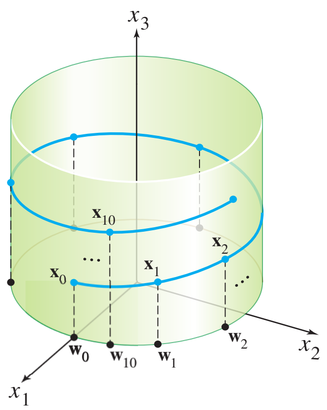

FIGURE 5
Iterates of two points under the action of a 3×3 matrix with a complex eigenvalue.

The phenomenon displayed in Example 7 persists in higher dimensions. For instance, if A is a 3×3 matrix with a complex eigenvalue, then there is a plane in $\mathbb{R}^3$ on which A acts as a rotation (possibly combined with scaling). Every vector in that plane is rotated into another point on the same plane. We say that the plane is **invariant** under A.

EXAMPLE 8 The matrix $A = \begin{bmatrix} .8 & -.6 & 0 \\ .6 & .8 & 0 \\ 0 & 0 & 1.07 \end{bmatrix}$ has eigenvalues $.8 \pm .6i$ and 1.07. Any vector $\mathbf{w}_0$ in the x_1x_2-plane (with third coordinate 0) is rotated by A into another point in the plane. Any vector $\mathbf{x}_0$ not in the plane has its x_3-coordinate multiplied by 1.07. The iterates of the points $\mathbf{w}_0 = (2, 0, 0)$ and $\mathbf{x}_0 = (2, 0, 1)$ under multiplication by A are shown in Figure 5. ■

PRACTICE PROBLEM

Show that if a and b are real, then the eigenvalues of $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ are $a \pm bi$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

5.5 EXERCISES

Let each matrix in Exercises 1–6 act on $\mathbb{C}^2$. Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$.

1. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ **2.** $\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$ **4.** $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$

In Exercises 7–12, use Example 6 to list the eigenvalues of A. In each case, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the composition of a rotation and a scaling. Give the angle φ of the rotation, where $-\pi < \varphi \le \pi$, and give the scale factor r.

7. $\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$ **8.** $\begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$

9. $\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$ **10.** $\begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$

11. $\begin{bmatrix} .1 & .1 \\ -.1 & .1 \end{bmatrix}$ **12.** $\begin{bmatrix} 0 & .3 \\ -.3 & 0 \end{bmatrix}$

In Exercises 13–20, find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = PCP^{-1}$. For Exercises 13–16, use information from Exercises 1–4.

13. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ **14.** $\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$ **16.** $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -.8 \\ 4 & -2.2 \end{bmatrix}$ **18.** $\begin{bmatrix} 1 & -1 \\ .4 & .6 \end{bmatrix}$

19. $\begin{bmatrix} 1.52 & -.7 \\ .56 & .4 \end{bmatrix}$ **20.** $\begin{bmatrix} -1.64 & -2.4 \\ 1.92 & 2.2 \end{bmatrix}$

21. In Example 2, solve the first equation in (2) for x_2 in terms of x_1, and from that produce the eigenvector $\mathbf{y} = \begin{bmatrix} 2 \\ -1 + 2i \end{bmatrix}$ for the matrix A. Show that this $\mathbf{y}$ is a (complex) multiple of the vector $\mathbf{v}_1$ used in Example 2.

22. Let A be a complex (or real) $n \times n$ matrix, and let $\mathbf{x}$ in $\mathbb{C}^n$ be an eigenvector corresponding to an eigenvalue λ in $\mathbb{C}$. Show that for each nonzero complex scalar μ, the vector $\mu\mathbf{x}$ is an eigenvector of A.

Chapter 7 will focus on matrices A with the property that $A^T = A$. Exercises 23 and 24 show that every eigenvalue of such a matrix is necessarily real.

23. Let A be an $n \times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q = \overline{\mathbf{x}}^T A\mathbf{x}$. The equalities below show that q is a real number by verifying that $\overline{q} = q$. Give a reason for each step.

$$\overline{q} \underset{(a)}{=} \overline{\overline{\mathbf{x}}^T A\mathbf{x}} \underset{(b)}{=} \mathbf{x}^T \overline{A\mathbf{x}} \underset{(c)}{=} \mathbf{x}^T A\overline{\mathbf{x}} \underset{(d)}{=} (\mathbf{x}^T A\overline{\mathbf{x}})^T \underset{(e)}{=} \overline{\mathbf{x}}^T A^T \mathbf{x} = q$$

24. Let A be an $n \times n$ real matrix with the property that $A^T = A$. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$ in $\mathbb{C}^n$, then, in fact, λ is real and the real part of $\mathbf{x}$ is an eigenvector of A. [*Hint:* Compute $\bar{\mathbf{x}}^T A\mathbf{x}$, and use Exercise 23. Also, examine the real and imaginary parts of $A\mathbf{x}$.]

25. Let A be a real $n \times n$ matrix, and let $\mathbf{x}$ be a vector in $\mathbb{C}^n$. Show that $\text{Re}(A\mathbf{x}) = A(\text{Re}\,\mathbf{x})$ and $\text{Im}(A\mathbf{x}) = A(\text{Im}\,\mathbf{x})$.

26. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$.

a. Show that $A(\text{Re}\,\mathbf{v}) = a\,\text{Re}\,\mathbf{v} + b\,\text{Im}\,\mathbf{v}$ and $A(\text{Im}\,\mathbf{v}) = -b\,\text{Re}\,\mathbf{v} + a\,\text{Im}\,\mathbf{v}$. [*Hint:* Write $\mathbf{v} = \text{Re}\,\mathbf{v} + i\,\text{Im}\,\mathbf{v}$, and compute $A\mathbf{v}$.]

b. Verify that if P and C are given as in Theorem 9, then $AP = PC$.

[**M**] In Exercises 27 and 28, find a factorization of the given matrix A in the form $A = PCP^{-1}$, where C is a block-diagonal matrix with 2×2 blocks of the form shown in Example 6. (For each conjugate pair of eigenvalues, use the real and imaginary parts of one eigenvector in $\mathbb{C}^4$ to create two columns of P.)

27. $\begin{bmatrix} .7 & 1.1 & 2.0 & 1.7 \\ -2.0 & -4.0 & -8.6 & -7.4 \\ 0 & -.5 & -1.0 & -1.0 \\ 1.0 & 2.8 & 6.0 & 5.3 \end{bmatrix}$

28. $\begin{bmatrix} -1.4 & -2.0 & -2.0 & -2.0 \\ -1.3 & -.8 & -.1 & -.6 \\ .3 & -1.9 & -1.6 & -1.4 \\ 2.0 & 3.3 & 2.3 & 2.6 \end{bmatrix}$

SOLUTION TO PRACTICE PROBLEM

Remember that it is easy to test whether a vector is an eigenvector. There is no need to examine the characteristic equation. Compute

$$A\mathbf{x} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} a + bi \\ b - ai \end{bmatrix} = (a + bi)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Thus $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector corresponding to $\lambda = a + bi$. From the discussion in this section, $\begin{bmatrix} 1 \\ i \end{bmatrix}$ must be an eigenvector corresponding to $\bar{\lambda} = a - bi$.

5.6 DISCRETE DYNAMICAL SYSTEMS

Eigenvalues and eigenvectors provide the key to understanding the long-term behavior, or *evolution*, of a dynamical system described by a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$. Such an equation was used to model population movement in Section 1.10, various Markov chains in Section 4.9, and the spotted owl population in the introductory example for this chapter. The vectors $\mathbf{x}_k$ give information about the system as time (denoted by k) passes. In the spotted owl example, for instance, $\mathbf{x}_k$ listed the numbers of owls in three age classes at time k.

The applications in this section focus on ecological problems because they are easier to state and explain than, say, problems in physics or engineering. However, dynamical systems arise in many scientific fields. For instance, standard undergraduate courses in control systems discuss several aspects of dynamical systems. The modern *state-space* design method in such courses relies heavily on matrix algebra.[1] The *steady-state response* of a control system is the engineering equivalent of what we call here the "long-term behavior" of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

[1] See G. F. Franklin, J. D. Powell, and A. Emami-Naeimi, *Feedback Control of Dynamic Systems*, 5th ed. (Upper Saddle River, NJ: Prentice-Hall, 2006). This undergraduate text has a nice introduction to dynamic models (Chapter 2). State-space design is covered in Chapters 7 and 8.