Also, the total flow into the network $(500 + 300 + 100 + 400)$ equals the total flow out of the network $(300 + x_3 + 600)$, which simplifies to $x_3 = 400$. Combine this equation with a rearrangement of the first four equations to obtain the following system of equations:

$$\begin{aligned}
x_1 + x_2 \phantom{{}- x_3 + x_4 + x_5} &= 800\\
x_2 - x_3 + x_4 \phantom{{}+ x_5} &= 300\\
x_4 + x_5 &= 500\\
x_1 \phantom{{}+ x_2 - x_3 + x_4} + x_5 &= 600\\
x_3 &= 400
\end{aligned}$$

Row reduction of the associated augmented matrix leads to

$$\begin{aligned}
x_1 \phantom{{}+ x_2 + x_3 + x_4} + x_5 &= 600\\
x_2 \phantom{{}+ x_3 + x_4} - x_5 &= 200\\
x_3 &= 400\\
x_4 + x_5 &= 500
\end{aligned}$$

The general flow pattern for the network is described by

$$\begin{cases}
x_1 = 600 - x_5\\
x_2 = 200 + x_5\\
x_3 = 400\\
x_4 = 500 - x_5\\
x_5 \text{ is free}
\end{cases}$$

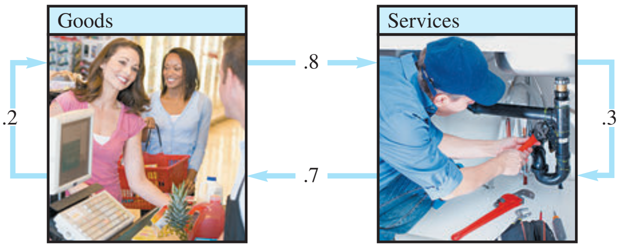
A negative flow in a network branch corresponds to flow in the direction opposite to that shown on the model. Since the streets in this problem are one-way, none of the variables here can be negative. This fact leads to certain limitations on the possible values of the variables. For instance, $x_5 \le 500$ because x_4 cannot be negative. Other constraints on the variables are considered in Practice Problem 2. ■

PRACTICE PROBLEMS

1. Suppose an economy has three sectors: Agriculture, Mining, and Manufacturing. Agriculture sells 5% of its output to Mining and 30% to Manufacturing, and retains the rest. Mining sells 20% of its output to Agriculture and 70% to Manufacturing, and retains the rest. Manufacturing sells 20% of its output to Agriculture and 30% to Mining, and retains the rest. Determine the exchange table for this economy, where the columns describe how the output of each sector is exchanged among the three sectors.
2. Consider the network flow studied in Example 2. Determine the possible range of values of x_1 and x_2. [*Hint:* The example showed that $x_5 \le 500$. What does this imply about x_1 and x_2? Also, use the fact that $x_5 \ge 0$.]

1.6 EXERCISES

1. Suppose an economy has only two sectors, Goods and Services. Each year, Goods sells 80% of its output to Services and keeps the rest, while Services sells 70% of its output to Goods and retains the rest. Find equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector's income match its expenditures.



2. Find another set of equilibrium prices for the economy in Example 1. Suppose the same economy used Japanese yen instead of dollars to measure the value of the various sectors' outputs. Would this change the problem in any way? Discuss.

3. Consider an economy with three sectors, Chemicals & Metals, Fuels & Power, and Machinery. Chemicals sells 30% of its output to Fuels and 50% to Machinery and retains the rest. Fuels sells 80% of its output to Chemicals and 10% to Machinery and retains the rest. Machinery sells 40% to Chemicals and 40% to Fuels and retains the rest.

a. Construct the exchange table for this economy.

b. Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.

c. [**M**] Find a set of equilibrium prices when the price for the Machinery output is 100 units.

4. Suppose an economy has four sectors, Agriculture (A), Energy (E), Manufacturing (M), and Transportation (T). Sector A sells 10% of its output to E and 25% to M and retains the rest. Sector E sells 30% of its output to A, 35% to M, and 25% to T and retains the rest. Sector M sells 30% of its output to A, 15% to E, and 40% to T and retains the rest. Sector T sells 20% of its output to A, 10% to E, and 30% to M and retains the rest.

a. Construct the exchange table for this economy.

b. [**M**] Find a set of equilibrium prices for the economy.

Balance the chemical equations in Exercises 5–10 using the vector equation approach discussed in this section.

5. Boron sulfide reacts violently with water to form boric acid and hydrogen sulfide gas (the smell of rotten eggs). The unbalanced equation is

$$B_2S_3 + H_2O \rightarrow H_3BO_3 + H_2S$$

[For each compound, construct a vector that lists the numbers of atoms of boron, sulfur, hydrogen, and oxygen.]

6. When solutions of sodium phosphate and barium nitrate are mixed, the result is barium phosphate (as a precipitate) and sodium nitrate. The unbalanced equation is

$$Na_3PO_4 + Ba(NO_3)_2 \rightarrow Ba_3(PO_4)_2 + NaNO_3$$

[For each compound, construct a vector that lists the numbers of atoms of sodium (Na), phosphorus, oxygen, barium, and nitrogen. For instance, barium nitrate corresponds to $(0, 0, 6, 1, 2)$.]

7. Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):

$$NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + H_2O + CO_2$$

8. The following reaction between potassium permanganate ($KMnO_4$) and manganese sulfate in water produces manganese dioxide, potassium sulfate, and sulfuric acid:

$$KMnO_4 + MnSO_4 + H_2O \rightarrow MnO_2 + K_2SO_4 + H_2SO_4$$

[For each compound, construct a vector that lists the numbers of atoms of potassium (K), manganese, oxygen, sulfur, and hydrogen.]

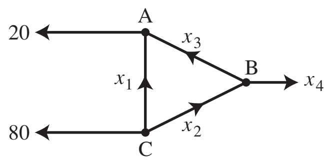
9. [**M**] If possible, use exact arithmetic or rational format for calculations in balancing the following chemical reaction:

$$PbN_6 + CrMn_2O_8 \rightarrow Pb_3O_4 + Cr_2O_3 + MnO_2 + NO$$

10. [**M**] The chemical reaction below can be used in some industrial processes, such as the production of arsene (AsH_3). Use exact arithmetic or rational format for calculations to balance this equation.

$$MnS + As_2Cr_{10}O_{35} + H_2SO_4$$
$$\rightarrow HMnO_4 + AsH_3 + CrS_3O_{12} + H_2O$$

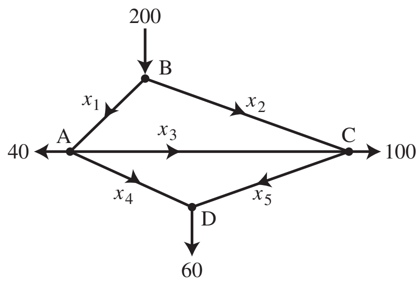
11. Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the largest possible value for x_3?



12. a. Find the general traffic pattern in the freeway network shown in the figure. (Flow rates are in cars/minute.)

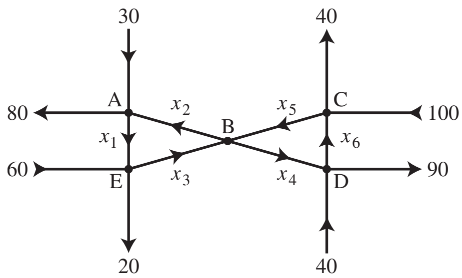
b. Describe the general traffic pattern when the road whose flow is x_4 is closed.

c. When $x_4 = 0$, what is the minimum value of x_1?

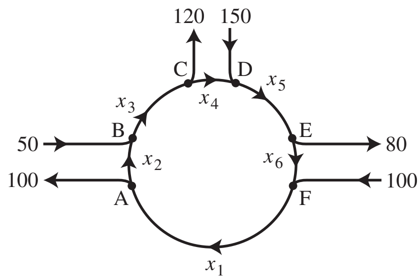


13. a. Find the general flow pattern in the network shown in the figure.

b. Assuming that the flow must be in the directions indicated, find the minimum flows in the branches denoted by x_2, x_3, x_4, and x_5.



14. Intersections in England are often constructed as one-way "roundabouts," such as the one shown in the figure. Assume that traffic must travel in the directions shown. Find the general solution of the network flow. Find the smallest possible value for x_6.



SOLUTIONS TO PRACTICE PROBLEMS

1. Write the percentages as decimals. Since all output must be taken into account, each column must sum to 1. This fact helps to fill in any missing entries.

Distribution of Output from:			
Agriculture	**Mining**	**Manufacturing**	**Purchased by:**
.65	.20	.20	Agriculture
.05	.10	.30	Mining
.30	.70	.50	Manufacturing

2. Since $x_5 \leq 500$, the equations D and A for x_1 and x_2 imply that $x_1 \geq 100$ and $x_2 \leq 700$. The fact that $x_5 \geq 0$ implies that $x_1 \leq 600$ and $x_2 \geq 200$. So, $100 \leq x_1 \leq 600$, and $200 \leq x_2 \leq 700$.

1.7 LINEAR INDEPENDENCE

The homogeneous equations in Section 1.5 can be studied from a different perspective by writing them as vector equations. In this way, the focus shifts from the unknown solutions of $A\mathbf{x} = \mathbf{0}$ to the vectors that appear in the vector equations.