When we add, the intermediate terms on the right cancel, and we get

$$\begin{aligned} F'(x_1)\Delta x &+ F'(x_2)\Delta x + F'(x_3)\Delta x + \cdots + F'(x_n)\Delta x \\ &\approx (\cancel{F(x_2)} - F(x_1)) + (\cancel{F(x_3)} - \cancel{F(x_2)}) \\ &\quad + (\cancel{F(x_4)} - \cancel{F(x_3)}) + \cdots + (F(x_n + \Delta x) - \cancel{F(x_n)}) \\ &= -F(x_1) + F(x_n + \Delta x) \\ &= F(b) - F(a), \end{aligned}$$

because $x_1 = a$ and $x_n + \Delta x = b$. Thus,

$$F'(x_1)\Delta x + F'(x_2)\Delta x + F'(x_3)\Delta x + \cdots + F'(x_n)\Delta x \approx F(b) - F(a).$$

Since the approximation improves as $\Delta x \to 0$, we see that the limit (5) and, thus, (4) must hold.

INCORPORATING TECHNOLOGY

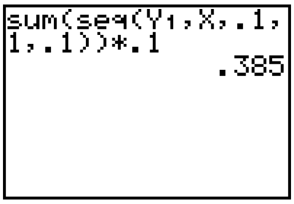


Figure 13

Riemann Sums When the points selected for a Riemann sum are all midpoints, all left endpoints, or all right endpoints, $[f(x_1) + f(x_2) + \cdots + f(x_n)]$ is the sum of the sequence of values $f(x_1), f(x_2), \ldots, f(x_n)$, where successive numbers $x_1, x_2, \ldots, x_n$ each differ by the value Δx. In such a case, this sum of a sequence can be evaluated on a graphing calculator. Figure 13 shows the computation of the Riemann sum asked for in Example 6. To do this, first set $Y_1 = X^2$. Return to the home screen and press $\boxed{\text{2nd}}$ [LIST], and move the cursor right to MATH. Press $\boxed{5}$ to display **sum(** and press $\boxed{\text{2nd}}$ [LIST]; then, move the cursor right to OPS. Press $\boxed{5}$ to display **seq(**. Now complete the expression as in Fig. 13. The numbers 0, 1, and .1 are determined as explained in Example 6.

Check Your Understanding 6.3

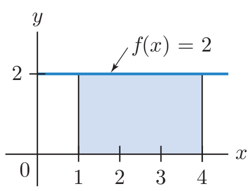
1. Determine Δx and the midpoints of the subintervals formed by partitioning the interval $-2 \le x \le 2$ into five subintervals.

2. Find the area under the curve $y = e^{x/2}$ from $x = -3$ to $x = 2$.

EXERCISES 6.3

In Exercises 1–6, compute the area of the shaded region in two different ways: (a) by using simple geometric formulas; (b) by Applying Theorem 1.

1.



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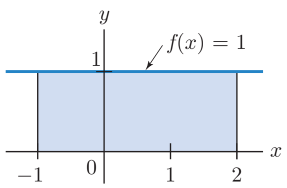

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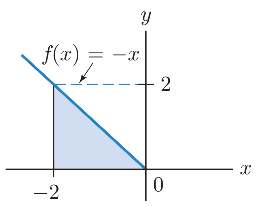

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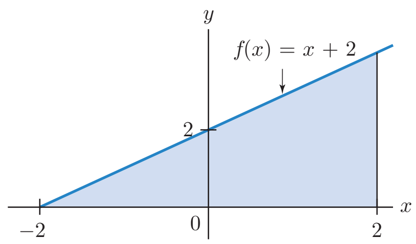

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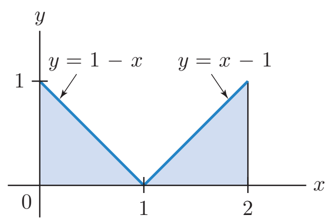

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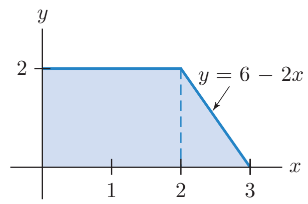
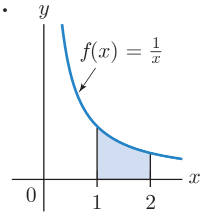


In Exercises 7–12, set-up the definite integral that gives the area of the shaded region. Do not evaluate the integral.

7.



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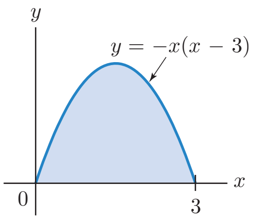

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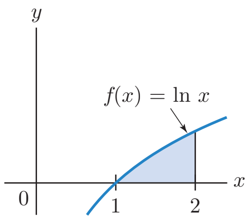

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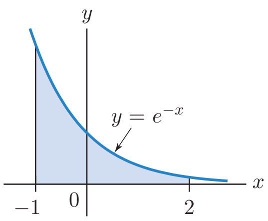

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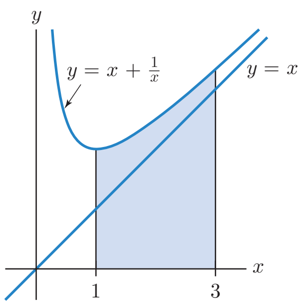

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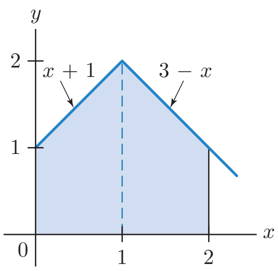

13. Use Theorem 1 to compute the shaded area in Exercise 7.

14. Use Theorem 1 to compute the shaded area in Exercise 8.

15. Use Theorem 1 to compute the shaded area in Exercise 11.

In Exercises 16–18, draw the region whose area is given by the definite integral.

16. $\int_2^4 x^2\, dx$ **17.** $\int_0^4 (8 - 2x)dx$ **18.** $\int_0^4 \sqrt{x}\, dx$

Find the area under each of the given curves.

19. $y = 4x$; $x = 2$ to $x = 3$

20. $y = 3x^2$; $x = -1$ to $x = 1$

21. $y = 3x^2 + x + 2e^{x/2}$; $x = 0$ to $x = 1$

22. $y = \sqrt{x}$; $x = 0$ to $x = 4$

23. $y = (x - 3)^4$; $x = 1$ to $x = 4$

24. $y = e^{3x}$; $x = -\frac{1}{3}$ to $x = 0$

25. Find the real number $b > 0$ so that the area under the graph of $y = x^3$ from 0 to b is equal to 4.

26. Find the real number $b > 0$ so that the area under the graph of $y = x^2$ from 0 to b is equal to the area under the graph of $y = x^3$ from 0 to b.

Determine Δx and the midpoints of the subintervals formed by partitioning the given interval into n subintervals.

27. $0 \le x \le 2$; $n = 4$ **28.** $0 \le x \le 3$; $n = 6$

29. $1 \le x \le 4$; $n = 5$ **30.** $3 \le x \le 5$; $n = 5$

In Exercises 31–36, use a Riemann sum to approximate the area under the graph of $f(x)$ on the given interval, with selected points as specified.

31. $f(x) = x^2$; $1 \le x \le 3$, $n = 4$, midpoints of subintervals

32. $f(x) = x^2$; $-2 \le x \le 2$, $n = 4$, midpoints of subintervals

33. $f(x) = x^3$; $1 \le x \le 3$, $n = 5$, left endpoints

34. $f(x) = x^3$; $0 \le x \le 1$, $n = 5$, right endpoints

35. $f(x) = e^{-x}$; $2 \le x \le 3$, $n = 5$, right endpoints

36. $f(x) = \ln x$; $2 \le x \le 4$, $n = 5$, left endpoints

In Exercises 37–40, use a Riemann sum to approximate the area under the graph of $f(x)$ in Fig. 14 on the given interval, with selected points as specified. Draw the approximating rectangles.

37. $0 \le x \le 8$, $n = 4$, midpoints of subintervals

38. $3 \le x \le 7$, $n = 4$, left endpoints

39. $4 \le x \le 9$, $n = 5$, right endpoints

40. $1 \le x \le 7$, $n = 3$, midpoints of subintervals

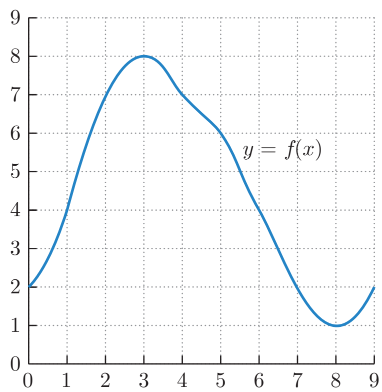

Figure 14

41. Use a Riemann sum with $n = 4$ and left endpoints to estimate the area under the graph of $f(x) = 4 - x$ on the interval $1 \le x \le 4$. Then repeat with $n = 4$ and midpoints. Compare the answers with the exact answer, 4.5, which can be computed from the formula for the area of a triangle.

42. Use a Riemann sum with $n = 4$ and right endpoints to estimate the area under the graph of $f(x) = 2x - 4$ on the interval $2 \le x \le 3$. Then, repeat with $n = 4$ and midpoints. Compare the answers with the exact answer, 1, which can be computed from the formula for the area of a triangle.

43. The graph of the function $f(x) = \sqrt{1 - x^2}$ on the interval $-1 \le x \le 1$ is a semicircle. The area under the graph is $\frac{1}{2}\pi(1)^2 = \pi/2 = 1.57080$, to five decimal places. Use a Riemann sum with $n = 5$ and midpoints to estimate the area under the graph. See Fig. 15. Carry out the calculations to five decimal places and compute the error (the difference between the estimate and 1.57080).

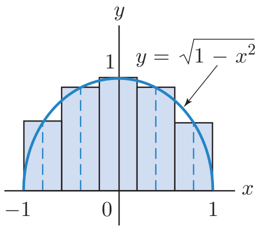

Figure 15

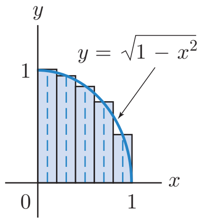

Figure 16

44. Use a Riemann sum with $n = 5$ and midpoints to estimate the area under the graph of $f(x) = \sqrt{1-x^2}$ on the interval $0 \le x \le 1$. The graph is a quarter circle, and the area under the graph is .78540, to five decimal places. See Fig. 16. Carry out the calculations to five decimal places and compute the error.

45. Estimate the area (in square feet) of the residential lot in Fig. 17.

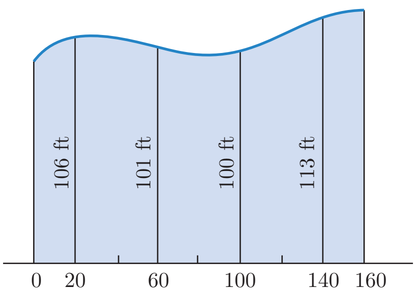

Figure 17

46. A farmer wants to divide the lot in Fig. 18 into two lots of equal area by erecting a fence that extends from the road to the river as shown. Determine the location of the fence.

In Exercises 47 and 48, we show that, as the number of subintervals increases indefinitely, the Riemann sum approximation of the area under the graph of $f(x) = x^2$ from 0 to 1 approaches the value $\frac{1}{3}$, which is the exact value of the area.

47. Verify the given formula for $n = 1, 2, 3, 4$:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

48. Partition the interval [0, 1] into n equal subintervals of length $\Delta x = 1/n$ each, and let $x_1, x_2, \ldots, x_n$ denote the right endpoints of the subintervals. Let

$$S_n = [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

denote the Riemann sum that estimates the area under the graph of $f(x) = x^2$ on the interval $0 \le x \le 1$.

(a) Show that $S_n = \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2)$.

(b) Using the previous exercise, conclude that

$$S_n = \frac{n(n+1)(2n+1)}{6n^3}.$$

(c) As n increases indefinitely, S_n approaches the area under the curve. Show that this area is 1/3.

Technology Exercises

49. The area under the graph of the function e^{-x^2} plays an important role in probability. Compute this area from -1 to 1.

50. Compute the area under the graph of $y = \frac{1}{1+x^2}$ from 0 to 5.

Evaluate a Riemann sum to approximate the area under the graph of $f(x)$ on the given interval, with points selected as specified.

51. $f(x) = x\sqrt{1+x^2}$; $1 \le x \le 3$, $n = 20$, midpoints of subintervals

52. $f(x) = \sqrt{1-x^2}$; $-1 \le x \le 1$, $n = 20$, left endpoints of subintervals

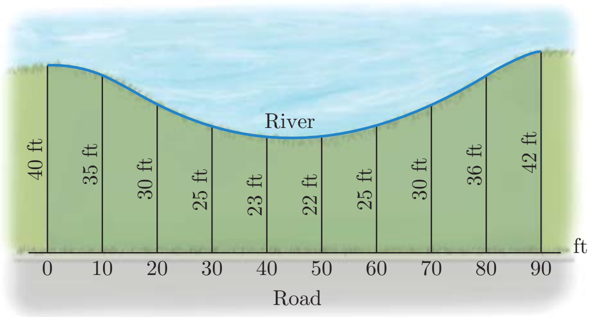

Figure 18

Solutions to Check Your Understanding 6.3

1. Since $n = 5$, $\Delta x = \frac{2-(-2)}{5} = \frac{4}{5} = .8$. The first midpoint is $x_1 = -2 + .8/2 = -1.6$. We find subsequent midpoints by successively adding .8, to obtain $x_2 = -.8$, $x_3 = 0$, $x_4 = .8$, and $x_5 = 1.6$.

2. The desired area is

$$\int_{-3}^{2} e^{x/2}\,dx = 2e^{x/2}\Big|_{-3}^{2} = 2e - 2e^{-3/2} \approx 4.99.$$