Nonlinear vortex-phonon interactions in a Bose-Einstein condensate

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Abstract

We consider the nonlinear coupling between an exact vortex solution in a Bose-Einstein condensate and a spectrum of elementary excitations in the medium. These excitations, or Bogoliubov-de Gennes modes, are indeed a special kind of phonons. We treat the spectrum of elementary excitations in the medium as a gas of quantum particles, sometimes also called *bogolons*. An exact kinetic equation for the bogolon gas is derived, and an approximate form of this equation, valid in the quasi-classical limit, is also obtained. We study the energy transfer between the vortex and the bogolon gas, and establish conditions for vortex instability and damping.

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I. INTRODUCTION

The area of Bose-Einstein condensates (BEC), specially those produced with laser cooled low density alkaline gases, has been explored in the last two decades in many different directions [1, 2]. And the formation of vortices can be considered as one of their most remarkable properties.

Vortices in BECs have been studied by many authors, in both experiments [3, 4] and theory [1, 5]. Similarities with Rossby waves, as those existing in the rotating atmosphere of planets, has also be explored [6]. Abrikosov arrays or latices of quantum vortices can be excited, and display oscillations called Tkachenko modes [7, 8], as first observed by [9]. Rossby-Tkachenco modes, corresponding to general class of lattice oscillations, can also be considered [10].

In this work, we consider the interaction of vortices with a spectrum of elementary excitations in the medium. These excitations, or Bogoliubov-de Gennes (BdG) modes, are indeed a special kind of phonons. We treat the spectrum of elementary excitations in the medium as a gas of quantum particles, sometimes also called *bogolons*. This could be relevant to the excitation of vortices in a turbulent BEC [11].

Starting from the usual BdG mode equations, we derive an equivalent wave-kinetic equation describing the evolution of an appropriate Wigner function. Wigner functions were used in the past to describe the condensate itself [10]. But here the Wigner functions are used to describe the BdG or bogolon modes, while both the background condensate and the vortex are described with the usual mean field wave functions.

The vortex dispersion relation in the presence of an arbitrary bogolon spectrum is derived. Conditions for the excitation and damping of vortices due to the presence of a BdG or bogolon spectrum are established. Special cases are considered explicitly. The structure of the paper is the following. In section II, we present the basic equations describing the BEC in the mean field approximation. In section III, using the the auto-correlation function for two distinct pairs of time and positions, the kinetic equation for the bogolon gas is derived. Section IV focuses on the evolution of a single vortex in a turbulent background described by the appropriate Wigner quasi-probability function. In section V the question of vortex amplitude increase or damping is addressed in terms of the new kinetic theory. Finally in section VI our conclusions are collected.

II. BASIC FORMULATION

We describe the evolution of BEC using the mean field approximation. For that purpose, we start with the GP equation, which can be written as

$$i\hbar\frac{\partial}{\partial t}\psi = \left(H_0 + g\,|\psi|^2\right)\psi\,, \quad H_0 = -\frac{\hbar^2\nabla^2}{2m} + U_0(\mathbf{r})\,. \tag{1}$$

Here we use the standard notation, where ψ is the condensate order parameter, $U_0(\mathbf{r})$ the confining potential, and g the coupling constant. Let us assume a generic solution of the form

$$\psi(\mathbf{r},t) = \left[\psi_0(\mathbf{r},t) + \tilde{\psi}(\mathbf{r},t)\right]\exp(-i\mu t/\hbar)\,, \tag{2}$$

where ψ_0 describes the slow condensate field, and $\tilde{\psi}$ is a superposition of BdG modes. Here, μ is the chemical potential. Replacing this in Eq. (1), and averaging over a time interval much longer than the typical BdG mode period, we get for the slow condensate field

$$i\hbar\frac{\partial\psi_0}{\partial t} = \left[H_0 + g(n_0 + 2n_T) - \mu\right]\psi_0\,, \tag{3}$$

where we have used the slow and fast densities, as defined by $n_0 = |\psi_0|^2$ and the average of turbulent fluctuations $n_T = \left\langle|\tilde{\psi}|^2\right\rangle$. Here, we should notice that $\left\langle\tilde{\psi}\right\rangle = 0$ and $\psi_0 \equiv \langle\psi\rangle$. Subtracting Eq. (3) from (1), we obtain for the fast component of the matter field

$$i\hbar\frac{\partial\tilde{\psi}}{\partial t} = \left(H_r + g|\tilde{\psi}|^2\right)\tilde{\psi} + 2g\left(|\tilde{\psi}|^2 - n_T\right)\psi_0 + g\left(\psi_0^2\tilde{\psi}^* + \psi_0^*\tilde{\psi}^2\right)\,. \tag{4}$$

with the new Hamiltonian $H_r = H_0 + 2gn_0(\mathbf{r}) - \mu$. We should notice that we have, in this equation, $|\tilde{\psi}|^2 = n_T + \delta|\tilde{\psi}|^2$, where n_T is the slow part and $\delta|\tilde{\psi}|^2$ contains the high frequency mixing of the BdG spectrum. Such spectrum can be explicitly described as

$$\tilde{\psi}(\mathbf{r},t) = \sum_k \left[u_k(\mathbf{r})\exp(-i\omega_k t) + v_k^*(\mathbf{r})\exp(i\omega_k t)\right]\,, \tag{5}$$

where each mode is identified by the quantity k, representing a set of discrete numbers of a continuum of wavevectors, ω_k are the eigenfrequencies, and the pair of functions $u_k(\mathbf{r})$ and $v_k(\mathbf{r})$ are the corresponding BdG field components. Replacing this in Eq. (4), and neglecting the nonlinear mode mixing terms, we get the usual BdG equations for each mode

$$(\hbar\omega_k - H_r)u_k = g\psi_0^2 v_k\,, \quad (\hbar\omega_k + H_r)v_k = -(g\psi_0^2)^* u_k\,. \tag{6}$$

In homogeneous condensates, and in a broad range of situations discussed in our previous work [12], we can assume solutions that satisfy the equations $\nabla^2(u_k, v_k) = -k^2(u_k, v_k)$. We can then easily solve eqs.(6) and derive the mode dispersion relation

$$\hbar\omega_k = \sqrt{(H_k + 2gn_0 - \mu)^2 - (gn_0)^2}\,, \tag{7}$$

with $H_k = \hbar^2 k^2/2m + U_0(\mathbf{r})$. In the limit of very short wavelengths, we can neglect the confining potential and set $U_0 = 0$, as well as $\mu = gn_0$. We are then reduced to the well known expression

$$\omega_k = \sqrt{c_s^2 k^2 + \frac{\hbar^2 k^4}{4m^2}}\,, \quad c_s = \sqrt{\frac{gn_0}{m}} \tag{8}$$

Here c_s is the Bogoliubov sound speed. A generalization of this dispersion relation to twisted BdG modes in homogeneous, cylindrical and toroidal geometries can be found in [12]. At this point we introduce $u_k = |u_k| \exp(i\varphi_u)$ and $v_k = |v_k| \exp(i\varphi_v)$. The mode energy, or density, can then be written as

$$n_k = |\psi_k|^2 = |u_k|^2 + |v_k|^2 + 2|u_k||v_k| \cos(\varphi_u - \varphi_v) \tag{9}$$

For mode components in quadrature, we have $\cos(\varphi_u - \varphi_v) = 0$, and symmetry arguments imply that $|u_k|^2 = |v_k|^2$. At this point, it should be noticed that eqs. (6) can be reduced to

$$\left[\hbar^2\omega_k^2 - (H_r^2 - |g\psi_0^2|^2)\right] u_k = 0\,. \tag{10}$$

It is useful to introduce the new quantities

$$U_k(\mathbf{r}, t) = u_k(\mathbf{r}) \exp(-i\omega_k t)\,, \quad V_k(\mathbf{r}, t) = v_k(\mathbf{r}) \exp(-i\omega_k t)\,. \tag{11}$$

They obviously satisfy the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \omega_k^2\right)(U_k, V_k) = 0\,, \tag{12}$$

where ω_k is determined by Eq. (8). This will be useful in the study of the nonlinear coupling between the BdG modes (or *bogolons*) and a vortex, as shown next.

III. KINETIC EQUATION FOR THE BOGOLON GAS

In order to study the energy transfer between a slowly varying perturbation, more specifically, a single vortex, and a background spectrum of turbulent fluctuations, we use for the slow component of the condensate wavefunction

$$\psi_0(\mathbf{r}, t) = \psi_{00}(\mathbf{r}) + \psi_v(\mathbf{r}, t)\,, \tag{13}$$

where the first term is the steady-state part of the condensate, and ψ_v is the disturbance associated with the vortex. We now have

$$|\psi_0|^2 = n_0 + \psi_{00}\psi_v^* + \psi_{00}^*\psi_v\,, \quad n_0 = |\psi_{00}|^2 + |\psi_v|^2\,. \tag{14}$$

We notice that, for $U_0 = 0$ and $\mu = gn_0$, we have

$$H_r^2 - |g\psi_0^2|^2 = -\frac{\hbar^2\nabla^2}{2m}\left(-\frac{\hbar^2\nabla^2}{2m} + gn_0\right) + 2g^2 n_0 \left(\psi_{00}\psi_v^* + \psi_{00}^*\psi_v\right)\,. \tag{15}$$

This means that, in a condensate perturbed by a vortex ψ_v, the BdG or bogolon mode can be described by

$$\left[\frac{\partial^2}{\partial t^2} - \left(gn_0 - \frac{\hbar^2\nabla^2}{2m}\right)\frac{\nabla^2}{2m} - G\right](U_k, V_k) = 0\,, \tag{16}$$

with the auxiliary function $G \equiv G(\mathbf{r}, t)$ defined by

$$G = 2\frac{g^2 n_0}{\hbar^2}\left(\psi_{00}\psi_v^* + \psi_{00}^*\psi_v\right)\,. \tag{17}$$

It can easily be seen that this mode equation reduces to be above Eq. (12) when the vortex disappears and $\psi_v = 0$. This perturbed mode equation has now to be coupled to the evolution equation for the vortex field ψ_v, which in turn will depend on the mode functions (U_k, V_k), as shown below.

But, before considering the vortex equation, it is useful to replace Eq. (16) by a wave-kinetic equation capable of describing an arbitrary superposition of BdG modes, or in other words, an arbitrary bogolon gas. For that purpose, we follow the standard Wigner-Moyal procedure [10], focusing on the field U_k, given the symmetry with V_k. We start by introducing the auto-correlation function for two distinct pairs of time and positions, as defined by

$$K_{12} = U_k(\mathbf{r}_1, t_1) U_k^*(\mathbf{r}_2, t_2) \equiv U_1 U_2^*\,. \tag{18}$$

From Eq. (16) we obtain the evolution equation for this quantity as

$$\left[\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} + c_s^2\left(\nabla_1^2 - \nabla_2^2\right) - \frac{\hbar^2}{4m^2}\left(\nabla_1^4 - \nabla_2^4\right) + G_1 - G_2\right] K_{12} = 0\,, \tag{19}$$

where we have used the obvious notation $G_j \equiv G(\mathbf{r}_j, t_j)$, for $(j = 1, 2)$. We now define new pairs of space variables $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, and $\mathbf{s} = \mathbf{r}_2 - \mathbf{r}_2$, and similarly for time $t = (t_1 + t_2)/2$, and $\tau = t_2 - t_1$, and introduce the double Fourier transformation

$$K_{12} \equiv K(\mathbf{r}, t, \mathbf{s}, \tau) = \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} W(\mathbf{r}, t, \mathbf{k}, \omega) \exp(i\mathbf{k} \cdot \mathbf{s} - i\omega\tau) \, . \tag{20}$$

The new function $W \equiv W(\mathbf{r}, t, \mathbf{k}, \omega)$ is the Wigner function for the BdG field. Replacing this in Eq. (19) we are then able to derive the following equation determining the evolution of W

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_k \cdot \nabla \right) W = \frac{1}{\omega} G \sin\left(\Lambda W\right) \, . \tag{21}$$

This is the wave-kinetic equation for the bogolon field, as described by the Wigner function W. Here, we have used the bogolon group velocity $\mathbf{v}_k$, as determined from the above dispersion relation

$$\mathbf{v}_k = \frac{\partial \omega_k}{\partial \mathbf{k}} = \left(c_s^2 + \frac{\hbar^2 k^2}{2m^2} \right) \frac{\mathbf{k}}{\omega} \, . \tag{22}$$

In Eq. (21) we have also used the double-sided operator

$$\Lambda = \frac{1}{2} \left(\overleftarrow{\nabla} \cdot \frac{\overrightarrow{\partial}}{\partial \mathbf{k}} - \frac{\overleftarrow{\partial}}{\partial t} \frac{\overrightarrow{\partial}}{\partial \omega} \right) \, , \tag{23}$$

where $\overleftarrow{\nabla}$ and $\overleftarrow{\partial}/\partial t$ act backwards on G, whereas $\overrightarrow{\partial}/\partial \mathbf{k}$ and $\overrightarrow{\partial}/\partial \omega$ act forward on W. We can considerably simplify this equation by assuming the quasi-classical or geometric optics approximation ,where the bogolons can be described as classical quasi-particles. In this limit we can use $\sin \Lambda \simeq \Lambda$. The wave-kinetic equation (21) can also be written in another equivalent form, as

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_k \cdot \nabla \right) W = \frac{1}{2\omega} \int \frac{d\mathbf{q}}{(2\pi)^3} \int \frac{d\Omega}{2\pi} G(\Omega, \mathbf{q}) [W^- - W^+] \exp(i\mathbf{q} \cdot \mathbf{r} - i\omega t) \, . \tag{24}$$

In this new equation we have used the quantities $W^\pm \equiv W(\omega \pm \Omega/2, \mathbf{k} \pm \mathbf{q}/2)$, and the Fourier components

$$G(\Omega, \mathbf{q}) = \int d\mathbf{r} \int dt \, G(\mathbf{r}, t) \exp(-i\mathbf{q} \cdot \mathbf{r} + i\omega t) \tag{25}$$

The quasi-classical approximation can be recovered by using the development

$$W^\pm \simeq W \pm \frac{\Omega}{2} \frac{\partial W}{\partial \omega} \pm \frac{\mathbf{k}}{2} \cdot \frac{\partial W}{\partial \mathbf{k}} \tag{26}$$

To complete our discussion of the wave-kinetic description of the BdG mode field, we assume that the frequency ω_k of each mode $\mathbf{k}$ is determined by its linear dispersion relation. This approximation is sometimes called the *particle approximation*, and justifies the use of a reduced Wigner function, defined as

$$W(\mathbf{r}, t, \mathbf{k}) = 2\pi W(\mathbf{r}, t, \mathbf{k}, \omega)\delta(\omega - \omega_k) \tag{27}$$

This reduced form of W will be used in the following. It is also useful to state that the wave-kinetic in the geometric optics limit reduces to a Vlasov-type of equation, which takes the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_k \cdot \nabla + \mathbf{F}_k \cdot \frac{\partial}{\partial \mathbf{k}}\right) W = 0\,, \tag{28}$$

where $\mathbf{F}_k = -\nabla(G/2\omega_k)$ plays the role of a force acting on the BdG quasi-particles.

IV. VORTEX IN A BOGOLON FIELD

We consider now the evolution of a vortex in a turbulent background, as described by the Wigner quasi-distribution $W(\mathbf{r}, t, \mathbf{k})$. For simplicity, we use the geometric optics approximation. Generalization to the exact wave-kinetic description involves an heavier description, but is straightforward. The turbulent gas of bogolons is then described by Eq. (28), while the vortex is described by Eq. (3). At this point, it should be noticed that

$$n_T \equiv \left\langle |\tilde{\psi}|^2 \right\rangle = \int W(\mathbf{r}, t, \mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^3}\,. \tag{29}$$

Using Eq. (13), this allows us to rewrite Eq. (3) in the new form

$$i\hbar \frac{\partial \psi_v}{\partial t} = [H_0 + g n_0 - \mu]\,\Psi_v + 2g\psi_{00} \int W(\mathbf{r}, t, \mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^3}\,. \tag{30}$$

We can also assume that the static mean field ψ_{00} is determined by the condition, $[H_0 + g n_0 - \mu]\,\psi_{00} = 0$. This will determine the Thomas-Fermi density profile. We have also neglected the ψ_v contribution to the last term of In Eq. (30), which is valid for the perturbative analysis to be discussed here. For the nonlinear saturation regime, this contribution would have to be included.

At this point, we assume a generic vortex solution of the form $\psi_v(\mathbf{r}, t) = \Psi(\mathbf{r}) \exp(-i\Omega t)$. We can also write, for the bogolon gas distribution, $W(\mathbf{r}, t, \mathbf{k}) = W_0(\mathbf{r}) + \delta W(\mathbf{r}, t, \mathbf{k})$, where

$\delta W(\mathbf{r},t,\mathbf{k}) = W_v(\mathbf{r},\mathbf{k})\exp(-i\Omega t)$ is the perturbation of the bogolon gas, induced by the presence of the vortex. Replacing this in Eq. (30), we get

$$\hbar\Omega\Psi = [H_0 + gn_0 - \mu]\,\Psi + 2g\psi_{00}\int W_v(\mathbf{r},\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^3}\;. \tag{31}$$

In order to derive a closed equation for the quantity Ψ, we need to relate W_v to Ψ, which can be done with the wave-kinetic equation (28). Noting that

$$\mathbf{F}_k = -\frac{g}{2\omega_k}\psi_{00}^*\nabla\Psi\,, \tag{32}$$

we obtain

$$(-i\Omega + \mathbf{v}_k\cdot\nabla)\,W_v = \frac{g}{2\omega_k}\psi_{00}^*\nabla\Psi\cdot\frac{\partial W_0}{\partial\mathbf{k}}\;. \tag{33}$$

To proceed further, we take the plausible assumption that the spatial structure of W_v has the same shape of the vortex itself, which allows us to write $W_v(\mathbf{r},\mathbf{k}) = \Psi(\mathbf{r})A_v(\mathbf{r},\mathbf{k})$. The last term in Eq. (31) becomes equal to

$$2g\psi_{00}\int W_v(\mathbf{r},\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^3} = -g^2|\psi_{00}|^2\Psi\mathbf{Q}\cdot\int\frac{(\partial W_0/\partial\mathbf{k})}{\omega_k(i\Omega - \mathbf{v}_k\cdot\mathbf{Q})}\frac{d\mathbf{k}}{(2\pi)^3}\;. \tag{34}$$

Here, we have introduced the new vector function $\mathbf{Q}\equiv\mathbf{Q}(\mathbf{r})$, such that $\mathbf{Q} = \nabla\Psi/\Psi$. Similarly, Eq. (33) can be written in terms of the quantity A_v, as

$$A_v(\mathbf{r},\mathbf{k}) = -\frac{g}{2\omega_k}\psi_{00}^*\mathbf{Q}\cdot\frac{(\partial W_0/\partial\mathbf{k})}{(i\Omega - \mathbf{v}_k\cdot\mathbf{Q})}\;. \tag{35}$$

It is useful to notice that, in the absence of bogolon turbulence, the vortex solution would imply that $\Omega = \mu/\hbar$. The vortex solution would therefore be determined by the simple equation

$$[H_0 + gn_0 - 2\mu]\,\Psi(\mathbf{r}) = 0. \tag{36}$$

For a vortex around the z-axis, the corresponding solution would therefore be of the form $\Psi(\mathbf{r}) = R(r)\Phi(z)\exp(il\theta)$, where the integer l is the vortex charge, and cylindrical coordinates were used. The presence of turbulence introduces an energy correction to the vortex, ϵ, as determined by

$$\hbar\Omega = \mu + \epsilon \tag{37}$$

Assuming that Eq. (36) is still satisfied, we can reduce Eq. (31) to

$$\epsilon\Psi = 2g\psi_{00}\int W_v(\mathbf{r},\mathbf{k})\frac{d\mathbf{k}}{(2\pi)^3}\;. \tag{38}$$

Finally, using Eq. (30), and integrating over the entire volume V of the condensate, we obtain

$$\epsilon = -\frac{g^2}{V}\int_V d\mathbf{r}\, n_{00}(\mathbf{r})\mathbf{Q}\cdot\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{(\partial W_0/\partial \mathbf{k})}{\omega_k(i\Omega - \mathbf{v}_k\cdot\mathbf{Q})}\,. \tag{39}$$

This is the main result of the present paper. It gives the energy correction to the vortex due to the presence of an arbitrary spectrum of BdG turbulence, as described by the unperturbed Wigner function W_0.

V. VORTEX STABILITY

We can now analyze the problem of vortex stability and the possible exchange of energy between the vortex and the bogolon spectrum of elementary excitations. First, we notice that the vector function $\mathbf{Q}(\mathbf{r})$ can be written in a more explicit form as

$$\mathbf{Q} \equiv \frac{\nabla\Psi}{\Psi} = \frac{\mathbf{e}_r}{L_r} + \frac{\mathbf{e}_z}{L_z} + il\frac{\mathbf{e}_\theta}{r}\,, \tag{40}$$

with

$$L_r^{-1} = \frac{1}{R}\frac{dR}{dr}\,,\quad L_z^{-1} = \frac{1}{\Phi}\frac{d\Phi}{dz}\,, \tag{41}$$

where $R(r)$ and $\Phi(z)$ define the unperturbed form of the vortex. As a simple example, let us consider the case where all the BdG modes propagate along the z-axis, as described by the simple Wigner function $W_0(\mathbf{k}) = (2\pi)^2 W_0(k_z)\delta(\mathbf{k}_\perp)$. Eq. (39) is then reduced to

$$\epsilon = -\frac{g^2}{V}\int_V d\mathbf{r}\, n_{00}(\mathbf{r})\int \frac{dk_z}{2\pi}\frac{(\partial W_0/\partial k_z)}{\omega_k(i\Omega L_z - v_k)}\,. \tag{42}$$

Let us now focus on the imaginary part of this energy correction, $\Gamma = \Im(\epsilon) = \Im(\Omega)$, which describes the possible occurrence of an instability. We get

$$\Gamma = \frac{g^2}{V}\int_V d\mathbf{r}\, n_{00}(\mathbf{r})\int \frac{dk_z}{2\pi}\frac{\mu L_z}{\omega_k}\frac{(\partial W_0/\partial k_z)}{(\Gamma L_z + v_k)^2 + \mu^2 L_z^2}\,. \tag{43}$$

For a nearly homogeneous condensate with $n_{00}(\mathbf{r}) \simeq n_0 = cte.$, and for $\mu L_z \gg (\Gamma L_z, v_k)$, this can be approximately written as

$$\Gamma \simeq \frac{g^2 n_0}{\mu V}\int_V \frac{d\mathbf{r}}{L_z}\int \frac{dk_z}{2\pi}\frac{1}{\omega_k}\frac{\partial W_0}{\partial k_z}\,. \tag{44}$$

And, using the Bogoliubov dispersion relation, we finally get

$$\Gamma \simeq \frac{g^2 n_0}{\mu V}c_s\int\int \frac{L_z(\mathbf{r})}{\omega_k^2}W_0(\mathbf{r},k_z)\frac{dk_z}{2\pi}d\mathbf{r}\,. \tag{45}$$

This result clearly shows that, in the condensate regions where the turbulence is present, and if $L_z < 0$, we have damping of the vortex due to its interaction with the phonons. It means that the phonons tend to gain energy. In contrast, if $L_z > 0$, the vortex grows at the expense of the turbulence energy. In a vortex with a finite size, this can be summarized by saying that the phonons will damp the vortex at the vortex front (where L_z decreases) and will excite the vortex at the rear. **In both cases, the vortex will become unstable, and eventually decay into other vortex solutions, with emission or absorption of bogolons. The final result of the instability cannot be described by the present linear stability analysis. Only in the case of $L_z = 0$ can we strictly say that the vortex remains stable in the presence of turbulence. This will be the case of a vortex aligned with the z-axis.**

But we can also consider vortices with finite curvature, as those discussed in detail by [22, 23], and shown in Fig. 1. It obviously has $L_z > 0$ for $z < 0$, and $L_z < 0$ for $z > 0$. We can see from eq. (45) that such a curved vortex will remain stable if immersed in homogeneous turbulence (Fig. 1a), and become unstable under the action of inhomogeneous turbulence, as illustrated in Fig. 1b. A variety of situations can therefore occur where the value and sign of Γ will depend on the configuration of the bogolon spectrum, as defined by $W_0(\mathbf{r}, k_z)$, and on the way it occupies the vortex volume.

Finally, we would like to notice that the vortex itself will be forced to move due to existence of turbulence. Adapting the analysis of [23] to the present problem, we can easily conclude that the local velocity $\mathbf{v}(\mathbf{r})$ of a vortex line in the presence of an arbitrary bogolon distribution $W_0(\mathbf{r}, k_z)$ is given by

$$\mathbf{v}(\mathbf{r}) = \frac{3l\hbar}{2m\mu} \ln\left(\frac{R_\perp}{|l|\xi}\right) (\hat{\mathbf{e}}_z \times \nabla_\perp) \int W_0(\mathbf{r}, \mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^3} \,. \tag{46}$$

Here $R_\perp$ represents the condensate dimensions in the perpendicular direction, and ξ is the healing length. This expression is valid for a vortex with week curvature, if we neglect the confining potential. The evolution of a vortex in the bogolon gas will eventually modify the present stability analysis.

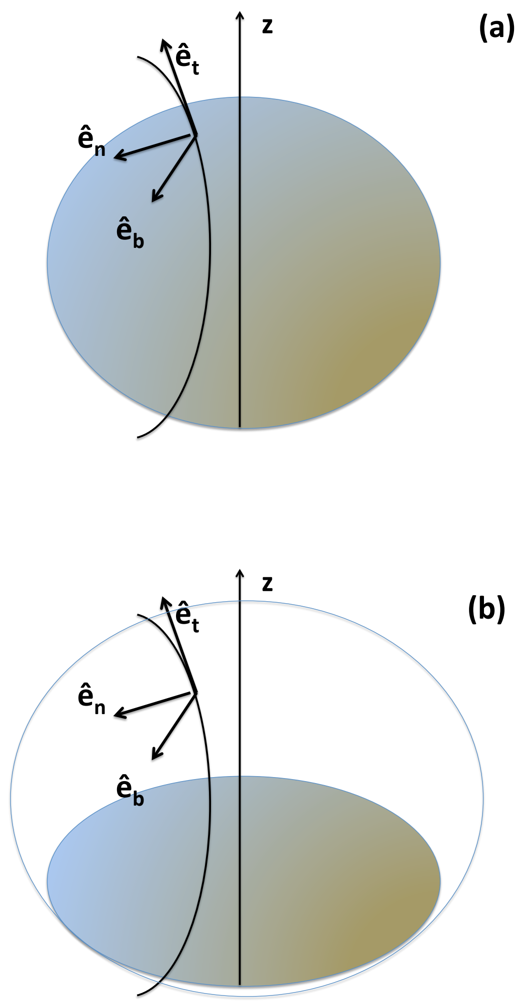

FIG. 1: *Vortex with a finite curvature in a bogolon gas. The vortex line is represented, as well as the condensate region occupied by turbulence: (a) stable configuration in homogeneous turbulence; (b) Unstable configuration when the bogolon gas only covers part of the condensate volume.*

VI. CONCLUSIONS

We have studied the vortex-phonon interactions in a Bose-Einstein condensate. We have considered the case where a single vortex interacts with an arbitrary spectrum of elementary excitations, or BdG modes, which we have associated with a bogolon field. Starting from a generic form of BdG equations, we have derived a wave-kinetic equation which determines the evolution of the bogolon field. The field is described by a Wigner function. Exact and approximate versions of the wave-kinetic equation where stated. In the quasi-classical approximation, the field can be described as a gas of quasi-particles, the bogolons, which correspond to phonons propagating in a condensed quantum gas. In this case, the wave-kinetic equation reduces to a Vlasov-type of equation, and the Wigner quasi-distribution reduces to a classical distribution function.

Using a perturbative analysis we were then able to derive the growth rate of a vortex in the turbulent field, and characterized the possible regimes where instability can eventually take place. The present analysis shows that, in general, a finite exchange of energy takes place between a vortex and the surrounding oscillations, which could be useful to future analysis of simulations and experiments. The present stability analysis is valid in the geometric optics or quasi-classical approximation, where the typical wavelength of the bologons is much smaller than the size of the vortex. But the same approach can be used for the general case, if instead of the Vlasov equation we use the exact wave-kinetic equations for the bogolon field. **We have also called the attention to the possible existence of vortex motion in the presence of turbulence, which will eventually modify the vortex stability. The possible existence of stable vortex-bogolon configurations is a very interesting but difficult problem, which will be analyzed elsewhere.**

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