As n gets larger and larger, this approximation becomes arbitrarily close to the true volume. The expression on the right is a Riemann sum for the definite integral of $f(x) = \pi[g(x)]^2$. Therefore, the volume of the solid equals the value of the definite integral.

> The volume of the *solid of revolution* obtained from revolving the region below the graph of $y = g(x)$ from $x = a$ to $x = b$ about the x-axis is
> $$\int_a^b \pi[g(x)]^2\, dx.$$

EXAMPLE 6 **Volume of a Solid** Find the volume of the solid of revolution we obtain by revolving the region of Fig. 6 about the x-axis.

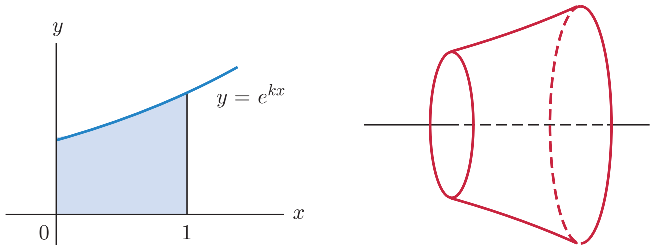

Figure 6

SOLUTION Here, $g(x) = e^{kx}$, and

$$[\text{volume}] = \int_0^1 \pi(e^{kx})^2\, dx = \int_0^1 \pi e^{2kx}\, dx = \frac{\pi}{2k}e^{2kx}\Big|_0^1 = \frac{\pi}{2k}(e^{2k} - 1).$$

▶ *Now Try Exercise 31*

EXAMPLE 7 **Volume of a Cone** Find the volume of a right circular cone of radius r and height h.

SOLUTION The cone in Fig. 7(a) is the solid of revolution swept out when the shaded region in Fig. 7(b) is revolved about the x-axis. With the formula developed previously, the volume of the cone is

$$\int_0^h \pi\left(\frac{r}{h}x\right)^2 dx = \frac{\pi r^2}{h^2}\int_0^h x^2\, dx = \frac{\pi r^2}{h^2}\frac{x^3}{3}\Big|_0^h = \frac{\pi r^2 h}{3}.$$

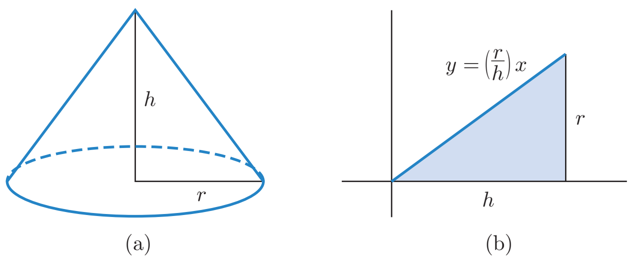

Figure 7

▶ *Now Try Exercise 35*

Check Your Understanding 6.5

1. A rock dropped from a bridge has a velocity of $-32t$ feet per second after t seconds. Find the average velocity of the rock during the first 3 seconds.

2. An investment yields \$300 per year compounded continuously for 10 years at 10% interest. What is the (future) value of this income stream at the end of 10 years?

EXERCISES 6.5

Determine the average value of $f(x)$ over the interval from $x = a$ to $x = b$, where

1. $f(x) = x^2$; $a = 0$, $b = 3$.

2. $f(x) = 1 - x$; $a = -1$, $b = 1$.

3. $f(x) = 100e^{-.5x}$; $a = 0$, $b = 4$.

4. $f(x) = 2$; $a = 0$, $b = 1$.

5. $f(x) = 1/x$; $a = 1/3$, $b = 3$.

6. $f(x) = \dfrac{1}{\sqrt{x}}$; $a = 1$, $b = 9$.

7. **Average Temperature** During a certain 12-hour period, the temperature at time t (measured in hours from the start of the period) was $T(t) = 47 + 4t - \frac{1}{3}t^2$ degrees. What was the average temperature during that period?

8. **Average Population** Assuming that a country's population is now 3 million and is growing exponentially with growth constant .02, what will be the average population during the next 50 years?

9. **Average Amount of Radium** One hundred grams of radioactive radium having a half-life of 1690 years is placed in a concrete vault. What will be the average amount of radium in the vault during the next 1000 years?

10. **Average Amount of Money** One hundred dollars is deposited in the bank at 5% interest compounded continuously. What will be the average value of the money in the account during the next 20 years?

Consumers' Surplus Find the consumers' surplus for each of the following demand curves at the given sales level x.

11. $p = 3 - \dfrac{x}{10}$; $x = 20$

12. $p = \dfrac{x^2}{200} - x + 50$; $x = 20$

13. $p = \dfrac{500}{x + 10} - 3$; $x = 40$

14. $p = \sqrt{16 - .02x}$; $x = 350$

Producers' Surplus Figure 8 shows a supply curve for a commodity. It gives the relationship between the selling price of the commodity and the quantity that producers will manufacture. At a higher selling price, a greater quantity will be produced. Therefore, the curve is increasing. If (A, B) is a point on the curve, then, to stimulate the production of A units of the commodity, the price per unit must be B dollars. Of course, some producers will be willing to produce the commodity even with a lower selling price. Since everyone receives the same price in an open efficient economy, most producers are receiving more than their minimal required price. The excess is called the *producers' surplus*. Using an argument analogous to that of the *consumers' surplus*, we can show that the total producers' surplus when the price is B is the area of the shaded region in Fig. 8. Find the producers' surplus for each of the following supply curves at the given sales level x.

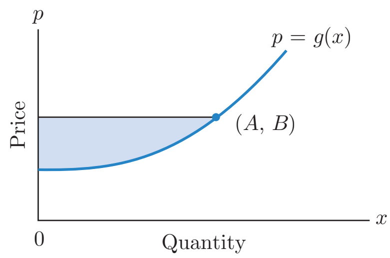

Figure 8 Producers' surplus.

15. $p = .01x + 3$; $x = 200$

16. $p = \dfrac{x^2}{9} + 1$; $x = 3$

17. $p = \dfrac{x}{2} + 7$; $x = 10$

18. $p = 1 + \frac{1}{2}\sqrt{x}$; $x = 36$

Consumers' and Producers' Surpluses For a particular commodity, the quantity produced and the unit price are given by the coordinates of the point where the supply and demand curves intersect. For each pair of supply and demand curves, determine the point of intersection (A, B) and the consumers' and producers' surplus. (See Fig. 9.)

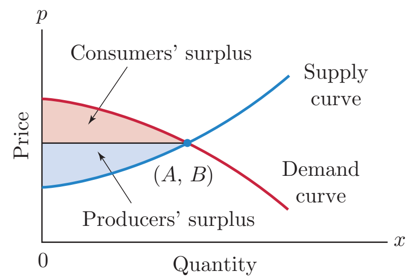

Figure 9

19. Demand curve: $p = 12 - (x/50)$; supply curve: $p = (x/20) + 5$.

20. Demand curve: $p = \sqrt{25 - .1x}$; supply curve: $p = \sqrt{.1x + 9} - 2$.

21. **Future Value** Suppose that money is deposited daily in a savings account at an annual rate of \$1000. If the account pays 5% interest compounded continuously, estimate the balance in the account at the end of 3 years.

22. **Future Value** Suppose that money is deposited daily in a savings account at an annual rate of \$2000. If the account pays 6% interest compounded continuously, approximately how much will be in the account at the end of 2 years?

23. **Future Value** Suppose that money is deposited steadily in a savings account so that \$16,000 is deposited each year. Determine the balance at the end of 4 years if the account pays 8% interest compounded continuously.

24. **Future Value** Suppose that money is deposited steadily in a savings account so that \$14,000 is deposited each year. Determine the balance at the end of 6 years if the account pays 4.5% interest compounded continuously.

25. **Future Value** An investment pays 10% interest compounded continuously. If money is invested steadily so that \$5000 is deposited each year, how much time is required until the value of the investment reaches \$140,000?

26. **Rate of a Savings Account** A savings account pays 4.25% interest compounded continuously. At what rate per year must money be deposited steadily in the account to accumulate a balance of \$100,000 after 10 years?

27. Present Value of an Income Stream Suppose that money is to be deposited daily for 5 years in a savings account at an annual rate of \$1000 and the account pays 4% interest compounded continuously. Let the interval from 0 to 5 be divided into daily subintervals, with each subinterval of duration $\Delta t = \frac{1}{365}$ year. Let $t_1, \ldots, t_n$ be points chosen from the subintervals.

(a) Show that the present value of a daily deposit at time t_i is $1000\,\Delta t\, e^{-.04 t_i}$.

(b) Find the Riemann sum corresponding to the sum of the present values of all the deposits.

(c) What is the function and interval corresponding to the Riemann sum in part (b)?

(d) Give the definite integral that approximates the Riemann sum in part (b).

(e) Evaluate the definite integral in part (d). This number is the *present value of a continuous income stream*.

28. Use the result of Exercise 27 to calculate the present value of a continuous income stream of \$5000 per year for 10 years at an interest rate of 5% compounded continuously.

Volume of Solids of Revolution Find the volume of the solid of revolution generated by revolving about the x-axis the region under each of the following curves.

29. $y = \sqrt{4 - x^2}$ from $x = -2$ to $x = 2$ (generates a sphere of radius 2)

30. $y = \sqrt{r^2 - x^2}$ from $x = -r$ to $x = r$ (generates a sphere of radius r)

31. $y = x^2$ from $x = 1$ to $x = 2$

32. $y = kx$ from $x = 0$ to $x = h$ (generates a cone)

33. $y = \sqrt{x}$ from $x = 0$ to $x = 4$ (The solid generated is called a *paraboloid*.)

34. $y = 2x - x^2$ from $x = 0$ to $x = 2$

35. $y = 2x + 1$ from $x = 0$ to $x = 1$ (The solid generated is called a *truncated cone*.)

36. $y = e^{-x}$ from $x = 0$ to $x = 1$

For the Riemann sums in Exercises 37–40, determine n, b, and $f(x)$.

37. $[(8.25)^3 + (8.75)^3 + (9.25)^3 + (9.75)^3]\,(.5);\ a = 8$

38. $\left[\frac{3}{1} + \frac{3}{1.5} + \frac{3}{2} + \frac{3}{2.5} + \frac{3}{3} + \frac{3}{3.5}\right](.5);\ a = 1$

39. $[(5 + e^5) + (6 + e^6) + (7 + e^7)]\,(1);\ a = 4$

40. $[3(.3)^2 + 3(.9)^2 + 3(1.5)^2 + 3(2.1)^2 + 3(2.7)^2]\,(.6);$ $a = 0$

41. Suppose that the interval $0 \le x \le 3$ is divided into 100 subintervals of width $\Delta x = .03$. Let $x_1, x_2, \ldots, x_{100}$ be points in these subintervals. Suppose that in a particular application we need to estimate the sum

$$(3 - x_1)^2\,\Delta x + (3 - x_2)^2\,\Delta x + \cdots + (3 - x_{100})^2\,\Delta x.$$

Show that this sum is close to 9.

42. Suppose that the interval $0 \le x \le 1$ is divided into 100 subintervals of width $\Delta x = .01$. Show that the following sum is close to 5/4.

$$\left[2(.01) + (.01)^3\right]\Delta x + \left[2(.02) + (.02)^3\right]\Delta x + \cdots + \left[2(1.0) + (1.0)^3\right]\Delta x$$

Technology Exercises

The following exercises ask for an unknown quantity x. After setting up the appropriate formula involving a definite integral, use the fundamental theorem to evaluate the definite integral as an expression in x. Because the resulting equation will be too complicated to solve algebraically, you must use a graphing utility to obtain the solution. (*Note:* If the quantity x is an interest rate paid by a savings account, it will most likely be between 0 and .10.)

43. A single deposit of \$1000 is to be made into a savings account and the interest (compounded continuously) is allowed to accumulate for 3 years. Therefore, the amount at the end of t years is $1000e^{rt}$.

(a) Find an expression (involving r) that gives the average value of the money in the account during the 3-year time period $0 \le t \le 3$.

(b) Find the interest rate r at which the average amount in the account during the 3-year period is \$1070.60.

44. A single deposit of \$100 is made into a savings account paying 4% interest compounded continuously. How long must the money be held in the account so that the average amount of money during that time period will be \$122.96?

45. Money is deposited steadily so that \$1000 is deposited each year into a savings account.

(a) Find the expression (involving r) that gives the (future) balance in the account at the end of 6 years.

(b) Find the interest rate that will result in a balance of \$6997.18 after 6 years.

46. Money is deposited steadily so that \$3000 is deposited each year into a savings account. After 10 years the balance is \$36,887. What interest rate, with interest compounded continuously, did the money earn?

Solutions to Check Your Understanding 6.5

1. By definition, the average value of the function $v(t) = -32t$ for $t = 0$ to $t = 3$ is

$$\frac{1}{3-0}\int_0^3 -32t\,dt = \frac{1}{3}(-16t^2)\Big|_0^3 = \frac{1}{3}(-16 \cdot 3^2)$$

$$= -48 \text{ feet per second.}$$

Note: There is another way to approach this problem:

$$[\text{Average velocity}] = \frac{[\text{distance traveled}]}{[\text{time elapsed}]}.$$

As we discussed in Section 6.4, distance traveled equals the area under the velocity curve. Therefore,

$$[\text{Average velocity}] = \frac{\int_0^3 -32t\,dt}{3}.$$