Nonlinear oscillations of non-neutral plasmas in a time-dependent harmonic trap

Fernando Haas* and Luiz Gustavo Ferreira Soares

Instituto de Física, Universidade Federal do Rio Grande do Sul,

Av. Bento Gonçalves 9500, 91501-970 Porto Alegre, RS, Brasil

(Dated: 06 February 2020)

Abstract

A non-neutral plasma is confined in a quasi-1D device and described by a fluid model. The use of the Lagrangian variables method together with a certain *Ansatz* for the velocity field reduces the problem essentially to ordinary differential equations satisfied by a scale function. In the case of thermal dominated plasma, the governing equation is the Pinney equation, having a close connection with the time-dependent harmonic oscillator. For a slowly varying frequency of the trap potential, an approximate solution is derived and shown to be accurate in the adiabatic limit. In the case of negligible thermal effects, the resulting non-homogeneous time-dependent oscillator equation for the scale function is also approximately solved, in the adiabatic limit. The validity conditions of the thermal dominated and Coulomb dominated cases are determined. The results are applied to a confined antiproton plasma, with implication on antimatter atoms experiments.

PACS numbers: 37.10.Rs, 47.10.Fg, 52.27.Jt

Keywords: non-neutral plasma, antiproton plasma, adiabatic cooling, exact nonlinear oscillations, Lagrangian variables method.

* Corresponding author. E-mail: fernando.haas@ufrgs.br.

I. INTRODUCTION

The analysis of exact or approximate nonlinear structures in plasmas is a traditional an honorable research field [1]-[3]. Recent developments include the influence of large amplitude electromagnetic waves on electron plasma waves [4], the nonlinear dynamics of cold magnetized non-relativistic plasma in the presence of electron-ion collisions [5], nonlinear structures in a degenerate one-dimensional electron gas [6], nonlinear waves in twirling plasmas [7], temperature effects on large amplitude electron oscillation [8], nonlinear standing waves in bounded plasmas [9], nonlinear surface waves on a plasma sphere [10], the dynamics of strongly nonlinear electrostatic waves in warm plasma [11] and nonlinear isothermal waves in degenerate plasma [12]. In the present work, we consider a non-relativistic non-neutral plasma confined in an one-dimensional trap. The method of Lagrangian variables [1]-[3] is applied, since it provides a systematic approach free of *ad hoc* assumptions, at least as much as possible.

The primary motivation of the work is the relevance of quasi-1D confined non-neutral plasmas for antimatter experiments. Namely, the creation of antihydrogen, composed of an antiproton and a positron, has necessarily an initial stage where antiprotons are confined in a potential well, which is only possible for low temperature of the order of a few kelvins [13]-[16]. These low temperatures of the antiproton gas can be achieved most efficiently by means of adiabatic cooling, where the external harmonic trap has a frequency slowly increasing with time, after the antiprotons have been pre-cooled collisionally by electrons. However, our methods apply to general non-neutral plasmas confined in 1D structures such as Penning-Malmberg devices. In such traps, a strong external magnetic field provides the radial confinement and hyperbolic electrodes create in the center an harmonic force that confines axially. In addition, we consider also hot trapped non-neutral plasmas, for the sake of generality.

This work is organized as follows. In Section II, the model fluid equations are presented and written in terms of Lagrangian variables. For a suitable equation of state and velocity field, the thermal dominated case is analyzed in Section III. In this case the main concern is the solution of the nonlinear ordinary differential equation known as Pinney's equation, which is described in Section IV for the case of a slowly varying external trap frequency. In Section V, the situation where thermal effects is dominant is discussed and reduced to an inhomogeneous time-dependent oscillator equation, which is approximately solved in the adiabatic limit. The solution is applied to the case of the cooling of an antiproton gas to temperatures near the absolute zero. Section V contains the conclusions. Finally, in Appendix A we describe the exact solution of the Pinney equation for a certain class of external trap frequencies, in terms of Bessel functions.

II. BASIC MODEL AND LAGRANGIAN VARIABLES METHOD

The hydrodynamic equations for the non-neutral plasma trapped in a one-dimensional well are

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv) = 0\,, \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{1}{mn}\frac{\partial p}{\partial z} - \omega^2(t) z - \frac{eE}{m}\,, \tag{2}$$

$$\frac{\partial E}{\partial z} = -\frac{e\sigma_\perp n}{\varepsilon_0} \tag{3}$$

where n is the 1D number density along the z axis, v is the fluid velocity, E is the electric field, $m, -e$ are resp. the ions (e.g. antiprotons) mass and charge, $\sigma_\perp$ is the 2D number density in the perpendicular plane, and ε_0 is the vacuum permittivity. Systems of antiprotons tend to be dilute and hence the collision frequency is low in such weakly coupled plasmas. Therefore, the entropy is nearly constant, so that an adiabatic equation of state is indicated. To avoid a choice of equation of state, one could extend the macroscopic analysis taking into account the time-evolution of the higher order moment (the pressure dyad) of the particle distribution function, or directly employ kinetic methods [13]. In both cases the treatment would be more involved. In the present case,

$$p = n_0 \kappa_B T_0 \left(\frac{n}{n_0}\right)^3 \tag{4}$$

is the isentropic equation of state with adiabatic index $\gamma = (d+2)/d = 3$ for dimensionality $d = 1$, where n_0, T_0 are a reference number density and a reference temperature (κ_B is the Boltzmann constant). Finally, with the motivation of the adiabatic ion cooling, following [13] we adopt a time-dependent trap frequency

$$\omega(t) = \frac{\omega_0}{(1+\Omega t)^\beta}\,, \tag{5}$$

where ω_0, Ω and β are positive constants. For slowly varying frequency one has $|\dot{\omega}|/\omega \ll \omega$, or $\beta\Omega \ll \omega_0(1+\Omega t)^{1-\beta}$, which holds [13] for all times $t \geq 0$ provided $\beta \leq 1$ and $\beta\Omega \ll \omega_0$. Notice that collisional effects are not included in the model equations, since a very collisional plasma would not keep its quasi-1D character for a long time.

The Lagrangian variables ξ, τ are given [1]-[3] by

$$\xi = z - \int_0^\tau v(\xi,\tau')\,d\tau'\,, \quad \tau = t\,, \tag{6}$$

so that

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial t} + v\,\frac{\partial}{\partial z}\,, \quad \frac{\partial}{\partial\xi} = \left(1 + \int_0^\tau \frac{\partial v(\xi,\tau')}{\partial\xi}\,d\tau'\right)\frac{\partial}{\partial z}\,. \tag{7}$$

The continuity equation (1) becomes

$$\frac{\partial}{\partial\tau}\left[\left(1 + \int_0^\tau \frac{\partial v(\xi,\tau')}{\partial\xi}\,d\tau'\right)n\right] = 0\,, \tag{8}$$

with solution

$$n = n(\xi,0)\left(1 + \int_0^\tau \frac{\partial v(\xi,\tau')}{\partial\xi}\,d\tau'\right)^{-1}\,, \tag{9}$$

where $n(\xi,0)$ is the initial 1D number density.

The Gauss law (3) becomes

$$\frac{\partial E}{\partial\xi} = -\frac{e\sigma_\perp}{\varepsilon_0}\,n(\xi,0)\,, \tag{10}$$

with solution

$$E = -\frac{e\sigma_\perp}{\varepsilon_0}\int_0^\xi n(\xi',0)\,d\xi' + E_0(\tau)\,. \tag{11}$$

where $E_0(\tau)$ is the electric field at $\xi = 0$.

Finally, the momentum equation (2) becomes

$$\begin{aligned}\frac{\partial v}{\partial\tau} = &- \frac{3\,\kappa_B T_0}{2\,m}\Big(1 + \int_0^\tau \frac{\partial v(\xi,\tau')}{\partial\xi}\,d\tau'\Big)^{-1}\frac{\partial}{\partial\xi}\Big[\Big(\frac{n}{n_0}\Big)^2\Big] \\ &- \omega^2(\tau)\Big(\xi + \int_0^\tau v(\xi,\tau')\,d\tau'\Big) + \frac{\omega_p^2}{n_0}\int_0^\xi n(\xi',0)d\xi' - \frac{eE_0(\tau)}{m}\,,\end{aligned} \tag{12}$$

where $\omega_p = \sqrt{n_0\sigma_\perp e^2/(m\varepsilon_0)}$.

As it stands, Eq. (12) is a difficult integro-differential equation. To proceed, we suppose the special form

$$v = \dot{Z}(\tau) + \dot{\rho}(\tau)\xi\,, \tag{13}$$

where for definiteness and without loss of generality $Z(0) = 0$ and $\rho(0) = 1$, so that in particular $n = n(\xi, 0)/\rho$. Equation (13) can be regarded as a first order expansion of the velocity field around $\xi = 0$. Equation (12) becomes

$$\ddot{Z} + \omega^2 Z + \frac{eE_0}{m} + (\ddot{\rho} + \omega^2\rho)\xi = -\frac{3\,k_B T_0}{2m\rho^3}\frac{\partial}{\partial\xi}\left[\left(\frac{n(\xi,0)}{n_0}\right)^2\right] + \frac{\omega_p^2}{n_0}\int_0^\xi n(\xi',0)d\xi' \,. \tag{14}$$

The left-hand side of Eq. (14), being a linear function of ξ, must be paired by the right-hand side, which can happens only under certain circumstances discussed in what follows.

III. LOW DENSITY PLASMAS

For sufficiently low densities, the self-consistent Coulomb interaction $\sim \omega_p^2$ can be neglected in Eq. (14). This imposes a specific initial number density so that the temperature term behaves linearly in ξ,

$$n(\xi, 0) = n_0 \left(1 - \left(\frac{\xi}{\xi_0}\right)^2\right)^{1/2} , \tag{15}$$

where $\xi_0 > 0$ is a constant, and where $n(\xi, 0) = 0$ outside the interval $-\xi_0 < \xi < \xi_0$. Equation (14) splits into

$$\ddot{\rho} + \omega^2\rho = \frac{\kappa^2}{\rho^3} \,, \quad \kappa^2 = \frac{3\kappa_B T_0}{m\xi_0^2} > 0 \,, \tag{16}$$

and

$$\ddot{Z} + \omega^2 Z = -\frac{eE_0}{m} \,, \tag{17}$$

making sense inside the plasma. The concavity of the quadratic form in Eq. (15) was chosen so that $\kappa^2 > 0$ assuring $\rho > 0$ for all times, avoiding any singular expressions. Also notice that n_0 is the maximal initial 1D number density, attained by definition and without loss of generality due to a choice of reference system at $\xi = 0$, as depicted in Fig. 1. Equations (16) and (17) are quite similar to Eqs. (16) and (17) of [17], also derived using a Lagrangian frame for oscillation modes of thin oblate non-neutral plasmas. The differences are that Eq. (16) above does not contain a Coulomb term due to the low density assumption, and has a time-dependent harmonic trap frequency.

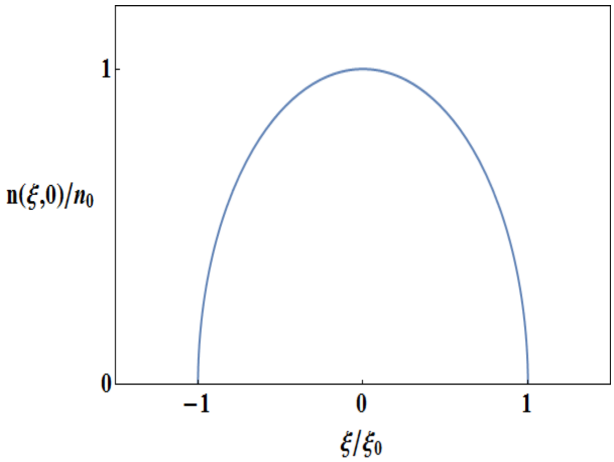

FIG. 1: Initial 1D number density from Eq. (15).

It is convenient to collect all results obtained so far within the plasma ($(\xi/\xi_0)^2 < 1$):

$$\begin{aligned} n &= \frac{n(\xi,0)}{\rho}\,, \quad v = \dot{Z} + \dot{\rho}\xi\,, \quad \xi = \frac{z-Z}{\rho}\,, \\ E &= -\frac{n_0 e \sigma_\perp \xi_0}{2\varepsilon_0}\left[\frac{\xi}{\xi_0}\left(1-\left(\frac{\xi}{\xi_0}\right)^2\right)^{1/2} + \arcsin^{-1}\left(\frac{\xi}{\xi_0}\right)\right] + E_0\,, \end{aligned} \tag{18}$$

where $n(\xi,0)$ is given by Eq. (15) and where ρ and Z satisfy resp. Eqs. (16) and (17). We are also able to exactly derive the number of confined particles,

$$N = \sigma_\perp A_\perp \int_{-\xi_0}^{\xi_0} n(\xi,0)d\xi = \frac{\pi}{2}\sigma_\perp A_\perp n_0 \xi_0\,. \tag{19}$$

where $A_\perp$ is the occupied area in the perpendicular plane, and the instantaneous temperature $T \sim n^{\gamma-1}$,

$$T = T_0(n/n_0)^2 = \frac{T_0}{\rho^2}\left(1-\left(\frac{\xi}{\xi_0}\right)^2\right)\,. \tag{20}$$

In physical coordinates, the nonlinear structure is centered at $z = Z(t)$, where the center of mass $Z(t)$ executes driven oscillations according to Eq. (17).

In addition, the condition for neglecting space charge effects can be analytically found. Assuming $E_0 = 0$, the thermal effects are dominating over the Coulomb term in Eq. (14) provided

$$\frac{2\kappa^2}{\omega_p^2\rho^3} \gg f\left(\frac{\xi}{\xi_0}\right) = \left(1-\left(\frac{\xi}{\xi_0}\right)^2\right)^{1/2} + \frac{\arcsin(\xi/\xi_0)}{\xi/\xi_0} \geq \pi/2\,, \quad -\xi_0 \leq \xi \leq \xi_0\,, \tag{21}$$

which can be checked against experimental conditions. Notice that since ρ tends to increase, the neglect of electrostatic effects becomes less justifiable as time goes on. Namely, as time increases, both thermal and electrostatic contributions become smaller, but at different rates so that eventually the thermal effects become negligible due to the cooling. As expected, space charge effects are enhanced at the center of the structure, as shown in Fig. (2).

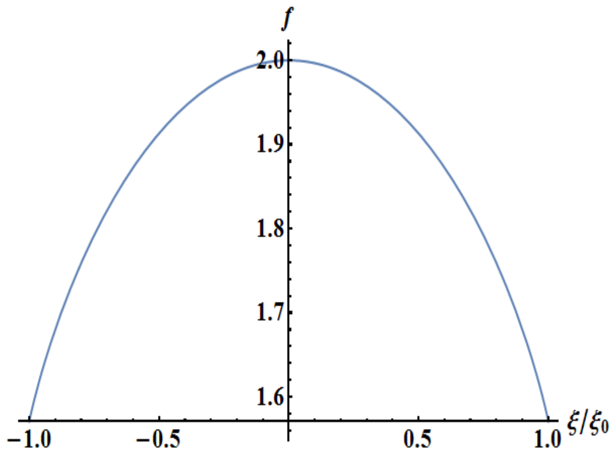

FIG. 2: Function $f = f(\xi/\xi_0)$ characteristic of the space charge effects, from Eq. (21).

The details of the exact solution are strongly dependent on the properties of the function ρ satisfying Eq. (16), known as Pinney equation [18], discussed in the next Section.

IV. EXACT AND APPROXIMATE SOLUTIONS OF THE PINNEY EQUATION

The Pinney equation (16) appears in many contexts such as cosmology [19, 20], magnetogasdynamics [21], Bose-Einstein condensates [22], dissipative quantum mechanics models [23], Hamiltonian Ermakov systems [24], limit cycles analysis [25] and Kepler-Ermakov systems in Riemannian spaces [26]. The solution [18] reads

$$\rho = (Ax_1^2 + 2Bx_1x_2 + Cx_2^2)^{1/2}\,, \tag{22}$$

where $x_{1,2}$ are independent solutions of the time-dependent harmonic oscillator equation,

$$\ddot{x} + \omega^2(t)x = 0\,, \tag{23}$$

and where A, B, C are constants such that $AC - B^2 = \kappa^2/W^2$, with $W = x_1\dot{x}_2 - x_2\dot{x}_1$ being the Wronskian.

In the case of the frequency (5), Eq. (23) is reducible to the Bessel equation, so that the Pinney equation is exactly solvable. Details are shown in the Appendix.

However, although presently the Pinney equation is exactly solvable, the general solution in terms of Bessel functions is a little too involved. Moreover, it is certainly of interest to have available approximate solutions, for the case where the solution of the time-dependent harmonic oscillator equation (23) is unknown. Supposing an arbitrary slowly varying frequency so that $\dot\omega/\omega^2 \sim \varepsilon \ll 1$, it is admissible to employ the WKB (Wentzel-Kramers-Brillouin) solutions

$$x_1 = \frac{\cos\mathcal{T}}{\sqrt{\omega(t)}}\,, \quad x_2 = \frac{\sin\mathcal{T}}{\sqrt{\omega(t)}}\,, \quad \mathcal{T} = \int_0^t \omega(t')dt'\,, \tag{24}$$

to be inserted into Eq. (22). In this context the Wronskian is $W = 1$ and

$$\rho = \frac{1}{\sqrt{\omega}}\left(A\,\cos^2\mathcal{T} + 2B\,\sin\mathcal{T}\cos\mathcal{T} + C\sin^2\mathcal{T}\right)^{1/2}\,, \quad AC - B^2 = \kappa^2\,. \tag{25}$$

Direct evaluation shows that (25) yields

$$\frac{1}{\rho^4\omega^2}\left(\rho^3(\ddot\rho + \omega^2\rho) - \kappa^2\right) = \frac{1}{4\omega^4}(3\dot\omega^2 - 2\omega\ddot\omega) = \mathcal{O}(\varepsilon^2)\,, \tag{26}$$

showing the adiabatic validity of the proposed expression. As far as we know, in spite of the long history of the Pinney equation and the associated invariant [27], the WKB solution (25) is new and potentially useful whenever the frequency is slowly varying in time. For the initial condition $\rho(0) = 1, \dot\rho(0) = 0$, one has from Eq. (25) the net result

$$\rho = \frac{1}{\sqrt{\omega_0\omega}}\left[\left(\omega_0\cos\mathcal{T} + \frac{\dot\omega_0}{2\omega_0}\sin\mathcal{T}\right)^2 + \kappa^2\sin^2\mathcal{T}\right]^{1/2}\,, \tag{27}$$

where $\omega_0 = \omega(0)$ and $\dot\omega_0 = \dot\omega(0)$, valid for arbitrary slowly varying trap frequency.

It is interesting to note that in addition to the oscillations the solution remains in a sense locked at the minimum $\rho_* = \sqrt{\kappa/\omega(t)}$ of the pseudo-potential $V = V(\rho) = \omega^2\rho^2/2 + \kappa^2/(2\rho^2)$ so that $\ddot\rho = -\partial V/\partial\rho$.

The condition (21) for dominance of thermal effects can be satisfied in available conceptual experimental setups [28, 29]. The corresponding solution (27) is shown in Figure 3, for typical parameters: $\kappa_B T_0 = 30\,\mathrm{eV}$ and $\xi_0 = 1\,\mathrm{cm}$, which for an antiproton gas yields $\kappa/2\pi = 1.5\times 10^6\,\mathrm{Hz}$. Supposing $N = 10^4$ confined antiprotons with a circular cross section of radius $2\,\mathrm{mm}$, from Eq. (19) one derives a number density $n_0\sigma_\perp = 5.1\times 10^{10}\,\mathrm{m}^{-3}$ so that $\omega_p/2\pi = 4.7\times 10^4\,\mathrm{Hz}$. In addition, the trap frequency (5) is considered, with $\beta = 1, \Omega = 0.02\omega_0$

and $\omega_0/2\pi = 100\,\mathrm{kHz}$. Actually, as the plasma expands due to the lowering trap frequency, the electrostatic effects become smaller in a slower rate as the thermal effects, so that the inequality (21) is attended in a less accurate way due to the overall increase of the scale function ρ. It should also be noted that the exact solution (for brevity not shown) in terms of Bessel functions yields almost the same results.

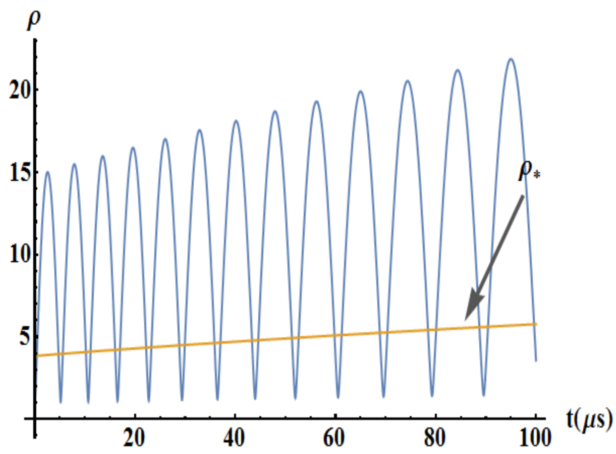

FIG. 3: WKB solution (27) of the Pinney equation, with $\omega(t)$ defined in Eq. (5). Initial conditions: $\rho(0) = 1, \dot{\rho}(0) = 0$. Parameters: $\kappa/2\pi = 4.7 \times 10^6\,\mathrm{Hz}$, $\beta = 1, \Omega = 0.02\omega_0$ and $\omega_0/2\pi = 100\,\mathrm{kHz}$. The monotonous curve shows ρ_* defined in the text.

Similarly, one can also track the time-evolution of the temperature, among other possibilities. Using Eq. (20), the temperature at the center of the structure ($\xi = 0$) is shown in Figure 4, using the WKB solution and the same parameters as for Figure 3.

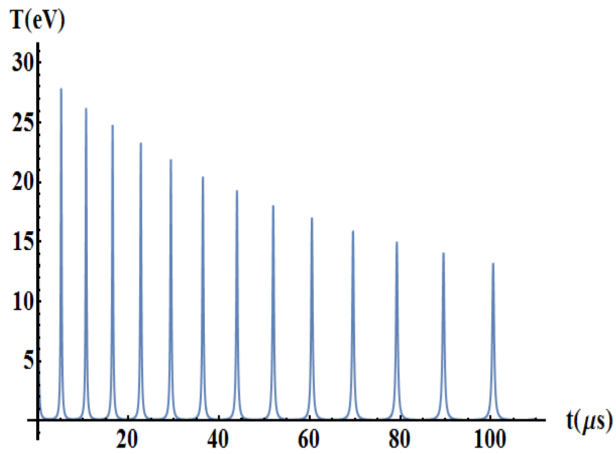

FIG. 4: Temperature $T(\xi = 0) = T_0/\rho^2$ at the center, measured in energy units, for $\kappa_B T_0 = 30\,\mathrm{eV}$, using the WKB solution (27) and the same remaining parameters of Figure 3.

V. NEGLIGIBLE THERMAL EFFECTS

If the thermal contribution $\sim T_0$ is neglected in Eq. (14), the only possibility is

$$n(\xi, 0) = n_0 \,, \quad -\xi_0 < \xi < \xi_0 \,, \tag{28}$$

a constant and symmetric (due to a choice of reference system) distribution, so that the pressure term identically vanishes. The center of mass still satisfies Eq. (17), together with

$$\ddot{\rho} + \omega^2 \rho = \omega_p^2 \,, \tag{29}$$

valid inside the plasma.

Within the plasma ($(\xi/\xi_0)^2 < 1$), we have the following results:

$$\begin{aligned} n &= \frac{n_0}{\rho} \,, \quad v = \dot{Z} + \dot{\rho}\xi \,, \quad \xi = \frac{z - Z}{\rho} \,, \\ E &= -n_0 e \sigma_\perp \xi / \varepsilon_0 + E_0 \,, \\ N &= 2\sigma_\perp A_\perp n_0 \xi_0 \,, \quad T = T_0/\rho^2 \,, \end{aligned} \tag{30}$$

with spatially homogeneous density and temperature.

From now on for simplicity we assume $E_0 = 0, Z = 0$. In this case the main concern is the solution of Eq. (29). Using the transformation described in Appendix A, when the trap frequency is given by Eq. (5) it can be mapped to an inhomogeneous Bessel equation, whenever $\beta \neq 1$. The exact solution is therefore available but a little too involved, which is also true in the case $\beta = 1$. For a slowly varying trap frequency, it is more useful to consider the WKB solutions of the homogeneous equation shown in the previous Section, together with the variation of parameters method. Following this approach, one gets for $\rho(0) = 1, \dot\rho(0) = 0$ the approximate solution

$$\begin{aligned}
\rho \;=\; & \sqrt{\frac{\omega_0}{\omega(t)}}\Big(\cos\mathcal{T}(t) + \frac{\dot\omega_0}{2\omega_0^2}\sin\mathcal{T}(t) + \\
& + \frac{\omega_p^2}{\sqrt{\omega_0}}\sin\mathcal{T}(t)\int_0^t \frac{\cos\mathcal{T}(t')dt'}{\sqrt{\omega(t')}} - \frac{\omega_p^2}{\sqrt{\omega_0}}\cos\mathcal{T}(t)\int_0^t \frac{\sin\mathcal{T}(t')dt'}{\sqrt{\omega(t')}}\Big)\,, \quad \mathcal{T}(t) = \int_0^t \omega(t')dt'\,,
\end{aligned} \tag{31}$$

which is valid for arbitrary frequency, with $\omega_0 = \omega(0), \dot\omega_0 = \dot\omega(0)$. We easily obtain

$$\frac{\ddot\rho + \omega^2\rho - \omega_p^2}{\omega^2\rho} = \frac{1}{4\omega^4}(3\dot\omega^2 - 2\omega\ddot\omega) = \mathcal{O}(\varepsilon^2)\,, \tag{32}$$

showing the adiabatic validity of the solution provided $\dot\omega/\omega^2 = \mathcal{O}(\varepsilon), \varepsilon \ll 1$.

According to Eq. (14), the neglect of thermal effects is justified provided

$$\frac{3\kappa_B T_0}{m\rho^3}\frac{\partial n(\xi,0)}{\partial\xi} \ll n_0\omega_p^2\xi \tag{33}$$

inside the plasma. It is not entirely straightforward to deal with the inequality (33), due to the discontinuous derivative of the initial number density. However, far from the frontiers the inequality is automatically satisfied, due to the flat character of $n(\xi,0)$. This holds true also at $\xi = 0$, where both thermal and electrostatic effects are negligible. On the other hand, near the frontier ($\xi \simeq \xi_0$) if one assumes the estimate $\partial n(\xi,0)/\partial\xi \simeq n_0/\xi_0$, similar to the case of negligible electrostatic effect, we then have from Eq. (33),

$$\frac{\kappa^2}{\rho^3} \ll \omega_p^2\,, \tag{34}$$

where κ is defined in Eq. (16). Notice that since ρ tends to increase, it is sufficient to attend Eq. (34) at $t = 0$, implying $\kappa^2 \ll \omega_p^2$. This is due to different scaling of temperature and number density, $T \sim \rho^{-2}, n \sim \rho^{-1}$, so that thermal effects become smaller in a fastest rate.

The condition (34) is safely satisfied in the experiments reported in [14], where $N = 5 \times 10^5$ antiprotons were confined in a cylindrical trap with $A_\perp = 12.5\text{mm}^2$ and length $2\xi_0 = 10\,\text{mm}$, with initial temperature $T_0 = 31\,\text{K}$, together with $\rho \simeq 1$ (which holds at least in the initial stages of the expansion). In this case one has $\kappa/2\pi = 2.8 \times 10^4\text{Hz}$ and $\omega_p/2\pi = 4.7 \times 10^5\,\text{Hz}$. In addition, we can consider the trap frequency (5), with

$\beta = 1, \Omega = 0.02\omega_0$ and $\omega_0/2\pi = 100\,\mathrm{kHz}$. The time-evolution of the scaling function ρ can be analytically found from Eq. (31), as shown in Figure 5. Similarly, the cooling is represented in Figure 6. It is interesting to note that in addition to the oscillations the solution remains in a sense locked at the minimum $\rho_* = \omega_p^2/\omega^2(t)$ of the pseudo-potential $V = V(\rho) = \omega^2\rho^2/2 - \omega_p^2\rho$ so that $\ddot{\rho} = -\partial V/\partial\rho$.

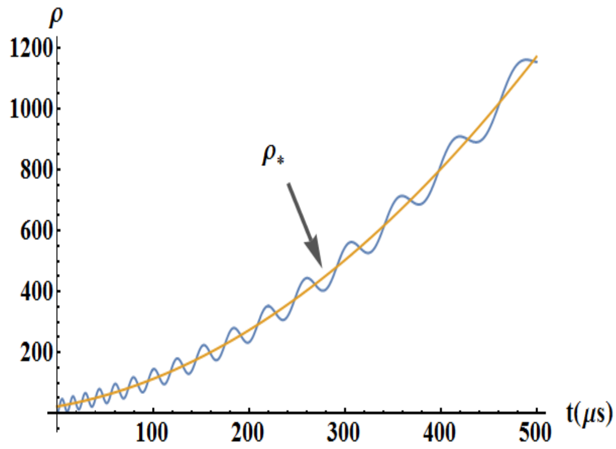

FIG. 5: Time-evolution of the scaling function ρ, according to the approximate solution (31), with $\omega(t)$ defined in Eq. (5). Initial conditions: $\rho(0) = 1, \dot{\rho}(0) = 0$. Parameters: $\omega_p/2\pi = 4.7 \times 10^5\,\mathrm{Hz}$, $\beta = 1, \Omega = 0.02\omega_0$ and $\omega_0/2\pi = 100\,\mathrm{kHz}$. The monotonous curve represents ρ_* defined in the text.

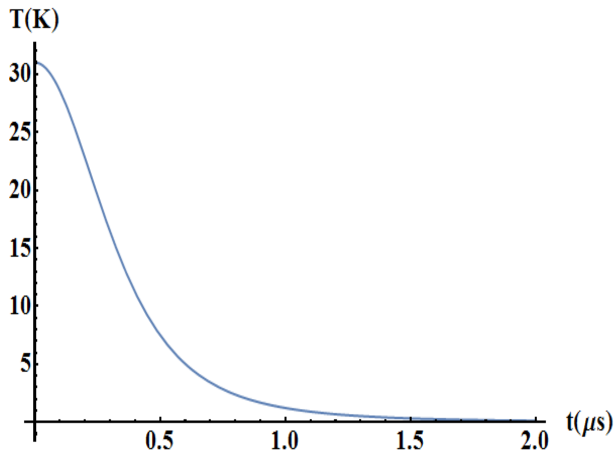

FIG. 6: Temperature $T = T_0/\rho^2$, for $T_0 = 31\,\mathrm{K}$, using the approximate solution (31) and the same remaining parameters of Figure 5.

Finally, one can wonder about the situation where thermal and electrostatic effects are both relevant. This case is not accessible to the present methods, due to the following reasoning. Multiplying all terms in Eq. (14) by ρ^3, deriving twice with respect to ξ and once with respect to τ gives $\omega_p^2 \times d(\rho^3)/d\tau \times \partial n(\xi,0)/\partial\xi = 0$. Since ρ is not a constant, this condition can be meet in only two cases: (a) setting $\omega_p \simeq 0$, which corresponds to negligible electrostatic effects; (b) setting $\partial n(\xi,0)/\partial\xi \simeq 0$ inside the plasma, which was followed in the Section on negligible thermal effects.

VI. CONCLUSION

In this work, we have derived nonlinear structures for trapped quasi-1D non-neutral plasmas, by means of the Lagrangian variables method. The findings are relevant for the experimental creation of antimatter atoms, such as antihydrogen, which involves the confinement of antiprotons as an initial step. For this purpose, we have considered a time-dependent harmonic trap, with a slowly decreasing frequency, which provides the adiabatic cooling of the trapped non-neutral plasma. In the dilute plasma case, the details of the nonlinear structure reduce to the solution of the so-called Pinney equation, which is ubiquitous in nonlinear studies. The solution of the Pinney equation valid in the adiabatic limit of slowly varying general trap frequencies was obtained for the first time. In addition, the exact solution of the Pinney equation for a certain class of external traps was also provided, in terms of Bessel functions. The case of Coulomb dominated systems is reducible to an inhomogeneous time-dependent harmonic oscillator equation, which was solved in the adiabatic limit of slowly varying frequencies. The conditions for the dominance of thermal or electrostatic effects were determined and compared to standard experimental parameters. The results will be relevant for trapped non-neutral plasmas under time varying harmonic potentials. In addition, the approximate solution of the Pinney equation in adiabatic conditions has an intrinsic impact on its own.

Acknowledgments

The authors acknowledge the support by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq). This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. Data Availability Statement: the data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: EXACT SOLUTION OF THE PINNEY EQUATION

The general solution (22) of the Pinney equation crucially depends on the solution of the time-dependent harmonic oscillator equation (23). Assuming the trap frequency (5) and $\beta \neq 1$, the transformation

$$x = (1+\Omega t)^{1/2} y(s)\,, \quad s = \frac{\omega_0 (1+\Omega t)^{1-\beta}}{\Omega|1-\beta|} \tag{A1}$$

maps (23) into Bessel's equation,

$$s^2 \frac{d^2 y}{ds^2} + s\frac{dy}{ds} + (s^2 - \nu^2) y = 0\,, \quad \nu = \frac{1}{2|1-\beta|}\,. \tag{A2}$$

Hence one can employ

$$x_1 = (1+\Omega t)^{1/2} J_\nu(s)\,, \quad x_2 = (1+\Omega t)^{1/2} Y_\nu(s) \tag{A3}$$

in Eq. (22), with Wronskian $W = 2(1-\beta)\Omega/\pi$, where J_ν, Y_ν are the Bessel functions of the first and second kind. The determination of the constants A, B, C in general is found after some simple algebra given the initial condition.

The marginal case $\beta = 1$ deserves a separate analysis, yielding

$$x_1 = \sqrt{1+\Omega t}\cos\mathcal{T}\,, \quad x_2 = (1+\Omega t)^{1/2}\sin\mathcal{T}\,, \tag{A4}$$

where now

$$\mathcal{T} = \left(\frac{\omega_0^2}{\Omega^2} - \frac{1}{4}\right)^{1/2} \ln(1+\Omega t)\,, \tag{A5}$$

and the Wronskian is $W = \Omega\sqrt{\omega_0^2/\Omega^2 - 1/4}$. The exact solution of the Pinney equation with $\rho(0) = 1, \dot\rho(0) = 0$ reads

$$\rho = \sqrt{1+\Omega t}\left[\left(\cos\mathcal{T} - \frac{\sin\mathcal{T}}{2\sqrt{\omega_0^2/\Omega^2 - 1/4}}\right)^2 + \frac{\kappa^2\sin^2\mathcal{T}}{\omega_0^2 - \Omega^2/4}\right]^{1/2}\,. \tag{A6}$$

---

[1] R. C. Davidson, *Methods in nonlinear plasma theory* (Academic Press, New York, 1972).

[2] E. Infeld, and G. Rowlands, *Nonlinear waves, solitons and chaos* 2nd. ed. (Cambridge University Press, Cambridge, 2000).

[3] B. K. Shivamoggi, *Introduction to nonlinear fluid-plasma waves* (Kluwer, Dordrecht, 1988).

[4] G. Brodin, and L. Stenflo, Phys. Lett. A **381**, 1033 (2017).

[5] B. Sahu, A. Sinha, and R. Roychouwdhury, Phys. Plasmas **22**, 092306 (2015).

[6] S. Ghosh, N. Chakrabarty and F. Haas, Europhys. Lett. **105**, 30006 (2014).

[7] A. R. Karimov, J. Plasma Phys. **75**, 817 (2009).

[8] L. Stenflo, and G. Brodin, Phys. Plasmas **23**, 074501 (2016).

[9] Sh. Amiranashvili, M. Y. Yu, L. Stenflo, G. Brodin, and M. Servin, Phys. Rev. E **66**, 046403 (2002).

[10] L. Stenflo, M. Y. Yu, and S. V. Vladimirov, Phys. Rev. E **48**, 4859 (1993).

[11] A. Asghari, S. Sobhanian, M. Ghoranneviss, M. K. Salem, and M. Kouhi, Eur. Phys. J. D **74**, 20 (2020).

[12] A. Dubinov, and A. A. Dubinova, Plasma Phys. Rep. **34**, 403 (2008).

[13] G. Manfredi, and P.-A. Hervieux, Phys. Rev. Lett. **109**, 255005 (2012).

[14] G. Gabrielse, W. S. Kolthammer, R. McConnell, P. Richerme, R. Kalra, E. Novitski, D. Grzonka, W. Oelert, T. Sefzick, M. Zielinski, D. Fitzakerley, M. C. George, E. A. Hessels, C. H. Storry, M. Weel, A. Müllers, and J. Walz (ATRAP Collaboration), Phys. Rev. Lett. **106**, 073002 (2011).

[15] M. Ahmadi, B. X. R. Alves, C. J. Baker, W. Bertsche, E. Butler, A. Capra, C. Carruth, C. L. Cesar, M. Charlton, S. Cohen, R. Collister, S. Eriksson, A. Evans, N. Evetts, J. Fajans, T. Friesen, M. C. Fujiwara, D. R. Gill, A. Gutierrez, J. S. Hangst, W. N. Hardy, M. E. Hayden, C. A. Isaac, A. Ishida, M. A. Johnson, S. A. Jones, S. Jonsell, L. Kurchaninov, N. Madsen, M. Mathers, D. Maxwell, J. T. K. McKenna, S. Menary, J. M. Michan, T. Momose, J. J. Munich, P. Nolan, K. Olchanski, A. Olin, P. Pusa, C. O. Rasmussen, F. Robicheaux, R. L. Sacramento, M. Sameed, E. Sarid, D. M. Silveira, S. Stracka, G. Stutter, C. So, T. D. Tharp, J. E. Thompson, R. I. Thompson, D. P. van der Werf, and J. S. Wurtele, Nature Comm. **8**, 681 (2017).

[16] G. B. Andresen, M. D. Ashkezari, M. Baquero-Ruiz, W. Bertsche, P. D. Bowe, E. Butler, C. L. Cesar, S. Chapman, M. Charlton, J. Fajans, T. Friesen, M. C. Fujiwara, D. R. Gill, J. S. Hangst, W. N. Hardy, R. S. Hayano, M. E. Hayden, A. Humphries, R. Hydomako,

S. Jonsell,, L. Kurchaninov, R. Lambo, N. Madsen, S. Menary, P. Nolan, K. Olchanski, A. Olin, A. Povilus, P. Pusa, F. Robicheaux, E. Sarid, D. M. Silveira, C. So, J. W. Storey, R. I. Thompson, D. P. van der Werf, D. Wilding, J. S. Wurtele, and Y. Yamazaki (ALPHA Collaboration), Phys. Rev. Lett. **105**, 013003 (2010).

[17] Sh. Amiranashvili, M. Y. Yu, and L. Stenflo, Phys. Plasmas **10**, 1239 (2003).

[18] E. Pinney, Proc. Amer. Math. Soc. **1**, 681 (1950).

[19] R. M. Hawkins, and J. E. Lidsey, Phys. Rev. D **66**, 023523 (2002).

[20] H. C. Rosu, S. C. Mancas, and P. Chen, Phys. Lett. A **379**, 882 (2015).

[21] C. Rogers, and W. K. Schief, J. Math. Phys. **52**, 083701 (2011).

[22] F. Haas, Phys. Rev. A **65**, 33603 (2002).

[23] F. Haas, Physica Scripta **81**, 025004 (2010).

[24] F. Haas, and J. Goedert, J. Phys. A: Math. Gen. **32**, 2835 (1999).

[25] J. Llibre, and E. Pérez-Chavela, Phys. Lett. A **375**, 1080 (2011).

[26] M. Tsamparlis, and A. Paliathanasis, J. Phys. A: Math. Theor. **45**, 275202 (2012).

[27] R. M. Morris, and P. G. L. Leach, Appl. Anal. Discrete Math. **11**, 62 (2017).

[28] Y. Enomoto, N. Kuroda, K. Michishio, C. H. Kim, H. Higaki, Y. Nagata, Y. Kanai, H. A. Torii, M. Corradini, M. Leali, E. Lodi-Rizzini, V. Mascagna, L. Venturelli, N. Zurlo, K. Fujii, M. Ohtsuka, K. Tanaka, H. Imao, Y. Nagashima, Y. Matsuda, B. Juhász, A. Mohri, and Y. Yamazaki, Phys. Rev. Lett. **105**, 243401 (2010).

[29] G. Gabrielse, N. S. Bowden, P. Oxley, A. Speck, C. H. Storry, J. N. Tan, M. Wessels, D. Grzonka, W. Oelert, G. Schepers, T. Sefzick, J. Walz, H. Pittner, T.W. Hänsch, and E. A. Hessels (ATRAP Collaboration), Phys. Rev. Lett. **89**, 213401 (2002).