

# Nonlinear hydrodynamic Langmuir waves in fully degenerate relativistic plasma

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The combined effect of special relativity and electron degeneracy on Langmuir waves is analyzed, by utilizing a rigorous fully relativistic hydrodynamic model. Assuming a traveling wave solution form, a set of conservation laws is identified, together with a pseudo-potential function depending on the relativistic parameter  $p_F/(mc)$  (where  $p_F$  is the Fermi momentum,  $m$  is the mass of the charge carriers, and  $c$  the speed of light), as well as on the amplitude of the electrostatic energy perturbation.

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## I. INTRODUCTION

In recent years, a number of special-relativistic hydrodynamic models of charged particle systems have appeared. In this respect, many works in the literature have already been properly criticized [1, 2], the main points namely being: (a) they do not take into account relativistic mass increase due to random, not bulk motion; (b) the pressure term is frequently written in a non-Lorentz-invariant form; (c) the majority of approaches are not consistent with the well known relativistic hydrodynamics of neutral fluids [3, 4]. The purpose of the present work is to perform an analysis of joint relativistic and Fermi-degenerate effects on Langmuir waves in one-species plasmas, using the accurate relativistic hydrodynamic model presented in [1, 2]. The interest in plasmas with relativistic effects arising precisely due to degeneracy, with Fermi momenta of the order of  $mc$ , where  $m$  is the mass of the charge carriers and  $c$  the speed of light, is increasing due to the large densities achieved nowadays in both laboratory and astrophysical settings [5]. Examples are provided by plasmas present in intense laser-solid interaction experiments, or in compact stars as white dwarfs. Hence a proper formulation of these quantum-relativistic plasmas in a simplified fluid framework is necessary, in benchmark problems like the propagation of Langmuir waves. Fluid-theoretical studies can be taken as a first step towards the understanding of the nonlinear features of these systems.

The propagation of relativistic Langmuir waves in plasmas is a traditional subject, but has never been explicitly considered in connection to Fermi degeneracy. Akhiezer and Polovin [6] and Dawson [7] considered the wave breaking problem in the cold case. Later, Coffey [8], in the non-relativistic case, and Katsouleas [9] in the relativistic case, addressed the same problem including thermal effects by means of a waterbag [10] model, which is formally analogous to a one-dimensional fully degenerate plasma model. This physical correspondence, however, was not recognized. In this context, one of the purposes of the present work is to fill this gap, starting from first principles and (in another distinctive approach) adopting the three-dimensional Fermi gas equation of state.

Admittedly, our formulation does not take into account quantum wave effects manifested in the dispersive, quantum Bohm potential term. The justification for this limitation is that for dense plasmas [11, 12] one can prove that quantum statistical effects are of a higher order relevance than quantum diffraction. Other quantum plasma systems, like those in ultra-small electronic devices depending on quantum wave-like effects such as quantum tunneling, would deserve an extended fluid treatment, whose rigorous form is unknown at present.

This work is organized as follows. In Section II, the appropriate relativistic hydrodynamic plasma model and equations of state for fully degenerate plasma are reviewed. In Section III, the analytical model for Langmuir waves is reduced to a set of ordinary differential equations, by seeking traveling-wave type solutions. The corresponding conservation laws are identified. Assuming a propagation speed of the order of the speed of light, linear and nonlinear solutions are addressed in Section IV. The conclusions are written in Section V.

## II. RELATIVISTIC HYDRODYNAMICS AND EQUATIONS OF STATE

Our starting point is the relativistic fluid model for one-species plasma, as proposed in [1, 2],

$$\frac{\partial(\gamma n)}{\partial t} + \frac{\partial(\gamma n u)}{\partial x} = 0, \quad (1)$$

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}\right)(\gamma u) = -\frac{\gamma}{H m n} \left(\frac{\partial}{\partial x} + \frac{u}{c^2} \frac{\partial}{\partial t}\right) P + \frac{e}{H m} \frac{\partial \phi}{\partial x}, \quad (2)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0} (\gamma n - n_0), \quad (3)$$

where  $n$  and  $u$  are respectively the number density and velocity field in the laboratory frame,  $\phi$  is the electrostatic field,  $\gamma = (1 - u^2/c^2)^{-1/2}$  is the relativistic dilation factor where  $c$  is the speed of light,  $-e$  is the electron's charge,  $\varepsilon_0$  is the vacuum's permittivity, and  $n_0$  is an equilibrium ionic background. In the momentum equation,  $P$  and  $H$  denote respectively the scalar pressure and a dimensionless enthalpy. These quantities should be specified by means of equations of state, ensuring the closure of the system, as described below.

The relativistic equations for charged fluids have been presented in several, nonequivalent ways in the literature; see e.g. [1, 2] for a critical review. The present form is manifestly consistent with the form(s) proposed in [3, 4] for neutral fluids, as obtained via a Lorentz transformation from the local proper reference system. The inclusion of the electrostatic field is obtained thanks to the covariant coupling with the electromagnetic stress-energy tensor, although here magnetic field generation is suppressed, for analytical simplicity. The enthalpy-like quantity  $H$  represents a non-trivial relativistic mass increase due to thermal motion only, and is frequently omitted in less rigorous relativistic treatments. On the other hand, the  $\gamma$  factor here representing the relativistic mass increase due to bulk motion appears in all possible and non-equivalent ways in fluid formulations in the literature, as remarked in [1, 2]. In addition, notice the manifestly Lorentz-invariant form of the pressure term in Eq. (2).

In the present treatment, changes are allowed along one spatial direction  $x$ , in a rectangular geometry with a full 3D velocity field  $\mathbf{u} = u(x, t)\hat{x}$ . Other, potentially interesting situations, would consider propagation of waves in truly one-degree-of-freedom systems, as can happen e.g. in the surroundings of neutron stars atmospheres, where the existence of super-strong magnetic fields effectively limit the dynamics to one dimension [13, 14]. Our choice is dictated by the interest in the analysis of quasi-1D wave structures propagating in otherwise isotropic media, such as in dense plasmas where relativistic and degeneracy effects come together. This scenario is likely to appear in the next generation of dense laser-plasma interaction experiments, or in astrophysical objects like white dwarfs. In this case, on physical grounds it is reasonable to assume energy spreading among the available degrees of freedom, so that a 3D equation of state is obeyed.

We note that, unlike non-degenerate plasmas, the equations of state for degenerate plasmas have a form which is strongly dependent on the number of available degrees of freedom. A more complete treatment, considering also 1D and 2D systems, will be reported elsewhere.

The equations of state for a fully degenerate electron gas are known since long, see Chandrasekhar [15] and also Oppenheimer and Volkoff [16]. Unlike these celebrated works devoted to the question of stellar structure, here we focus on the electrostatic and not the gravitational aspects. For completeness, we briefly review the derivation of the relevant equations of state, assuming in equilibrium a completely degenerate Fermi-Dirac distribution  $f = f(\mathbf{r}, \mathbf{p})$  in phase space,

$$f(\mathbf{r}, \mathbf{p}) = 2 \left(\frac{1}{2\pi\hbar}\right)^3, \quad |\mathbf{p}| < p_F; \quad f(\mathbf{r}, \mathbf{p}) = 0, \quad |\mathbf{p}| > p_F, \quad (4)$$

where  $\hbar$  is the scaled Planck's constant and  $p_F$  is the Fermi momentum. By means of  $\int f d\mathbf{p} = n_0$  one derive  $p_F = (3\pi^2 n_0)^{1/3} \hbar$ . The factor 2 in Eq. (4) is due to the electron spin.

For the scalar pressure, consider the average momentum flux

$$P_0 = \frac{1}{3} \int f \mathbf{v} \cdot \mathbf{p} d\mathbf{p}, \quad (5)$$

where  $\mathbf{v} = \mathbf{p}/(m\gamma)$ ,  $\gamma = [1 + p^2/(m^2 c^2)]^{1/2}$ . The result is

$$P_0 = \frac{\pi m^4 c^5}{3(2\pi\hbar)^3} \left[ \zeta_0 (2\zeta_0^2 - 3) \sqrt{1 + \zeta_0^2} + 3 \sinh^{-1} \zeta_0 \right], \quad (6)$$

where

$$\zeta_0 = \frac{p_F}{mc} = \frac{\hbar}{mc} (3\pi^2 n_0)^{1/3} \quad (7)$$

is the relativistic parameter [17]. Regarding the enthalpy-like function  $H$ , it is defined in equilibrium as

$$H_0 = \frac{\rho_0 + P_0}{n_0 mc^2} = \sqrt{1 + \zeta_0^2}, \quad (8)$$

where

$$\begin{aligned} \rho_0 &= \int f(p^2 c^2 + m^2 c^4)^{1/2} d\mathbf{p} \\ &= \frac{\pi m^4 c^5}{(2\pi\hbar)^3} \left[ \zeta_0 (2\zeta_0^2 + 1) \sqrt{1 + \zeta_0^2} - \sinh^{-1} \zeta_0 \right] \end{aligned} \quad (9)$$

is the equilibrium mass-energy density.

The above results have been found for a strict equilibrium. Assuming these relations to be valid in a local quasi-equilibrium, we replace  $n_0 \rightarrow n$  and derive the Chandrasekhar equations of state, to be inserted in the hydrodynamic equations,

$$\frac{P}{n_0 mc^2} = \frac{1}{8\zeta_0^3} \left[ \zeta (2\zeta^2 - 3) \sqrt{1 + \zeta^2} + 3 \sinh^{-1} \zeta \right], \quad (10)$$

$$\frac{\rho}{n_0 mc^2} = \frac{3}{8\zeta_0^3} \left[ \zeta (2\zeta^2 + 1) \sqrt{1 + \zeta^2} - \sinh^{-1} \zeta \right], \quad (11)$$

$$H = \sqrt{1 + \zeta^2}, \quad \zeta = \frac{\hbar}{mc} (3\pi^2 n)^{1/3} = \zeta_0 \left( \frac{n}{n_0} \right)^{1/3}. \quad (12)$$

In passing, note that  $P$  and  $\rho$  satisfy Taub's inequality [18]. Having at hand a closed system for the evolution of  $n$ ,  $u$  and  $\phi$ , in the following we shall investigate the behavior of a particular class of possible solutions, namely traveling-waves.

### III. TRAVELING-WAVE SOLUTIONS AND CONSERVATION LAWS

Assuming all fields to be dependent only on the travelling coordinate

$$X = x - Vt, \quad (13)$$

where  $V$  is the propagation speed, it is straightforward to see that the continuity, momentum and Poisson equations can be integrated once, thus yielding the first integrals  $J$ ,  $I$  and  $K$  given respectively by

$$J = \gamma n (V - u), \quad (14)$$

$$I = Hmc^2 \left( 1 - \frac{Vu}{c^2} \right) \gamma - e\phi, \quad (15)$$

$$K = \frac{\varepsilon_0}{2} \left( \frac{d\phi}{dX} \right)^2 + n_0 Hmc^2 \gamma - P + Hm(J - n_0 V) \gamma u. \quad (16)$$

These conservation laws have a general applicability, as long as the thermodynamic relation

$$H = \frac{1}{mc^2} \int \frac{dP}{n} \quad (17)$$

holds.

It is convenient to postulate the existence of a position  $X = X_0$  such that

$$n(X = X_0) = n_0, \quad u(X = X_0) = 0, \quad \frac{d\phi}{dX}(X = X_0) = -E_0, \quad (18)$$

where  $E_0$  is the electric field at the reference point  $X = X_0$ . This corresponds to a fluid element at rest and with the equilibrium density, possibly under the action of a non-zero acceleration due to a launched electric field. In this way, and for a convenient gauge choice for the scalar potential, one may evaluate

$$J = n_0 V, \quad (19)$$

$$e\phi = Hmc^2 \left(1 - \frac{Vu}{c^2}\right) \gamma - H_0 mc^2, \quad (20)$$

$$K = \frac{\varepsilon_0}{2} \left(\frac{d\phi}{dX}\right)^2 + n_0 Hmc^2 \gamma - P = \frac{\varepsilon_0 E_0^2}{2} + \rho_0, \quad (21)$$

We assume  $V \neq 0$  in the present context, to avoid trivial solutions. Without loss of generality, it is supposed that  $V > 0$ .

To proceed, a number of possibilities can be considered. Since in our case the scalar pressure is a complicated expression of the density, it is appropriate to express the conservation equations in terms of  $n$ . Furthermore, significant simplification appear in the physically interesting case of very large value of the propagation speed  $V \approx c$ , so that the simple relations

$$\frac{u}{c} = \frac{1 - (n_0/n)^2}{1 + (n_0/n)^2}, \quad e\phi = Hmc^2 \frac{n_0}{n} - H_0 mc^2 \quad (22)$$

are approximately valid. In this way, the  $K$  conservation law after a few algebraic steps takes the form

$$\frac{1}{2} \left(\frac{d\bar{n}}{d\bar{X}}\right)^2 + U(\bar{n}) = 0, \quad (23)$$

where

$$\bar{n} = \frac{n}{n_0}, \quad \bar{X} = \frac{\omega_p X}{c}, \quad \omega_p = \left(\frac{n_0 e^2}{m \varepsilon_0}\right)^{1/2}, \quad (24)$$

with the pseudo-potential

$$U(\bar{n}) = \frac{H^2 \bar{n}^4}{(1 + 2\zeta^2/3)^2} \left[ \frac{H(\bar{n}^2 + 1)}{2\bar{n}} - \frac{1}{8\zeta_0^3} (\zeta(2\zeta^2 - 3)H + 3 \sinh^{-1} \zeta) \right. \\ \left. - \frac{3}{8\zeta_0^3} (\zeta_0(2\zeta_0^2 + 1)H_0 - \sinh^{-1} \zeta_0) - \delta \right], \quad (25)$$

In this expression,  $H$  and  $\zeta$  are functions of  $\bar{n}$  according to Eq. (12), and

$$\delta = \frac{\varepsilon_0 E_0^2}{2n_0 mc^2} \quad (26)$$

is a measure of the electrostatic energy perturbation amplitude.

#### IV. LINEAR AND NONLINEAR OSCILLATIONS

Given a non-zero input  $\delta$ , the pseudo-potential in Eq. (25) admits linear oscillations around a minimum  $\bar{n} = \bar{n}^*$  such that  $(dU/d\bar{n})(\bar{n}^*) = 0$ ,  $(d^2U/d\bar{n}^2)(\bar{n}^*) > 0$ . Expanding around the equilibrium using  $\bar{n} = \bar{n}^* + \bar{n}_1$  gives

$$\frac{1}{2} \left(\frac{d\bar{n}_1}{d\bar{X}}\right)^2 + \frac{k^2}{2} (\bar{n}_1)^2 = -U(\bar{n}^*), \quad (27)$$

where, as can be found up to first order in  $\delta$ ,

$$\bar{n}^* = 1 + \frac{2\delta(18 + 29\zeta_0^2 + 10\zeta_0^4)}{H_0(3 + 2\zeta_0^2)^2}, \quad (28)$$

$$k^2 = \left(\frac{d^2U}{d\bar{n}^2}\right)(\bar{n}^*) = \\ = \frac{3H_0}{3 + 2\zeta_0^2} + \frac{8\delta(243 + 789\zeta_0^2 + 923\zeta_0^4 + 456\zeta_0^6 + 80\zeta_0^8)}{H_0^2(3 + 2\zeta_0^2)^4}, \quad (29)$$

$$U(\bar{n}^*) = -\frac{9\delta H_0^2}{(3 + 2\zeta_0^2)^2}. \quad (30)$$

For negligible relativistic parameter  $\zeta_0$  and perturbation amplitude  $\delta$  one has  $k^2 = 1$ . After restoring physical coordinates this recovers the usual Langmuir oscillations at the plasma skin depth scale  $c/\omega_p$ . On the other hand, larger degeneracy effects and a larger  $\zeta_0$  gives a smaller  $k$ , as shown in Fig. 1 for parameters representative of dense laser-solid plasma interaction experiments and white dwarfs. Finally, nonlinear oscillations can be predicted from the form of the pseudo-potential, which is depicted in Fig. 2, for a number density typical of white dwarfs and for different values of  $\delta$  (the parameter associated with the electrostatic energy perturbation). Notice that from Eq. (23) one needs  $U(\bar{n}) < 0$  to have a dynamical behavior. In view of the special class of initial conditions in Eq. (18), this requires  $\delta > 0$ , so that the fluid element at the reference point is acted by a launched electric field.

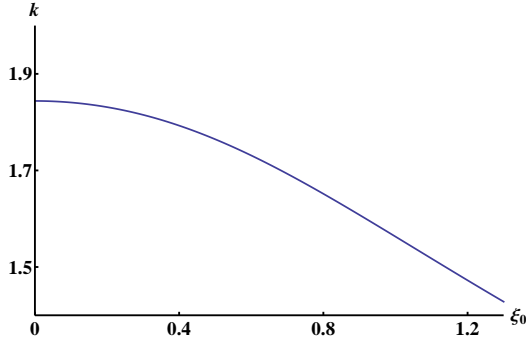


FIG. 1: Illustrative plots of the wave number  $k$  from Eq. (29), for  $\delta = 0.1$  and  $0 \leq \zeta_0 \leq 1.2$ . For reference, for  $n_0 = 10^{32} m^{-3}$  (current intense laser plasma experiments) one has  $\zeta_0 = 0.05$ , while for  $n_0 = 10^{36} m^{-3}$  (white dwarfs) one has  $\zeta_0 = 1.20$ .

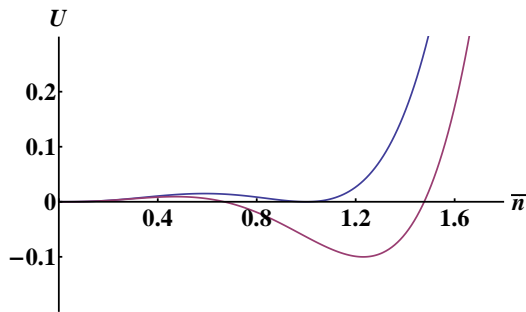


FIG. 2: Pseudo-potential  $U(\bar{n})$  from Eq. (25) for  $\zeta_0 = 1.20$  (white dwarfs) and either  $\delta = 0$  (upper) or  $\delta = 0.1$  (lower).

For initial conditions in the numerical simulations, we set  $X_0 = 0$  and take into account the fact that

$$\left( \frac{d\bar{n}}{d\bar{X}} \right) (0) = \pm \frac{H_0 \sqrt{2\delta}}{1 + 2\zeta_0^2/3}, \quad (31)$$

as follows from the initial condition in Eq. (18) and after managing the conservation laws. Perturbation from equilibrium needs  $\delta \neq 0$ , which is also consistent with a deeper minimum of the pseudo-potential, as can be inferred from Eq. (30). Moreover, using the conservation laws it can be shown that

$$\frac{d^2}{d\bar{X}^2} \left( \frac{H}{\bar{n}} \right) = \frac{1}{2}(\bar{n}^2 - 1), \quad (32)$$

with initial conditions  $\bar{n}(0) = 1$ ,  $(d\bar{n}/d\bar{X})(0)$  given by the relation (31). Typically one get oscillations like shown in Fig. 3, where the dimensionless density  $\bar{n}$  oscillates between the turning points where  $V(\bar{n}) = 0$ . Wave breaking is not observed, due to the propagation speed  $V \approx c$ .

## V. CONCLUSIONS

In this work, a consistent special-relativistic fluid treatment of Langmuir waves in dense plasmas is carried out. The presented model takes into account the relativistic mass increase due to thermal, degeneracy pressure effects, which

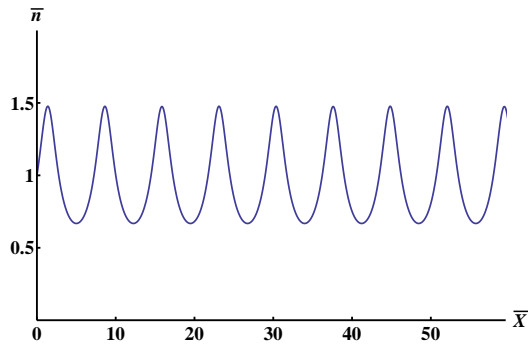


FIG. 3: Oscillations of the dimensionless density  $\bar{n}$  from Eq. (32) with  $\bar{n}(0) = 1$ ,  $\zeta_0 = 1.195$  (white dwarfs) and  $\delta = 0.1$ , which gives  $(d\bar{n}/d\bar{X})(0) = 0.36$ , consistent with Eq. (31). For these parameters one finds the turning points  $\bar{n} = 0.67$ ,  $\bar{n} = 1.48$ , so that  $U(\bar{n}) = 0$  in Eq. (25).

is an often neglected feature. This common over-simplification tends to be dangerous when the Fermi momentum  $p_F$  becomes of the order of  $mc$ , as happens in dense astrophysical objects, for instance. Moreover, our model equations have the pressure in terms of a Lorentz-covariant derivative expression, in accordance with rigorous relativistic hydrodynamic theories [1–4]. Some properties of linear and nonlinear traveling wave solutions have been explored, with emphasis on the role of the relativistic parameter  $\zeta_0$ . A more detailed account is reserved to future work, with a thorough examination of the role of dimensionality, different initial conditions and propagation speed.

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- [1] V. I. Bereziani and S. M. Mahajan, *Phys. Rev. Lett.* **73**, 1110 (1994).
  - [2] N. C. Lee and C. R. Choi, *Phys. Plasmas* **14**, 022307 (2007).
  - [3] S. Weinberg, *Gravitation and Cosmology* (John Wiley and Sons, New York, 1972).
  - [4] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1959).
  - [5] F. Haas, *Quantum Plasmas: An Hydrodynamic Approach* (Springer, N.Y., 2011).
  - [6] A. I. Akhiezer and R. V. Polovin, *Sov. Phys. JETP* **3**, 696 (1956).
  - [7] J. Dawson, *Phys. Rev.* **113**, 383 (1959).
  - [8] T. P. Coffey, *Phys. Fluids* **14**, 1402 (1971).
  - [9] T. Katsouleas and W. B. Mori, *Phys. Rev. Lett.* **61**, 90 (1988).
  - [10] P. Bertrand and M. R. Feix, *Phys. Lett. A* **28**, 68 (1968).
  - [11] B. Eliasson and P. K. Shukla, *J. Plasma Phys.* **76**, 7 (2009).
  - [12] M. McKerr, I. Kourakis and F. Haas, *Plasma Phys. Controll. Fusion* **56**, 035007 (2014).
  - [13] V. Canuto and J. Ventura, *Astrophys. Space Sci.* **18**, 104 (1972).
  - [14] P. Carbonaro, *Il Nuovo Cimento* **103**, 485 (1989).
  - [15] S. Chandrasekhar, *Mon. Not. R. Astron. Soc.* **95**, 207 (1935).
  - [16] J. R. Oppenheimer and G. M. Volkoff, *Phys. Rev.* **55**, 374 (1939).
  - [17] E. E. Salpeter, *Astrophys. J.* **134**, 669 (1961).
  - [18] A. H. Taub, *Phys. Rev.* **74**, 328 (1948).