## NUMERICAL NOTES

1. The fastest way to obtain $AB$ on a computer depends on the way in which the computer stores matrices in its memory. The standard high-performance algorithms, such as in LAPACK, calculate $AB$ by columns, as in our definition of the product. (A version of LAPACK written in C++ calculates $AB$ by rows.)
2. The definition of $AB$ lends itself well to parallel processing on a computer. The columns of $B$ are assigned individually or in groups to different processors, which independently and hence simultaneously compute the corresponding columns of $AB$.

## PRACTICE PROBLEMS

1. Since vectors in $\mathbb{R}^n$ may be regarded as $n \times 1$ matrices, the properties of transposes in Theorem 3 apply to vectors, too. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$, and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?

2. Let $A$ be a $4 \times 4$ matrix and let $\mathbf{x}$ be a vector in $\mathbb{R}^4$. What is the fastest way to compute $A^2\mathbf{x}$? Count the multiplications.
3. Suppose $A$ is an $m \times n$ matrix, all of whose rows are identical. Suppose $B$ is an $n \times p$ matrix, all of whose columns are identical. What can be said about the entries in $AB$?

# 2.1 EXERCISES

In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1. $-2A$, $B - 2A$, $AC$, $CD$
2. $A + 2B$, $3C - E$, $CB$, $EB$

In the rest of this exercise set and in those to follow, you should assume that each matrix expression is defined. That is, the sizes of the matrices (and vectors) involved "match" appropriately.

3. Let $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$. Compute $3I_2 - A$ and $(3I_2)A$.
4. Compute $A - 5I_3$ and $(5I_3)A$, when

$$A = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}.$$

In Exercises 5 and 6, compute the product $AB$ in two ways: (a) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately, and (b) by the row–column rule for computing $AB$.

5. $A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$
6. $A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$
7. If a matrix $A$ is $5 \times 3$ and the product $AB$ is $5 \times 7$, what is the size of $B$?
8. How many rows does $B$ have if $BC$ is a $3 \times 4$ matrix?
9. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of $k$, if any, will make $AB = BA$?
10. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.
11. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute $AD$ and $DA$. Explain how the columns or rows of $A$ change when $A$ is multiplied by $D$ on the right or on the left. Find a $3 \times 3$ matrix $B$, not the identity matrix or the zero matrix, such that $AB = BA$.

**12.** Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a $2 \times 2$ matrix $B$ such that $AB$ is the zero matrix. Use two different nonzero columns for $B$.

**13.** Let $\mathbf{r}_1, \dots, \mathbf{r}_p$ be vectors in $\mathbb{R}^n$, and let $Q$ be an $m \times n$ matrix. Write the matrix $[\,Q\mathbf{r}_1 \cdots Q\mathbf{r}_p\,]$ as a *product* of two matrices (neither of which is an identity matrix).

**14.** Let $U$ be the $3 \times 2$ cost matrix described in Example 6 of Section 1.8. The first column of $U$ lists the costs per dollar of output for manufacturing product $B$, and the second column lists the costs per dollar of output for product $C$. (The costs are categorized as materials, labor, and overhead.) Let $\mathbf{q}_1$ be a vector in $\mathbb{R}^2$ that lists the output (measured in dollars) of products B and C manufactured during the first quarter of the year, and let $\mathbf{q}_2, \mathbf{q}_3$, and $\mathbf{q}_4$ be the analogous vectors that list the amounts of products B and C manufactured in the second, third, and fourth quarters, respectively. Give an economic description of the data in the matrix $UQ$, where $Q = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4]$.

Exercises 15 and 16 concern arbitrary matrices $A$, $B$, and $C$ for which the indicated sums and products are defined. Mark each statement True or False. Justify each answer.

**15.** a. If $A$ and $B$ are $2 \times 2$ with columns $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}_1, \mathbf{b}_2$, respectively, then $AB = [\mathbf{a}_1\mathbf{b}_1 \quad \mathbf{a}_2\mathbf{b}_2]$.

b. Each column of $AB$ is a linear combination of the columns of $B$ using weights from the corresponding column of $A$.

c. $AB + AC = A(B + C)$

d. $A^T + B^T = (A + B)^T$

e. The transpose of a product of matrices equals the product of their transposes in the same order.

**16.** a. If $A$ and $B$ are $3 \times 3$ and $B = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3]$, then $AB = [A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3]$.

b. The second row of $AB$ is the second row of $A$ multiplied on the right by $B$.

c. $(AB)C = (AC)B$

d. $(AB)^T = A^T B^T$

e. The transpose of a sum of matrices equals the sum of their transposes.

**17.** If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, determine the first and second columns of $B$.

**18.** Suppose the first two columns, $\mathbf{b}_1$ and $\mathbf{b}_2$, of $B$ are equal. What can you say about the columns of $AB$ (if $AB$ is defined)? Why?

**19.** Suppose the third column of $B$ is the sum of the first two columns. What can you say about the third column of $AB$? Why?

**20.** Suppose the second column of $B$ is all zeros. What can you say about the second column of $AB$?

**21.** Suppose the last column of $AB$ is entirely zero but $B$ itself has no column of zeros. What can you say about the columns of $A$?

**22.** Show that if the columns of $B$ are linearly dependent, then so are the columns of $AB$.

**23.** Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why $A$ cannot have more columns than rows.

**24.** Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. [*Hint:* Think about the equation $AD\mathbf{b} = \mathbf{b}$.] Explain why $A$ cannot have more rows than columns.

**25.** Suppose $A$ is an $m \times n$ matrix and there exist $n \times m$ matrices $C$ and $D$ such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$. [*Hint:* Think about the product $CAD$.]

**26.** Suppose $A$ is a $3 \times n$ matrix whose columns span $\mathbb{R}^3$. Explain how to construct an $n \times 3$ matrix $D$ such that $AD = I_3$.

In Exercises 27 and 28, view vectors in $\mathbb{R}^n$ as $n \times 1$ matrices. For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the matrix product $\mathbf{u}^T\mathbf{v}$ is a $1 \times 1$ matrix, called the **scalar product**, or **inner product**, of $\mathbf{u}$ and $\mathbf{v}$. It is usually written as a single real number without brackets. The matrix product $\mathbf{u}\mathbf{v}^T$ is an $n \times n$ matrix, called the **outer product** of $\mathbf{u}$ and $\mathbf{v}$. The products $\mathbf{u}^T\mathbf{v}$ and $\mathbf{u}\mathbf{v}^T$ will appear later in the text.

**27.** Let $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, $\mathbf{u}\mathbf{v}^T$, and $\mathbf{v}\mathbf{u}^T$.

**28.** If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$, how are $\mathbf{u}^T\mathbf{v}$ and $\mathbf{v}^T\mathbf{u}$ related? How are $\mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$ related?

**29.** Prove Theorem 2(b) and 2(c). Use the row–column rule. The $(i, j)$-entry in $A(B + C)$ can be written as

$$a_{i1}(b_{1j} + c_{1j}) + \cdots + a_{in}(b_{nj} + c_{nj}) \text{ or } \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

**30.** Prove Theorem 2(d). [*Hint:* The $(i, j)$-entry in $(rA)B$ is $(ra_{i1})b_{1j} + \cdots + (ra_{in})b_{nj}$.]

**31.** Show that $I_m A = A$ when $A$ is an $m \times n$ matrix. You can assume $I_m\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^m$.

**32.** Show that $AI_n = A$ when $A$ is an $m \times n$ matrix. [*Hint:* Use the (column) definition of $AI_n$.]

**33.** Prove Theorem 3(d). [*Hint:* Consider the $j$th row of $(AB)^T$.]

**34.** Give a formula for $(AB\mathbf{x})^T$, where $\mathbf{x}$ is a vector and $A$ and $B$ are matrices of appropriate sizes.

**35.** **[M]** Read the documentation for your matrix program, and write the commands that will produce the following matrices (without keying in each entry of the matrix).

a. A $5 \times 6$ matrix of zeros

b. A $3 \times 5$ matrix of ones

c. The $6 \times 6$ identity matrix

d. A $5 \times 5$ diagonal matrix, with diagonal entries $3, 5, 7, 2, 4$

A useful way to test new ideas in matrix algebra, or to make conjectures, is to make calculations with matrices selected at random. Checking a property for a few matrices does not prove that the property holds in general, but it makes the property more believable. Also, if the property is actually false, you may discover this when you make a few calculations.

**36.** [**M**] Write the command(s) that will create a $6 \times 4$ matrix with random entries. In what range of numbers do the entries lie? Tell how to create a $3 \times 3$ matrix with random integer entries between $-9$ and $9$. [*Hint:* If $x$ is a random number such that $0 < x < 1$, then $-9.5 < 19(x - .5) < 9.5$.]

**37.** [**M**] Construct a random $4 \times 4$ matrix $A$ and test whether $(A + I)(A - I) = A^2 - I$. The best way to do this is to compute $(A + I)(A - I) - (A^2 - I)$ and verify that this difference is the zero matrix. Do this for three random matrices. Then test $(A + B)(A - B) = A^2 - B^2$ the same way for three pairs of random $4 \times 4$ matrices. Report your conclusions.

**38.** [**M**] Use at least three pairs of random $4 \times 4$ matrices $A$ and $B$ to test the equalities $(A + B)^T = A^T + B^T$ and $(AB)^T = A^T B^T$. (See Exercise 37.) Report your conclusions. [*Note:* Most matrix programs use $A'$ for $A^T$.]

**39.** [**M**] Let

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute $S^k$ for $k = 2, \dots, 6$.

**40.** [**M**] Describe in words what happens when you compute $A^5$, $A^{10}$, $A^{20}$, and $A^{30}$ for

$$A = \begin{bmatrix} 1/6 & 1/2 & 1/3 \\ 1/2 & 1/4 & 1/4 \\ 1/3 & 1/4 & 5/12 \end{bmatrix}$$

## SOLUTIONS TO PRACTICE PROBLEMS

**1.** $A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. So $(A\mathbf{x})^T = \begin{bmatrix} -4 & 2 \end{bmatrix}$. Also,

$$\mathbf{x}^T A^T = \begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \end{bmatrix}.$$

The quantities $(A\mathbf{x})^T$ and $\mathbf{x}^T A^T$ are equal, by Theorem 3(d). Next,

$$\mathbf{x}\mathbf{x}^T = \begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}$$

$$\mathbf{x}^T\mathbf{x} = \begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 25 + 9 \end{bmatrix} = 34$$

A $1 \times 1$ matrix such as $\mathbf{x}^T\mathbf{x}$ is usually written without the brackets. Finally, $A^T\mathbf{x}^T$ is not defined, because $\mathbf{x}^T$ does not have two rows to match the two columns of $A^T$.

**2.** The fastest way to compute $A^2\mathbf{x}$ is to compute $A(A\mathbf{x})$. The product $A\mathbf{x}$ requires 16 multiplications, 4 for each entry, and $A(A\mathbf{x})$ requires 16 more. In contrast, the product $A^2$ requires 64 multiplications, 4 for each of the 16 entries in $A^2$. After that, $A^2\mathbf{x}$ takes 16 more multiplications, for a total of 80.

**3.** First observe that by the definition of matrix multiplication,

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_1 \quad \cdots \quad A\mathbf{b}_1],$$

so the columns of $AB$ are identical. Next, recall that $\text{row}_i(AB) = \text{row}_i(A) \cdot B$. Since all the rows of $A$ are identical, all the rows of $AB$ are identical. Putting this information about the rows and columns together, it follows that all the entries in $AB$ are the same.

# 2.2 THE INVERSE OF A MATRIX

Matrix algebra provides tools for manipulating matrix equations and creating various useful formulas in ways similar to doing ordinary algebra with real numbers. This section