4.5 EXERCISES

For each subspace in Exercises 1–8, (a) find a basis, and (b) state the dimension.

1. $\left\{\begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s, t \text{ in } \mathbb{R}\right\}$ **2.** $\left\{\begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} : s, t \text{ in } \mathbb{R}\right\}$

3. $\left\{\begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a, b, c \text{ in } \mathbb{R}\right\}$ **4.** $\left\{\begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} : a, b \text{ in } \mathbb{R}\right\}$

5. $\left\{\begin{bmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R}\right\}$

6. $\left\{\begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} : a, b, c \text{ in } \mathbb{R}\right\}$

7. $\{(a, b, c) : a - 3b + c = 0, b - 2c = 0, 2b - c = 0\}$

8. $\{(a, b, c, d) : a - 3b + c = 0\}$

9. Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.

10. Find the dimension of the subspace H of $\mathbb{R}^2$ spanned by $\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}$.

In Exercises 11 and 12, find the dimension of the subspace spanned by the given vectors.

11. $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$

Determine the dimensions of Nul A and Col A for the matrices shown in Exercises 13–18.

13. $A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$

16. $A = \begin{bmatrix} 3 & 4 \\ -6 & 10 \end{bmatrix}$

17. $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{bmatrix}$ **18.** $A = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In Exercises 19 and 20, V is a vector space. Mark each statement True or False. Justify each answer.

19. a. The number of pivot columns of a matrix equals the dimension of its column space.

b. A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.

c. The dimension of the vector space $\mathbb{P}_4$ is 4.

d. If $\dim V = n$ and S is a linearly independent set in V, then S is a basis for V.

e. If a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly dependent.

20. a. $\mathbb{R}^2$ is a two-dimensional subspace of $\mathbb{R}^3$.

b. The number of variables in the equation $A\mathbf{x} = \mathbf{0}$ equals the dimension of Nul A.

c. A vector space is infinite-dimensional if it is spanned by an infinite set.

d. If $\dim V = n$ and if S spans V, then S is a basis of V.

e. The only three-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

21. The first four Hermite polynomials are $1, 2t, -2+4t^2$, and $-12t+8t^3$. These polynomials arise naturally in the study of certain important differential equations in mathematical physics.[2] Show that the first four Hermite polynomials form a basis of $\mathbb{P}_3$.

22. The first four Laguerre polynomials are $1, 1-t, 2-4t+t^2$, and $6-18t+9t^2-t^3$. Show that these polynomials form a basis of $\mathbb{P}_3$.

23. Let $\mathcal{B}$ be the basis of $\mathbb{P}_3$ consisting of the Hermite polynomials in Exercise 21, and let $\mathbf{p}(t) = 7 - 12t - 8t^2 + 12t^3$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathcal{B}$.

24. Let $\mathcal{B}$ be the basis of $\mathbb{P}_2$ consisting of the first three Laguerre polynomials listed in Exercise 22, and let $\mathbf{p}(t) = 7 - 8t + 3t^2$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathcal{B}$.

25. Let S be a subset of an n-dimensional vector space V, and suppose S contains fewer than n vectors. Explain why S cannot span V.

26. Let H be an n-dimensional subspace of an n-dimensional vector space V. Show that $H = V$.

27. Explain why the space $\mathbb{P}$ of all polynomials is an infinite-dimensional space.

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[2] See *Introduction to Functional Analysis*, 2nd ed., by A. E. Taylor and David C. Lay (New York: John Wiley & Sons, 1980), pp. 92–93. Other sets of polynomials are discussed there, too.

28. Show that the space $C(\mathbb{R})$ of all continuous functions defined on the real line is an infinite-dimensional space.

In Exercises 29 and 30, V is a nonzero finite-dimensional vector space, and the vectors listed belong to V. Mark each statement True or False. Justify each answer. (These questions are more difficult than those in Exercises 19 and 20.)

29. a. If there exists a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ that spans V, then $\dim V \leq p$.

b. If there exists a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then $\dim V \geq p$.

c. If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.

30. a. If there exists a linearly dependent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then $\dim V \leq p$.

b. If every set of p elements in V fails to span V, then $\dim V > p$.

c. If $p \geq 2$ and $\dim V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

Exercises 31 and 32 concern finite-dimensional vector spaces V and W and a linear transformation $T : V \to W$.

31. Let H be a nonzero subspace of V, and let $T(H)$ be the set of images of vectors in H. Then $T(H)$ is a subspace of W, by Exercise 35 in Section 4.2. Prove that $\dim T(H) \leq \dim H$.

32. Let H be a nonzero subspace of V, and suppose T is a one-to-one (linear) mapping of V into W. Prove that $\dim T(H) = \dim H$. If T happens to be a one-to-one mapping of V *onto* W, then $\dim V = \dim W$. Isomorphic finite-dimensional vector spaces have the same dimension.

33. [**M**] According to Theorem 11, a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ can be expanded to a basis for $\mathbb{R}^n$. One way to do this is to create $A = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_k \ \ \mathbf{e}_1 \ \cdots \ \mathbf{e}_n\,]$, with $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the columns of the identity matrix; the pivot columns of A form a basis for $\mathbb{R}^n$.

a. Use the method described to extend the following vectors to a basis for $\mathbb{R}^5$:

$$\mathbf{v}_1 = \begin{bmatrix} -9 \\ -7 \\ 8 \\ -5 \\ 7 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 6 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 7 \\ -8 \\ 5 \\ -7 \end{bmatrix}$$

b. Explain why the method works in general: Why are the original vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ included in the basis found for Col A? Why is Col $A = \mathbb{R}^n$?

34. [**M**] Let $\mathcal{B} = \{1, \cos t, \cos^2 t, \ldots, \cos^6 t\}$ and $\mathcal{C} = \{1, \cos t, \cos 2t, \ldots, \cos 6t\}$. Assume the following trigonometric identities (see Exercise 37 in Section 4.1).

$$\begin{aligned}
\cos 2t &= -1 + 2\cos^2 t \\
\cos 3t &= -3\cos t + 4\cos^3 t \\
\cos 4t &= 1 - 8\cos^2 t + 8\cos^4 t \\
\cos 5t &= 5\cos t - 20\cos^3 t + 16\cos^5 t \\
\cos 6t &= -1 + 18\cos^2 t - 48\cos^4 t + 32\cos^6 t
\end{aligned}$$

Let H be the subspace of functions spanned by the functions in $\mathcal{B}$. Then $\mathcal{B}$ is a basis for H, by Exercise 38 in Section 4.3.

a. Write the $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$, and use them to show that $\mathcal{C}$ is a linearly independent set in H.

b. Explain why $\mathcal{C}$ is a basis for H.

SOLUTIONS TO PRACTICE PROBLEMS

1. a. False. Consider the set $\{\mathbf{0}\}$.

b. True. By the Spanning Set Theorem, S contains a basis for V; call that basis S'. Then T will contain more vectors than S'. By Theorem 9, T is linearly dependent.

2. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a basis for $H \cap K$. Notice $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly independent subset of H, hence by Theorem 11, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ can be expanded, if necessary, to a basis for H. Since the dimension of a subspace is just the number of vectors in a basis, it follows that $\dim\,(H \cap K) = p \leq \dim H$.

4.6 RANK

With the aid of vector space concepts, this section takes a look *inside* a matrix and reveals several interesting and useful relationships hidden in its rows and columns.

For instance, imagine placing 2000 random numbers into a 40×50 matrix A and then determining both the maximum number of linearly independent columns in A and the maximum number of linearly independent columns in A^T (rows in A). Remarkably,