**NUMERICAL NOTE**

To optimize a computer algorithm to compute $A\mathbf{x}$, the sequence of calculations should involve data stored in contiguous memory locations. The most widely used professional algorithms for matrix computations are written in Fortran, a language that stores a matrix as a set of columns. Such algorithms compute $A\mathbf{x}$ as a linear combination of the columns of $A$. In contrast, if a program is written in the popular language C, which stores matrices by rows, $A\mathbf{x}$ should be computed via the alternative rule that uses the rows of $A$.

**PROOF OF THEOREM 4** As was pointed out after Theorem 4, statements (a), (b), and (c) are logically equivalent. So, it suffices to show (for an arbitrary matrix $A$) that (a) and (d) are either both true or both false. This will tie all four statements together.

Let $U$ be an echelon form of $A$. Given $\mathbf{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $[\,A \quad \mathbf{b}\,]$ to an augmented matrix $[\,U \quad \mathbf{d}\,]$ for some $\mathbf{d}$ in $\mathbb{R}^m$:

$$[\,A \quad \mathbf{b}\,] \sim \cdots \sim [\,U \quad \mathbf{d}\,]$$

If statement (d) is true, then each row of $U$ contains a pivot position and there can be no pivot in the augmented column. So $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$, and (a) is true. If (d) is false, the last row of $U$ is all zeros. Let $\mathbf{d}$ be any vector with a 1 in its last entry. Then $[\,U \quad \mathbf{d}\,]$ represents an *inconsistent* system. Since row operations are reversible, $[\,U \quad \mathbf{d}\,]$ can be transformed into the form $[\,A \quad \mathbf{b}\,]$. The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false. ■

## PRACTICE PROBLEMS

1. Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$. It can be shown that $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Use this fact to exhibit $\mathbf{b}$ as a specific linear combination of the columns of $A$.
2. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. Verify Theorem 5(a) in this case by computing $A(\mathbf{u} + \mathbf{v})$ and $A\mathbf{u} + A\mathbf{v}$.
3. Construct a $3 \times 3$ matrix $A$ and vectors $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ so that $A\mathbf{x} = \mathbf{b}$ has a solution, but $A\mathbf{x} = \mathbf{c}$ does not.

## 1.4 EXERCISES

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$

2. $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.

5. $\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

6. $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$

**7.** $x_1\begin{bmatrix}4\\-1\\7\\-4\end{bmatrix} + x_2\begin{bmatrix}-5\\3\\-5\\1\end{bmatrix} + x_3\begin{bmatrix}7\\-8\\0\\2\end{bmatrix} = \begin{bmatrix}6\\-8\\0\\-7\end{bmatrix}$

**8.** $z_1\begin{bmatrix}4\\-2\end{bmatrix} + z_2\begin{bmatrix}-4\\5\end{bmatrix} + z_3\begin{bmatrix}-5\\4\end{bmatrix} + z_4\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}4\\13\end{bmatrix}$

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

**9.** $\begin{aligned} 3x_1 + x_2 - 5x_3 &= 9 \\ x_2 + 4x_3 &= 0 \end{aligned}$

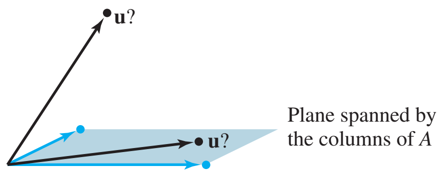
**10.** $\begin{aligned} 8x_1 - x_2 &= 4 \\ 5x_1 + 4x_2 &= 1 \\ x_1 - 3x_2 &= 2 \end{aligned}$

Given $A$ and $\mathbf{b}$ in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

**11.** $A = \begin{bmatrix}1 & 2 & 4\\0 & 1 & 5\\-2 & -4 & -3\end{bmatrix}, \mathbf{b} = \begin{bmatrix}-2\\2\\9\end{bmatrix}$

**12.** $A = \begin{bmatrix}1 & 2 & 1\\-3 & -1 & 2\\0 & 5 & 3\end{bmatrix}, \mathbf{b} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$

**13.** Let $\mathbf{u} = \begin{bmatrix}0\\4\\4\end{bmatrix}$ and $A = \begin{bmatrix}3 & -5\\-2 & 6\\1 & 1\end{bmatrix}$. Is $\mathbf{u}$ in the plane $\mathbb{R}^3$ spanned by the columns of $A$? (See the figure.) Why or why not?



Where is **u**?

**14.** Let $\mathbf{u} = \begin{bmatrix}2\\-3\\2\end{bmatrix}$ and $A = \begin{bmatrix}5 & 8 & 7\\0 & 1 & -1\\1 & 3 & 0\end{bmatrix}$. Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of $A$? Why or why not?

**15.** Let $A = \begin{bmatrix}2 & -1\\-6 & 3\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$. Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$, and describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ *does* have a solution.

**16.** Repeat Exercise 15: $A = \begin{bmatrix}1 & -3 & -4\\-3 & 2 & 6\\5 & -1 & -8\end{bmatrix}, \mathbf{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$.

Exercises 17–20 refer to the matrices $A$ and $B$ below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

$$A = \begin{bmatrix}1 & 3 & 0 & 3\\-1 & -1 & -1 & 1\\0 & -4 & 2 & -8\\2 & 0 & 3 & -1\end{bmatrix} \qquad B = \begin{bmatrix}1 & 3 & -2 & 2\\0 & 1 & 1 & -5\\1 & 2 & -3 & 7\\-2 & -8 & 2 & -1\end{bmatrix}$$

**17.** How many rows of $A$ contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$?

**18.** Do the columns of $B$ span $\mathbb{R}^4$? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each $\mathbf{y}$ in $\mathbb{R}^4$?

**19.** Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix $A$ above? Do the columns of $A$ span $\mathbb{R}^4$?

**20.** Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix $B$ above? Do the columns of $B$ span $\mathbb{R}^3$?

**21.** Let $\mathbf{v}_1 = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$.

Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^4$? Why or why not?

**22.** Let $\mathbf{v}_1 = \begin{bmatrix}0\\0\\-2\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\-3\\8\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}4\\-1\\-5\end{bmatrix}$.

Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$? Why or why not?

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

**23.** a. The equation $A\mathbf{x} = \mathbf{b}$ is referred to as a *vector equation*.

b. A vector $\mathbf{b}$ is a linear combination of the columns of a matrix $A$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

c. The equation $A\mathbf{x} = \mathbf{b}$ is consistent if the augmented matrix $[\,A \quad \mathbf{b}\,]$ has a pivot position in every row.

d. The first entry in the product $A\mathbf{x}$ is a sum of products.

e. If the columns of an $m \times n$ matrix $A$ span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^m$.

f. If $A$ is an $m \times n$ matrix and if the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^m$, then $A$ cannot have a pivot position in every row.

**24.** a. Every matrix equation $A\mathbf{x} = \mathbf{b}$ corresponds to a vector equation with the same solution set.

b. Any linear combination of vectors can always be written in the form $A\mathbf{x}$ for a suitable matrix $A$ and vector $\mathbf{x}$.

c. The solution set of a linear system whose augmented matrix is $[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}\,]$ is the same as the solution set of $A\mathbf{x} = \mathbf{b}$, if $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3\,]$.

d. If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ is not in the set spanned by the columns of $A$.

e. If the augmented matrix $[\,A \quad \mathbf{b}\,]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

f. If $A$ is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^m$.

**25.** Note that $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$. Use this fact (and no row operations) to find scalars $c_1, c_2, c_3$ such that $\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$.

**26.** Let $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$.

It can be shown that $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$. Use this fact (and no row operations) to find $x_1$ and $x_2$ that satisfy the equation

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}.$$

**27.** Let $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, and $\mathbf{v}$ represent vectors in $\mathbb{R}^5$, and let $x_1, x_2$, and $x_3$ denote scalars. Write the following vector equation as a matrix equation. Identify any symbols you choose to use.

$$x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + x_3\mathbf{q}_3 = \mathbf{v}$$

**28.** Rewrite the (numerical) matrix equation below in symbolic form as a vector equation, using symbols $\mathbf{v}_1, \mathbf{v}_2, \ldots$ for the vectors and $c_1, c_2, \ldots$ for scalars. Define what each symbol represents, using the data given in the matrix equation.

$$\begin{bmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \\ 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

**29.** Construct a $3 \times 3$ matrix, not in echelon form, whose columns span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.

**30.** Construct a $3 \times 3$ matrix, not in echelon form, whose columns do *not* span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.

**31.** Let $A$ be a $3 \times 2$ matrix. Explain why the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. Generalize your argument to the case of an arbitrary $A$ with more rows than columns.

**32.** Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about $n$ vectors in $\mathbb{R}^m$ when $n$ is less than $m$?

**33.** Suppose $A$ is a $4 \times 3$ matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about the reduced echelon form of $A$? Justify your answer.

**34.** Suppose $A$ is a $3 \times 3$ matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^3$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of $A$ must span $\mathbb{R}^3$.

**35.** Let $A$ be a $3 \times 4$ matrix, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be vectors in $\mathbb{R}^3$, and let $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Suppose $\mathbf{y}_1 = A\mathbf{x}_1$ and $\mathbf{y}_2 = A\mathbf{x}_2$ for some vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^4$. What fact allows you to conclude that the system $A\mathbf{x} = \mathbf{w}$ is consistent? (*Note:* $\mathbf{x}_1$ and $\mathbf{x}_2$ denote vectors, not scalar entries in vectors.)

**36.** Let $A$ be a $5 \times 3$ matrix, let $\mathbf{y}$ be a vector in $\mathbb{R}^3$, and let $\mathbf{z}$ be a vector in $\mathbb{R}^5$. Suppose $A\mathbf{y} = \mathbf{z}$. What fact allows you to conclude that the system $A\mathbf{x} = 4\mathbf{z}$ is consistent?

**[M]** In Exercises 37–40, determine if the columns of the matrix span $\mathbb{R}^4$.

**37.** $\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix}$

**38.** $\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix}$

**39.** $\begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix}$

**40.** $\begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix}$

**41.** **[M]** Find a column of the matrix in Exercise 39 that can be deleted and yet have the remaining matrix columns still span $\mathbb{R}^4$.

**42.** **[M]** Find a column of the matrix in Exercise 40 that can be deleted and yet have the remaining matrix columns still span $\mathbb{R}^4$. Can you delete more than one column?

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## SOLUTIONS TO PRACTICE PROBLEMS

**1.** The matrix equation

$$\begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$