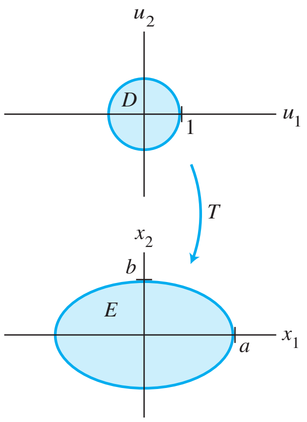
> The conclusions of Theorem 10 hold whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

EXAMPLE 5 Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$



SOLUTION We claim that E is the image of the unit disk D under the linear transformation T determined by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, because if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{x} = A\mathbf{u}$, then

$$u_1 = \frac{x_1}{a} \quad \text{and} \quad u_2 = \frac{x_2}{b}$$

It follows that $\mathbf{u}$ is in the unit disk, with $u_1^2 + u_2^2 \leq 1$, if and only if $\mathbf{x}$ is in E, with $(x_1/a)^2 + (x_2/b)^2 \leq 1$. By the generalization of Theorem 10,

$$\begin{aligned} \{\text{area of ellipse}\} &= \{\text{area of } T(D)\} \\ &= |\det A| \cdot \{\text{area of } D\} \\ &= ab \cdot \pi(1)^2 = \pi ab \end{aligned}$$

■

PRACTICE PROBLEM

Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, and let $A = \begin{bmatrix} 1 & -.1 \\ 0 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

3.3 EXERCISES

Use Cramer's rule to compute the solutions of the systems in Exercises 1–6.

1. $\begin{aligned} 5x_1 + 7x_2 &= 3 \\ 2x_1 + 4x_2 &= 1 \end{aligned}$

2. $\begin{aligned} 4x_1 + x_2 &= 6 \\ 3x_1 + 2x_2 &= 7 \end{aligned}$

3. $\begin{aligned} 3x_1 - 2x_2 &= 3 \\ -4x_1 + 6x_2 &= -5 \end{aligned}$

4. $\begin{aligned} -5x_1 + 2x_2 &= 9 \\ 3x_1 - x_2 &= -4 \end{aligned}$

5. $\begin{aligned} x_1 + x_2 &= 3 \\ -3x_1 + 2x_3 &= 0 \\ x_2 - 2x_3 &= 2 \end{aligned}$

6. $\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ -x_1 + 2x_3 &= 2 \\ 3x_1 + x_2 &= 2 \end{aligned}$

In Exercises 7–10, determine the values of the parameter s for which the system has a unique solution, and describe the solution.

7. $\begin{aligned} 6sx_1 + 4x_2 &= 5 \\ 9x_1 + 2sx_2 &= -2 \end{aligned}$

8. $\begin{aligned} 3sx_1 + 5x_2 &= 3 \\ 12x_1 + 5sx_2 &= 2 \end{aligned}$

9. $\begin{aligned} sx_1 + 2sx_2 &= -1 \\ 3x_1 + 6sx_2 &= 4 \end{aligned}$

10. $\begin{aligned} sx_1 - 2x_2 &= 1 \\ 4sx_1 + 4sx_2 &= 2 \end{aligned}$

In Exercises 11–16, compute the adjugate of the given matrix, and then use Theorem 8 to give the inverse of the matrix.

11. $\begin{bmatrix} 0 & -2 & -1 \\ 5 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 & 3 \\ -2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 4 \end{bmatrix}$

15. $\begin{bmatrix} 5 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

17. Show that if A is 2×2, then Theorem 8 gives the same formula for A^{-1} as that given by Theorem 4 in Section 2.2.

18. Suppose that all the entries in A are integers and $\det A = 1$. Explain why all the entries in A^{-1} are integers.

In Exercises 19–22, find the area of the parallelogram whose vertices are listed.

19. $(0, 0), (5, 2), (6, 4), (11, 6)$

20. $(0, 0), (-2, 4), (4, -5), (2, -1)$

21. $(-2, 0), (0, 3), (1, 3), (-1, 0)$

22. $(0, -2), (5, -2), (-3, 1), (2, 1)$

23. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -3)$, $(1, 2, 4)$, and $(5, 1, 0)$.

24. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 3, 0)$, $(-2, 0, 2)$, and $(-1, 3, -1)$.

25. Use the concept of volume to explain why the determinant of a 3×3 matrix A is zero if and only if A is not invertible. Do not appeal to Theorem 4 in Section 3.2. [*Hint:* Think about the columns of A.]

26. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation, and let $\mathbf{p}$ be a vector and S a set in $\mathbb{R}^m$. Show that the image of $\mathbf{p} + S$ under T is the translated set $T(\mathbf{p}) + T(S)$ in $\mathbb{R}^n$.

27. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

28. Repeat Exercise 27 with $\mathbf{b}_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}$.

29. Find a formula for the area of the triangle whose vertices are $\mathbf{0}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ in $\mathbb{R}^2$.

30. Let R be the triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that

$$\{\text{area of triangle}\} = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

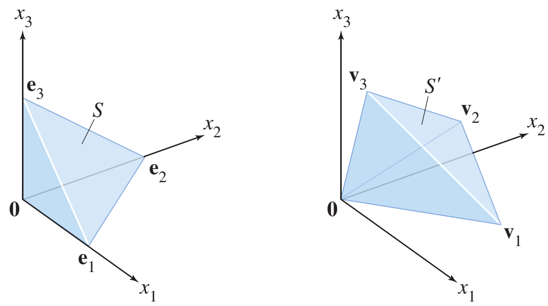
[*Hint:* Translate R to the origin by subtracting one of the vertices, and use Exercise 29.]

31. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b, and c are positive numbers. Let S be the unit ball, whose bounding surface has the equation $x_1^2 + x_2^2 + x_3^2 = 1$.

a. Show that $T(S)$ is bounded by the ellipsoid with the equation $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} + \dfrac{x_3^2}{c^2} = 1$.

b. Use the fact that the volume of the unit ball is $4\pi/3$ to determine the volume of the region bounded by the ellipsoid in part (a).

32. Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $\mathbf{0}$, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, and let S' be the tetrahedron with vertices at vectors $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. See the figure.



a. Describe a linear transformation that maps S onto S'.

b. Find a formula for the volume of the tetrahedron S' using the fact that

$$\{\text{volume of } S\} = (1/3) \cdot \{\text{area of base}\} \cdot \{\text{height}\}$$

33. [**M**] Test the inverse formula of Theorem 8 for a random 4×4 matrix A. Use your matrix program to compute the cofactors of the 3×3 submatrices, construct the adjugate, and set $B = (\text{adj}\, A)/(\det A)$. Then compute $B - \text{inv}(A)$, where $\text{inv}(A)$ is the inverse of A as computed by the matrix program. Use floating point arithmetic with the maximum possible number of decimal places. Report your results.

34. [**M**] Test Cramer's rule for a random 4×4 matrix A and a random 4×1 vector $\mathbf{b}$. Compute each entry in the solution of $A\mathbf{x} = \mathbf{b}$, and compare these entries with the entries in $A^{-1}\mathbf{b}$. Write the command (or keystrokes) for your matrix program that uses Cramer's rule to produce the second entry of $\mathbf{x}$.

35. [**M**] If your version of MATLAB has the `flops` command, use it to count the number of floating point operations to compute A^{-1} for a random 30×30 matrix. Compare this number with the number of flops needed to form $(\text{adj}\, A)/(\det A)$.

SOLUTION TO PRACTICE PROBLEM

The area of S is $\left|\det \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}\right| = 14$, and $\det A = 2$. By Theorem 10, the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is

$$|\det A| \cdot \{\text{area of } S\} = 2 \cdot 14 = 28$$