So p_2 is already orthogonal to q_1, and we can take $q_2 = p_2$. For the projection of p_3 onto $W_2 = \text{Span}\{q_1, q_2\}$, compute

$$\langle p_3, q_1\rangle = \int_0^1 12t^2 \cdot 1\, dt = 4t^3\Big|_0^1 = 4$$

$$\langle q_1, q_1\rangle = \int_0^1 1 \cdot 1\, dt = t\Big|_0^1 = 1$$

$$\langle p_3, q_2\rangle = \int_0^1 12t^2(2t-1)\, dt = \int_0^1 (24t^3 - 12t^2)\, dt = 2$$

$$\langle q_2, q_2\rangle = \int_0^1 (2t-1)^2\, dt = \frac{1}{6}(2t-1)^3\Big|_0^1 = \frac{1}{3}$$

Then

$$\text{proj}_{W_2} p_3 = \frac{\langle p_3, q_1\rangle}{\langle q_1, q_1\rangle} q_1 + \frac{\langle p_3, q_2\rangle}{\langle q_2, q_2\rangle} q_2 = \frac{4}{1} q_1 + \frac{2}{1/3} q_2 = 4q_1 + 6q_2$$

and

$$q_3 = p_3 - \text{proj}_{W_2} p_3 = p_3 - 4q_1 - 6q_2$$

As a function, $q_3(t) = 12t^2 - 4 - 6(2t-1) = 12t^2 - 12t + 2$. The orthogonal basis for the subspace W is $\{q_1, q_2, q_3\}$. ■

PRACTICE PROBLEMS

Use the inner product axioms to verify the following statements.

1. $\langle \mathbf{v}, \mathbf{0}\rangle = \langle \mathbf{0}, \mathbf{v}\rangle = 0$.
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$.

6.7 EXERCISES

1. Let $\mathbb{R}^2$ have the inner product of Example 1, and let $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (5, -1)$.
   a. Find $\|\mathbf{x}\|$, $\|\mathbf{y}\|$, and $|\langle \mathbf{x}, \mathbf{y}\rangle|^2$.
   b. Describe all vectors (z_1, z_2) that are orthogonal to $\mathbf{y}$.
2. Let $\mathbb{R}^2$ have the inner product of Example 1. Show that the Cauchy–Schwarz inequality holds for $\mathbf{x} = (3, -2)$ and $\mathbf{y} = (-2, 1)$. [*Suggestion:* Study $|\langle \mathbf{x}, \mathbf{y}\rangle|^2$.]

Exercises 3–8 refer to $\mathbb{P}_2$ with the inner product given by evaluation at $-1, 0$, and 1. (See Example 2.)

3. Compute $\langle p, q\rangle$, where $p(t) = 4 + t$, $q(t) = 5 - 4t^2$.
4. Compute $\langle p, q\rangle$, where $p(t) = 3t - t^2$, $q(t) = 3 + 2t^2$.
5. Compute $\|p\|$ and $\|q\|$, for p and q in Exercise 3.
6. Compute $\|p\|$ and $\|q\|$, for p and q in Exercise 4.
7. Compute the orthogonal projection of q onto the subspace spanned by p, for p and q in Exercise 3.
8. Compute the orthogonal projection of q onto the subspace spanned by p, for p and q in Exercise 4.
9. Let $\mathbb{P}_3$ have the inner product given by evaluation at $-3, -1, 1$, and 3. Let $p_0(t) = 1$, $p_1(t) = t$, and $p_2(t) = t^2$.
   a. Compute the orthogonal projection of p_2 onto the subspace spanned by p_0 and p_1.
   b. Find a polynomial q that is orthogonal to p_0 and p_1, such that $\{p_0, p_1, q\}$ is an orthogonal basis for $\text{Span}\{p_0, p_1, p_2\}$. Scale the polynomial q so that its vector of values at $(-3, -1, 1, 3)$ is $(1, -1, -1, 1)$.
10. Let $\mathbb{P}_3$ have the inner product as in Exercise 9, with p_0, p_1, and q the polynomials described there. Find the best approximation to $p(t) = t^3$ by polynomials in $\text{Span}\{p_0, p_1, q\}$.
11. Let p_0, p_1, and p_2 be the orthogonal polynomials described in Example 5, where the inner product on $\mathbb{P}_4$ is given by evaluation at $-2, -1, 0, 1$, and 2. Find the orthogonal projection of t^3 onto $\text{Span}\{p_0, p_1, p_2\}$.
12. Find a polynomial p_3 such that $\{p_0, p_1, p_2, p_3\}$ (see Exercise 11) is an orthogonal basis for the subspace $\mathbb{P}_3$ of $\mathbb{P}_4$. Scale the polynomial p_3 so that its vector of values is $(-1, 2, 0, -2, 1)$.

13. Let A be any invertible $n \times n$ matrix. Show that for $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$, the formula $\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{u}) \cdot (A\mathbf{v}) = (A\mathbf{u})^T(A\mathbf{v})$ defines an inner product on $\mathbb{R}^n$.

14. Let T be a one-to-one linear transformation from a vector space V into $\mathbb{R}^n$. Show that for $\mathbf{u}, \mathbf{v}$ in V, the formula $\langle \mathbf{u}, \mathbf{v} \rangle = T(\mathbf{u}) \cdot T(\mathbf{v})$ defines an inner product on V.

Use the inner product axioms and other results of this section to verify the statements in Exercises 15–18.

15. $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ for all scalars c.

16. If $\{\mathbf{u}, \mathbf{v}\}$ is an orthonormal set in V, then $\|\mathbf{u} - \mathbf{v}\| = \sqrt{2}$.

17. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$.

18. $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

19. Given $a \geq 0$ and $b \geq 0$, let $\mathbf{u} = \begin{bmatrix} \sqrt{a} \\ \sqrt{b} \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \sqrt{b} \\ \sqrt{a} \end{bmatrix}$. Use the Cauchy–Schwarz inequality to compare the geometric mean $\sqrt{ab}$ with the arithmetic mean $(a + b)/2$.

20. Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Use the Cauchy–Schwarz inequality to show that

$$\left(\frac{a + b}{2}\right)^2 \leq \frac{a^2 + b^2}{2}$$

Exercises 21–24 refer to $V = C[0, 1]$, with the inner product given by an integral, as in Example 7.

21. Compute $\langle f, g \rangle$, where $f(t) = 1 - 3t^2$ and $g(t) = t - t^3$.

22. Compute $\langle f, g \rangle$, where $f(t) = 5t - 3$ and $g(t) = t^3 - t^2$.

23. Compute $\|f\|$ for f in Exercise 21.

24. Compute $\|g\|$ for g in Exercise 22.

25. Let V be the space $C[-1, 1]$ with the inner product of Example 7. Find an orthogonal basis for the subspace spanned by the polynomials $1, t$, and t^2. The polynomials in this basis are called *Legendre polynomials*.

26. Let V be the space $C[-2, 2]$ with the inner product of Example 7. Find an orthogonal basis for the subspace spanned by the polynomials $1, t$, and t^2.

27. **[M]** Let $\mathbb{P}_4$ have the inner product as in Example 5, and let p_0, p_1, p_2 be the orthogonal polynomials from that example. Using your matrix program, apply the Gram–Schmidt process to the set $\{p_0, p_1, p_2, t^3, t^4\}$ to create an orthogonal basis for $\mathbb{P}_4$.

28. **[M]** Let V be the space $C[0, 2\pi]$ with the inner product of Example 7. Use the Gram–Schmidt process to create an orthogonal basis for the subspace spanned by $\{1, \cos t, \cos^2 t, \cos^3 t\}$. Use a matrix program or computational program to compute the appropriate definite integrals.

SOLUTIONS TO PRACTICE PROBLEMS

1. By Axiom 1, $\langle \mathbf{v}, \mathbf{0} \rangle = \langle \mathbf{0}, \mathbf{v} \rangle$. Then $\langle \mathbf{0}, \mathbf{v} \rangle = \langle 0\mathbf{v}, \mathbf{v} \rangle = 0\langle \mathbf{v}, \mathbf{v} \rangle$, by Axiom 3, so $\langle \mathbf{0}, \mathbf{v} \rangle = 0$.
2. By Axioms 1, 2, and then 1 again, $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.

6.8 APPLICATIONS OF INNER PRODUCT SPACES

The examples in this section suggest how the inner product spaces defined in Section 6.7 arise in practical problems. The first example is connected with the massive least-squares problem of updating the North American Datum, described in the chapter's introductory example.

Weighted Least-Squares

Let $\mathbf{y}$ be a vector of n observations, $y_1, \ldots, y_n$, and suppose we wish to approximate $\mathbf{y}$ by a vector $\hat{\mathbf{y}}$ that belongs to some specified subspace of $\mathbb{R}^n$. (In Section 6.5, $\hat{\mathbf{y}}$ was written as $A\mathbf{x}$ so that $\hat{\mathbf{y}}$ was in the column space of A.) Denote the entries in $\hat{\mathbf{y}}$ by $\hat{y}_1, \ldots, \hat{y}_n$. Then the *sum of the squares for error*, or SS(E), in approximating $\mathbf{y}$ by $\hat{\mathbf{y}}$ is

$$\text{SS(E)} = (y_1 - \hat{y}_1)^2 + \cdots + (y_n - \hat{y}_n)^2 \tag{1}$$

This is simply $\|\mathbf{y} - \hat{\mathbf{y}}\|^2$, using the standard length in $\mathbb{R}^n$.