{1}{2}} \subset X$ *such that*

$$C_0 \subset U_{\frac{1}{2}} \subset C_{\frac{1}{2}} \subset U_1.$$

Similarly, one constructs

$$C_0 \subset U_{\frac{1}{4}} \subset C_{\frac{1}{4}} \subset U_{\frac{1}{2}} \subset C_{\frac{1}{2}} \subset U_{\frac{3}{4}} \subset C_{\frac{3}{4}} \subset U_1.$$

Inductively, one constructs, for all dyadic *rationals, open sets* U_r, *with the property that*

$$s < r \qquad \Longrightarrow \qquad \overline{U_s} \subset U_r.$$

Define

$$f : X \longrightarrow [0, 1], \quad f(x) = \begin{cases} 1 & x \notin \cup_r U_r; \\ \inf\{r \mid x \in U_r\} & x \in \cup_r U_r. \end{cases}$$

Because dyadic fractions are dense and $\overline{U_s} \subset U_r$ *if* $s < r$, *we have*

$$x \in \overline{U_r} \implies f(x) \leqslant r, \qquad f(x) < r \implies x \in U_r,$$

and therefore

$$f^{-1}[0, t) = \bigcup_{r<t} U_r, \qquad f^{-1}(t, 1] = \bigcup_{t<r} (X \setminus \overline{U_r}).$$

An **open cover** $\mathcal{U}$ of topological space $(X, \mathscr{T}_X)$ is a set of open subsets, and we say that $\mathcal{U}$ **covers** a subset $Y \subset X$ if $Y \subset \cup_{U \in \mathcal{U}} U$. An open cover is a **basis** if, for all $V \subset X$ open and $x \in V$, there is $U \in \mathscr{U}$ with $x \in U \subset V$. A topology $\mathscr{T}$ on X is **second-countable** if there is it has a countable basis.

EXERCISE 72. *A second-countable regular space is normal.*

Let $A, B \subset X$ *be disjoint closed subsets, and let* $\mathscr{B}$ *be a countable basis. Let*

$$\mathscr{U} = \{U \in \mathscr{B} \mid x \in U \subset \overline{U} \subset X \setminus B, \ x \in A\}, \qquad \mathscr{V} = \{V \in \mathscr{B} \mid x \in V \subset \overline{V} \subset X \setminus A, x \in B\}$$

These are countable sets, and define

$$U'_r := U_r \setminus \cup_1^r \overline{V_i}, \qquad V'_r := V_r \setminus \cup_1^r \overline{U_i}$$

Because $A \subset \cup_r U_r$ *and* $A \cap \overline{V_r} = \varnothing$ *for all* r, *it follows that* $A \subset U := \cup_r U'_r$. *Similarly,* $B \subset V := \cup_r V'_r$. *The open sets* U, V *are disjoint because* $x \in U'_r \cap V'_s$ *would imply (if, say,* $r \geqslant s$*) that* $x \in (U_r \setminus \overline{V_s}) \cap V_s$.

EXERCISE 73. *A topological space is* **metrizable** *if its topology is that induced by a metric. Second-countable regular topological spaces are metrizable (Urysohn's metrization theorem).*

A subset $\mathcal{U}' \subset \mathcal{U}$ is a **subcover** if $\mathcal{U}'$ is itself an open cover of Y. A subset $Y \subset X$ is **compact** if every open cover of Y has a finite subcover, and it is **precompact** if its closure is compact. A topological space is **locally compact** if every open neighborhood of a point contains a compact neighborhood of it.

EXERCISE 74. $\mathbb{R}^m$, *with the usual topology, is normal, second-countable and locally compact.*

If Y is a subset of a topological space $(X, \mathscr{T}_X)$, the initial topology $\mathscr{T}_Y$ with respect to the inclusion map $\mathrm{i}_Y : Y \to X$ is called the **subspace** topology, in which $U \in \mathscr{T}_Y$ exactly when $U = U' \cap Y$ for some $U' \in \mathscr{T}_X$.

Exercise 75. *A subspace of a Hausdorff (resp., second-countable) space is again Hausdorff (resp., second-countable).*

Exercise 76. *The image of a compact set under a continuous map is compact. A closed subset of a compact space is compact. Every compact set in a Hausdorff space is closed.*

a) Let $f : X \to Y$ be continuous, and $K \subset X$ be compact. Let $\mathscr{U}$ be an open cover of fK. Then $f^{-1}\mathscr{U}$ is an open cover of K; hence there is a finte subcover $\mathscr{V}$, in which case $f\mathscr{V}$ is a finite subcover of $\mathscr{U}$. b) If X is compact and C is closed, and $\mathscr{U}$ is an open cover of C, then $\mathscr{U} \cup \{X \setminus C\}$ is an open cover of X. c) X is Hausdorff iff $\Delta_X \subset X \times X$ is closed. Hence for all $x \neq x'$, there are open neighborhoods

$$x \in U_{x,x'}, \qquad x' \in V_{x,x'}, \qquad U_{x,x'} \cap V_{x,x'} = \varnothing.$$

Let $K \subset X$ be compact. Let $z \notin K$. Then $\{U_{x,z} \mid x \in K\}$ is an open cover of K. Hence $K \subset U_{x_1,z} \cup \cdots \cup U_{x_r,z}$. Then $V = \cup_1^r V_{x_r,z}$ is an open neighborhood of z disjoint from K.

Exercise 77. *Let X be a compact space, and Y a Hausdorff space. Then a continuous bijection $f : X \to Y$ is a homeomorphism.*

It suffices to show that f^{-1} is continuous, i.e., that f is open. Equivalently, we show that f is closed: if $C \subset X$ is closed, then it is compact, hence $fC \subset Y$ is compact, hence closed, because Y is Hausdorff.

Exercise 78. *Let Y be second-countable, locally compact and Hausdorff, and $f : X \to Y$ a proper map. Then f is closed.*

Let $Y_n \subset Y$ be precompact open sets, such that $\overline{Y_n} \subset Y_{n+1}$ and $Y = \cup_n Y_n$. Because f is continuous, $X = \cup_n X_n$, where $X_n := f^{-1}(Y_n)$ are open and $\overline{X_n} \subset X_{n+1}$. Because f is proper,the X_n's are all precompact.

Let now $F \subset X$ be closed. Then $f|_{\overline{X_n}} : \overline{X_n} \to \overline{Y_n}$ is a continuous map from a compact space to a Hausdorff space, so (as in Exercise 77) it is a closed map. Hence $f(X) \cap \overline{Y_n}$ is closed in $\overline{Y_n}$ for each n.

Let $y \in \overline{Y_{n_0}} \setminus fF$. Then for each $n \geqslant n_0$, there are open sets $y \in U_n \subset Y$, such that $U_n \cap fF \cap Y_n = \varnothing$. Because Y is locally compact, we may take such U_n to be precompact; this implies that we can define a function $\lambda : \mathbb{N} \to \mathbb{N}$ by

$$\lambda(n) = \min\{m \mid \overline{U_n} \subset \overline{Y_m}\}$$

Then set $V := U_{n_0} \cap U_{\lambda(n_0)}$. It is an open, precompact neighborhood of y, whose closure is entirely contained in $\overline{Y_{\lambda(n_0)}}$, and which does not meet fF. Hence $Y \setminus fF$ is open — i.e., f is a closed map.

An open cover $\mathcal{U}$ is **locally finite** if each $x \in X$ lying in a subset $U \in \mathcal{U}$ lies in finitely many such subsets. Given open covers $\mathcal{U}, \mathcal{V}$ of Y, we say that $\mathcal{V}$ **refines** $\mathcal{U}$ if every $V \in \mathcal{V}$ is a subset of some $U \in \mathcal{U}$. A topological space is **paracompact** if every open cover $\mathcal{U}$ of X has a locally finite refinement.

Exercise 79. *A Hausdorff, second countable, locally compact topological space $(X, \mathscr{T})$ is paracompact.*

1) *The subset consisting of precompact open sets in a basis of X is again a basis. For if $\mathscr{B}$ is a basis of X, let $\mathscr{B}_c = \{U \in \mathscr{B} \mid \overline{U}$ is compact$\}$. Let $O \subset X$ be open, and let $x \in K_x \subset X$ be a compact neighborhood of x. Then we can find $U \in \mathscr{B}$ such that $x \in U_x \subset O \cap K_x$. Then $U_x \in \mathscr{B}_c$, and $O = \cup_{x \in O} U_x$.*
2) *There is a countable precompact open cover $\mathscr{V} = \{V_i \mid i \in \mathbb{N}\}$ satisfying $\overline{V_i} \subset V_{i+1}$. Indeed, if (U_i) is a countable precompact cover, and $O_i := \cup_1^i U_i$, then because $\overline{O_i}$ is compact, there is a smallest $m_i > m$ such that $\overline{O_i} \subset O_{m(i)}$. Then $V_i := \cup_1^{m_i} U_i$ is the desired cover.*
3) *Let $\mathscr{U}$ be an open cover of X, and $\mathscr{V}$ the countable, precompact open cover of item 2). Then for each $i \in \mathbb{N}$, $K_i := \overline{V_i} \setminus V_{i-1}$ is a compact set, and because $\overline{V_i} \subset V_{i+1}$, it follows that $W_i := V_{i+1} \setminus \overline{V_{i-2}}$ is an open neighborhood of K_i. Hence*

$$\mathcal{U}_i = \{U \cap W_i \mid U \in \mathcal{U}\}$$

is an open cover of the compact set K_i, and therefore has a finite subcover Λ_i. Then $\Lambda := \cup_{i \in \mathbb{N}} \Lambda_i$ is an at most countable open cover of X, which refines $\mathcal{U}$. It consists of precompact sets, and is locally finite because by construction, no $x \in V_i$ belongs to a set in Λ_j if $i + 2 \leqslant j$; in particular, x lies in finitely many sets in Λ.

2. Calculus

Let V and W be vector spaces. A map $f : U \to W$ from an open set $U \subset V$ is called **differentiable** if, for all $x \in U$ and $v \in V$, the limits

$$D_v f(x) := \lim_{t \to 0} \tfrac{f(x+tv) - f(x)}{t}$$

exist. f is **continuously differentiable** if the ensuing map $Df : U \times V \to W$ is continuous. It is **smooth** if all iterated derivatives

$$D^r f : U \times \underbrace{V \times \cdots \times V}_{r} \longrightarrow W, \qquad D^r f(x; v_1, ..., v_r) := (D_{v_r} D_{v_{r-1}} \cdots D_{v_1} f)(x)$$

exist and are continuously differentiable.

Theorem. *For a smooth function $f : U \to W$, $x \in U$ and $v, v_1, ..., v_r \in V$:*
a) $Df(x) : V \to W$ *is linear;*
b) If $x + [0,1]v \subset U$, *then*

$$f(x+v) = f(x) + \int_0^1 Df(x+tv)(v)\mathrm{d}t; \tag{4}$$

c) f *is locally constant iff* $Df = 0$*;*
d) $D^r f(x)$ *is symmetric: for any permutation in* r *letters* σ,

$$D^r f(x)(v_1, ..., v_r) = D^r f(x)(v_{\sigma(1)}, ..., v_{\sigma(r)});$$

e) If $f(U)$ *lies in an open* $O \subset W$, *and* $g : O \to Z$ *is another smooth map, then*

$$D(g \circ f)(x) = D(g)(f(x)) \circ D(f)(x). \tag{5}$$

Theorem. *For a smooth function* $f : U \to W$, *define*

$$\mathrm{Tayl}^r f : U \longrightarrow \mathrm{Pol}^r(V, W), \qquad \mathrm{Tayl}^r f(x)v = \sum_{i=0}^{r} \tfrac{1}{i!} D_v^i f(x). \tag{6}$$

If $g : W \supset U' \to Z$ *is a further smooth map, and* $fU \subset U'$, *then*

$$\mathrm{Tayl}^r(g \circ f)(x) = \mathrm{Tayl}^r g(f(x)) \cdot \mathrm{Tayl}^r f(x),$$

where $\cdot$ *denotes truncation of the composition. Moreover, if* $x + [0,1]v \subset U$, *then*

$$f(x+v) = \mathrm{Tayl}^r f(x)v + \tfrac{1}{r!}\int_0^1 (1-t)^r D^{r+1} f(x+tv)(v, ..., v)\mathrm{d}t. \tag{7}$$

Theorem (Implicit Function Theorem). *Let* $F : V \times W \to W$ *be a smooth map, and* $(x_0, y_0) \in V \times W$ *be such that* $D^2 f : T_{y_0}W \to T_{F(x_0,y_0)}W$ *is an isomorphism. Then around* (x_0, y_0), *the preimage* $F^{-1}F(x_0, y_0)$ *is the graph of a smooth map. That is: there are open neighborhoods* $x_0 \in U \subset V$ *and* $y_0 \in U' \subset W$ *and a smooth map* $\phi : U \to U'$, *such that*

$$F(x, y) = F(x_0, y_0) \qquad \Longleftrightarrow \qquad y = \phi(x)$$

for all $(x, y) \in U \times U'$.

Theorem (Inverse Function Theorem). *Let* $f : U \to W$ *be a smooth map, such that* $Df(x)$ *is an isomorphism. Then there is an open neighborhood* $U' \subset W$ *of* $f(x)$, *together with a smooth map* $g : U' \to U$, *such that* $f \circ g = \mathrm{id}_{U'}$ *and* $g \circ f|_{gU'} = \mathrm{id}_{gU'}$.

Consider a smooth map $v : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$. An equation of the form

$$\tfrac{d}{dt}c(t) = v(t, c(t))$$

on curves $c : (a, b) \to \mathbb{R}^m$ is called an **ordinary differential equation (ODE)**.

Theorem (Fundamental Theorem of ODEs). *Through every point* $x \in \mathbb{R}^m$ *there passes a unique maximal solution* $c_x : (a_x, b_x) \to \mathbb{R}^m$ *of the ODE* $\frac{d}{dt}c(t) = v(t, c(t))$, *with* $c_x(0) = x$. *Moreover, the map*

$$\phi : \mathbb{R} \times \mathbb{R}^m \supset U := \bigcup_{x \in \mathbb{R}^m} (a_x, b_x) \times \{x\} \longrightarrow \mathbb{R}^m, \qquad \phi(t, x) = c_x(t),$$

is smooth.

Given points $a, b \in \mathbb{R}^m$, we construct the rectangle

$$\Box_a^b = \{x \in \mathbb{R}^m \mid a_i \leqslant x_i < b_i,\ 1 \leqslant i \leqslant m\}, \qquad \Box_0^1 := \Box_{(0,...,0)}^{(1,...,1)}.$$

Consider a collection $\mathscr{D} \subset \mathscr{P}(\mathbb{R}^m)$, such that
$\mathscr{D}1)$ $A, B \in \mathscr{D}$ implies that $A \cup B, A \cap B, A \setminus B$ lie in $\mathscr{D}$;
$\mathscr{D}2)$ If $A \in \mathscr{D}$ and T is a translation, then $TA \in \mathscr{D}$;
$\mathscr{D}3)$ $\Box_0^1 \in \mathscr{D}$.

We consider functions $\mu : \mathscr{D} \to \mathbb{R}$ satisfying:
$\mu 1)$ $\mu(A) \geqslant 0$;
$\mu 2)$ If $A \cap B = \varnothing$, then $\mu(A \cup B) = \mu(A) + \mu(B)$;
$\mu 3)$ For any translation T, $\mu(A) = \mu(TA)$;
$\mu 4)$ $\mu(\square_0^1) = 1$.

For example, the collection $\mathscr{D}_{\text{pav}}$ of all **paved sets** $S \subset \mathbb{R}^m$, i.e., disjoint unions of finitely many rectangles satisfies $\mathscr{D}1)$ - $\mathscr{D}3)$, and the assignment $\mu : \mathscr{D}_{\text{pav}} \to \mathbb{R}$

$$\mu(\square_a^b) := \begin{cases} 0, & \square_a^b = \varnothing; \\ \prod_{i=1}^r (b_i - a_i), & \square_a^b \neq \varnothing. \end{cases}$$

satisfies $\mu 1)$-$\mu 4)$. Define the **inner-** and **outer content** of a subset $A \subset \mathbb{R}^m$ by

$$\mu_-(A) := \sup_{S \subset A} \mu(S), \qquad \mu_+(A) := \inf_{S \supset A} \mu(S). \tag{8}$$

A set A is **contented** if $\mu_-(A) = \mu_+(A)$, in which case we call this quantity the **content** $\mu(A)$ of A. The collection $\mathscr{D}_{\text{cont}}$ of contented sets satisfies $\mathscr{D}1)$ - $\mathscr{D}3)$, and the assignment $\mu : \mathscr{D}_{\text{cont}} \to \mathbb{R}$ of (8) satisfies $\mu 1)$-$\mu 4)$.

A set A is contented iff its boundary ∂A is contented and has content zero. Any subset of a set of content zero has content zero, and a set has content zero iff for all $\epsilon > 0$, it is a subset of a paved set of content at most ϵ. If $\phi : U \to \mathbb{R}^m$ is smooth and $A \in \mathscr{D}_{\text{cont}}$ is bounded of content zero and $\overline{A} \subset U$, then ϕA has content zero.

Exercise 80. *Let $v_1, ..., v_m \in \mathbb{R}^m$. Then*

$$A = \{\sum_1^n t_i v_i \mid t_i \in [0,1]\} \qquad \Longrightarrow \qquad \mu(A) = \sqrt{|\det\langle v_i, v_j \rangle|}.$$

A function is **paved** if it is a finite sum $f = \sum_\square a_\square \chi_\square$, where $a_\square \in \mathbb{R}$ and $\chi_\square$ is the characteristic function of the square $\square$. For a such function we define

$$\int f = \sum a_\square \mu(\square).$$

A function f is **contented** if, for all $\epsilon > 0$, there exist paved functions h, k, such that

$$h \leqslant f \leqslant k, \qquad \int (k - h) < \epsilon.$$

For example, the characteristic function χ_A of a set A is contented exactly when A is contented.

Exercise 81. *A bounded function of compact support which is continuous except at a set of content zero is contented.*

We denote by $\mathscr{C}(\mathbb{R}^m)$ the set of contented functions, and define a function $\int \mathrm{d}\mu : \mathscr{C} \to \mathbb{R}$ by:

$$\int f := \sup\{\int h \mid h \text{ is paved and } h \leqslant f\} = \inf\{\int k \mid k \text{ is paved and } f \leqslant k\}$$

Then:
$\int 1)$ $\int$ is a linear function;
$\int 2)$ $\int Tf = \int f$ for every translation T;
$\int 3)$ $\int f \geqslant 0$ if $f \geqslant 0$;
$\int 4)$ $\int \chi_{\square_0^1} = 1$

For a contented set A and a contented function f, we define the integral of f **over** A by:

$$\int_A f := \int \chi_A f.$$

Theorem. *Let f, g be contented functions, and A, A_1, A_2 contented sets. Then:*

a) If f and g coincide outside A, and $\mu(A)=0$, then $\int f=\int g$;
b) $\int_{A_1\cup A_2} f=\int_{A_1} f+\int_{A_2} f-\int_{A_1\cap A_2} f$;
c) $|\int_A f|\leqslant \sup_{x\in A}|f(x)|\mu(A)$;
d) If $\phi : U\to U'$ is a diffeomorphism between bounded open sets in $\mathbb{R}^m$, and $\operatorname{supp}(f)\subset U'$, then $f\circ\phi$ is contented and

$$\int_{U'} f=\int_U (f\circ\phi)|\det D\phi|. \tag{9}$$

3. Algebra

§1. Let $\Bbbk$ be a field. A $\Bbbk$-*algebra* is a vector space A over $\Bbbk$, endowed with a $\Bbbk$-bilinear map

$$\bullet : A\times A\longrightarrow A, (a,b)\mapsto a\bullet b.$$

A *derivation* of $(A,\bullet)$ is a linear map $D : A\to A$, such that

$$D(a\bullet b)=(Da)\bullet b+a\bullet(Db).$$

§2. A $\mathbb{Z}$-*grading* on a $\Bbbk$-vector space A is the data of a direct sum decomposition $A=\oplus_{n\in\mathbb{Z}}A^n$. Each $a\in A^n$ is said to have *degree* $|a|=n$. If $(A,\bullet)$ is a $\Bbbk$-algebra, and A has a $\mathbb{Z}$-grading, we say that $\bullet$ has *degree* k if $A^n\bullet A^m\subset A^{n+m+k}$, in which case we say that it is a *graded algebra* of degree k. A such graded algebra of degree k is *commutative* (resp., *anticommutative*) if

$$a\bullet b=(-1)^{(|a|-k)(|b|-k)}b\bullet a,\quad \text{resp.,}\quad a\bullet b=-(-1)^{(|a|-k)(|b|-k)}b\bullet a$$

§3. A linear endomorphism $D : A\to A$ of a graded vector space is said to be *graded* of degree $d\in\mathbb{Z}$ if $D(A^m)\subset A^{d+m}$ for all $m\in\mathbb{Z}$, and we denote by $\operatorname{End}^d(A)$ the $\Bbbk$-vector space of graded endomorphisms of degree d. A linear endomorphism $D\in\operatorname{End}^d(A)$ is a *graded derivation* of a graded algebra $(A,\bullet)$ of degree k if

$$D(a\bullet b)=(Da)\bullet b+(-1)^{d|a|}a\bullet(Db).$$

§4. A *pre-Lie algebra* of degree k is an anticommutative, graded algebra of degree k. In that case, $a\bullet\in\operatorname{End}^{|a|+k}(A)$. A pre-Lie algebra of degree k is a *Lie algebra* of degree k if each $a\bullet$ is a graded derivation:

$$a\bullet(b\bullet c)=((a\bullet b)\bullet c)+(-1)^{(|a|+k)|b|}b\bullet(a\bullet c).$$

EXERCISE 82. *If V is a graded vector space, then $A:=\oplus_{d\in\mathbb{Z}}\operatorname{End}^d(V)$ is a graded algebra of degree zero under composition:*

$$\circ : A^p\times A^q\longrightarrow A^{p+q},\qquad (D,D')\mapsto D\circ D',$$

and a Lie algebra of degree zero under the graded commutator*:*

$$[D,D']_c:=D\circ D'-(-1)^{dd'}D'\circ D.$$

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