Li(e)nearity

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Abstract

We illustrate the fact that linearity is a meaningful symmetry in the sense of Lie and Noether. The linearity symmetry role in the quadrature of second-order ordinary differential equations is reviewed, by means of the use of canonical coordinates and identifying a Wronskian-like conserved quantity as a first-order Lie invariant. The Jacobi last multiplier associated to two independent linearity symmetries is applied, in order to derive the Kanai-Caldirola Lagrangian from symmetry principles. In this context, the linearity symmetry is recognized to be also a Noether one. Extensions to higher-order linear ordinary differential equations are derived, where the linearity symmetry are shown to be also Lie and Noether symmetries. The corresponding invariants generalize the traditional Wronskian-like first integrals to higher-order systems.

Keywords: linearity, Lie and Noether symmetries, Jacobi last multiplier, higher-order Lagrangian system

1 Introduction

Let us consider the most general second order linear differential equation (LDE), with independent variable t and dependent one q, put in the standard form

$$\Delta_2(t,q,\dot{q},\ddot{q}) := \ddot{q} + a(t)\dot{q} + b(t)q + c(t) = 0 \tag{1}$$

where the overdot denotes differentiation with respect to t. In classical mechanics, such an equation describes a driven damped harmonic oscillator with *a priori* time-dependent frequency, dissipation rate and excitation. In undergraduate textbooks on mathematics [1–3], one learns that once a nonzero solution $s(t)$ of the homogeneous equation

$$\Delta_{2\mathrm{h}}(t,q,\dot{q},\ddot{q}) := \ddot{q} + a(t)\dot{q} + b(t)q = 0 \tag{2}$$

is known, the differential equation may be reduced to a first order one in the derivative of the dependent variable $z = q/s(t)$. The linearity[1] of Eq. (1) is the key-ingredient of the

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[1]The adjective 'linear' frequently used to designate a differential equation such as (1) is somewhat regrettable in the inhomogeneous case where $c(t) \neq 0$. Although the independent variable q and its derivatives appear linearly in (1), the solution space is generally affine and it would have been preferable to speak in terms of affine differential equations. We have chosen to follow the accepted terminology.

validity of this traditional recipe. However, the introduction of the variable z is commonly presented as a lucky and hence unsatisfactory ansatz. It overlooks the symmetry origin of the method of reduction, as was observed by Sophus Lie himself, the father of the theory of continuous transformation groups. In his classical book entitled *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen* [4], he details some examples of applications of his theory to differential equations for their reduction. In particular, he treats the case of Eq. (1) and exhibits the symmetry responsible of its reduction, a symmetry stemming only from the linearity property. Actually, in place of z, he uses the more convenient variable $w = \dot{q}s(t) - q\dot{s}(t)$ that will be named 'Wronskian variable' in this article[2].

Evidently, since the seminal works of Lie, the referred 'linearity symmetry' has been well recognized in the long history concerning the general second order LDE, where it has been treated mainly as a marginal result. An exception is [7], where the linearity symmetry is used to the reduction of order and quadrature of the homogeneous second order LDE – but without connection to Noether invariance. The existence of linear in velocity conserved quantities for linear time-dependent systems and its connection to invariance principles has been obtained in the literature [8–14]. Most importantly in our context, the Noetherian character of the Wronskian has been recognized [15, 16], at least in the case of the time-dependent harmonic oscillator where $a(t) = c(t) = 0$ in (1). Generalization to linear in velocity invariants for arbitrary multidimensional quadratic Hamiltonian and Lagrangian systems has been also provided [17]. More frequently, there is the emphasis [18] put on the quadratic invariants of the Ermakov-Lewis class [19–22], also known as Courant-Snyder invariant in the context of accelerator physics [16, 23].

If the above is possibly not an exhaustive list, it is enough for our purposes. In spite of the referred literature, recent textbooks [1]- [3] still treat the Wronskian conservation (in the case where $a(t) = c(t) = 0$) as an accidental fact, not linked to symmetries. Therefore, it is the purpose of the present article, to focus centrally on the role of the linearity symmetry in the context of Lie and Noether invariance, specially concerning the Wronskian type first integral to be discussed in the remaining Sections.

It might be mentioned that we consider only point, or geometric, symmetries. For instance, Noether herself [24] followed shortly after by Bessel-Hagen [25] has already formulated the conservation theorem allowing for velocities and higher-order derivatives dependence in the transformation equations. However, at these early times they have not investigated the consequences of this extra freedom, as remarked e.g. by Sarlet and Cantrijn [26]. One could even argue that all conservation laws from regular Lagrangian systems are Noetherian, by direct application of the converse of Noether's theorem. More exactly, the converse of Noether's theorem [26] provides a recipe for the determination of an action symmetry, in principle of a dynamical character, associated to any conserved quantity of the system. For instance, the conservation law of the Ermakov-Lewis and Laplace-Runge-Lenz invariants are well known to be associated to Noether symmetries of non-point character [26, 27]. To conclude, unless otherwise stated, our Noether symmetries are assumed to be point symmetries (pourquoi cette hypothèse?).

This work is organized as follows. In Section 2, the Lie point symmetry associated to the linearity of Eq. (1) is fully discussed. In Section 3 the same symmetry is shown to be a Noether point symmetry for the celebrated Caldirola-Kanai Lagrangian associated to Eq. (1). In Section 4, we consider the extension of the linearity symmetry and therefore the concept of Wronskian to third-order LDEs as well as fourth-order ones derivable from an action principle. Section 5 is devoted to a complete generalization and finally section 6 shows our conclusions.

[2]Our notations differ from Lie's ones. To obtain the latter, make the replacements $t \to x$, $q \to y$, $s(t) \to z(x)$, $w \to v$.

2 Linearity as a Lie point symmetry

2.1 The preliminary case of the first order linear differential equations

Before considering the second-order LDE (1), let us first have a look at the first-order one whose general form is

$$\Delta_1(t,q,\dot q) \coloneqq \dot q + a(t)q + b(t) = 0. \tag{3}$$

If $s(t)$ is a nonzero solution of the homogeneous equation

$$\Delta_{1\mathrm{h}}(t,q,\dot q) \coloneqq \dot q + a(t)q = 0, \tag{4}$$

then the transformation $(t,q) \to (t, q+s(t))$ leaves Δ_1 invariant in the sense that

$$\Delta_1(t, q+s(t), \dot q + \dot s(t)) = \Delta_1(t,q,\dot q) + \Delta_{1\mathrm{h}}(t, s(t), \dot s(t)) = \Delta_1(t,q,\dot q).$$

Alternatively stated, it is a finite symmetry of Δ_1. By linearity, the function $\varepsilon s(t)$ remains a solution of (4) for any value of a real parameter ε thus the family of transformations

$$\mathscr{L}(\varepsilon): \qquad (t,q) \longrightarrow (t, q+\varepsilon s(t)) \tag{5}$$

is a continuous one-parameter symmetry group of Δ_1. *A fortiori*, it leaves Δ_1 invariant on-shell, i.e. it is a Lie (point) symmetry of Eq. (3). Moreover, the time t is left unaffected while $z = q/s(t)$ merely undergoes a translation by ε:

$$z = \frac{q}{s(t)} \longrightarrow \frac{q+\varepsilon s(t)}{s(t)} = z + \varepsilon.$$

Hence, (t,z) is a couple of variables for which the continuous transformation reads simply

$$\mathscr{L}(\varepsilon): \quad (t,z) \longrightarrow (t, z+\varepsilon).$$

These so-called canonical variables of $\mathscr{L}(\varepsilon)$ have the great advantage of leaving the derivatives of z unchanged. Let us exploit this property by first expressing Δ_1 in terms of the new variables through

$$\Delta_1'(t,z,\dot z) \coloneqq \Delta_1(t,q,\dot q) = \Delta_1(t, s(t)z, s(t)\dot z + \dot s(t) z).$$

Equation (3) is equivalent to $\Delta_1' = 0$ and the invariance under $\mathscr{L}(\varepsilon)$ reads now

$$\Delta_1'(t, z+\varepsilon, \dot z) = \Delta_1'(t,z,\dot z).$$

This equality means simply that Δ_1' does not depend on z and that the original LDE (3) is reduced to an equation in t and $\dot z$ only. Explicitly:

$$\Delta_1' = s(t)\dot z + b(t) = 0.$$

Being linear in $\dot z$, this equation is easily integrated to provide the general solution $q(t)$:

$$q(t) = s(t)\left[C - \int \frac{b(t)}{s(t)}\,\mathrm{d}t\right] = e^{-\int a(t)\mathrm{d}t}\left[C - \int b(t)\, e^{\int a(t)\mathrm{d}t}\mathrm{d}t\right], \tag{6}$$

where C is a constant of integration and where one has specified $s(t)$ to be the obvious solution $\exp(-\int a(t)\mathrm{d}t)$ of Eq. (4). Hence, thanks to the linearity symmetry, Eq. (3) is solved by a single quadrature. This fact was seen and discussed by Lie as an application of his theory to first order differential equations, at the end of chapter 8 in Ref. [4].

Actually, the most important result regarding Lie's theory in the realm of first order differential equations is certainly the association of an integrating factor (also named Euler's multiplier) with any Lie point symmetry [5]. In Ref. [4], this association is stated and made explicit in chapter 6, theorem 8. With our notations, it can be restated as follows: if a first order differential equation, put in form

$$T(t,q)\dot{q} - Q(t,q) = 0, \tag{7}$$

is invariant under a transformation

$$(t,q) \longrightarrow (t + \varepsilon\,\eta(t,q), q + \varepsilon\,\xi(t,q))$$

then it admits as integrating factor

$$\mu(t,q) = \frac{1}{T\,\eta - Q\xi}\,.$$

In other words, there exists some function $I(t,q)$ such that

$$\mu(t,q)\big[T(t,q)\dot{q} - Q(t,q)\big] = \dot{I}.$$

Clearly, I keeps a constant value C along the solutions and the equality $I = C$ is said to be a first integral of Eq. (7). In the case of the LDE (3), the linearity symmetry (5) gives rise to the integrating factor $1/s(t) = \exp(\int a(t)\mathrm{d}t)$ and to the first integral

$$I(t,q) \coloneqq \frac{q}{s(t)} + \int \frac{b(t)}{s(t)}\,\mathrm{d}t = q\,e^{\int a(t)\mathrm{d}t} + \int b(t)\,e^{\int a(t)\mathrm{d}t}\mathrm{d}t = C \tag{8}$$

which amounts to (6). Note that I transforms like z under $\mathscr{L}(\varepsilon)$. If one had luckily chosen I instead of z as canonical variable, the reduced equation would have been simply $\dot{I} = 0$.

In conclusion, the linearity symmetry (5) is the source of the well-known integrating factor $\exp(\int a(t)\mathrm{d}t)$ of Eq. (3). On the other hand, it is worth noting that an homogeneous LDE such as (2) or (4) is evidently also invariant under a rescaling of q (here, the scale invariance is a Lie symmetry of the LDE and not merely an invariance of its left-hand side). However, we will not be concerned with this eventual extra symmetry and will remain focused on the invariance under the addition of solutions of the associated homogeneous LDE.

2.2 The case of second order differential equations

Now let us consider the second-order LDE (1) and let $s(t)$ be a nonzero solution of the homogeneous equation (2). Here again, the continuous transformation

$$\mathscr{L}(\varepsilon): \quad (t,q) \longrightarrow (t, q + \varepsilon s(t))$$

leaves evidently Δ_2 invariant. Exactly for the same reasons as in the previous paragraph, the expression of (1), in terms of the canonical variables t and $z = q/s(t)$, becomes a first order LDE in $\dot{z}$. Explicitly:

$$\Delta_2' = s(t)\ddot{z} + \big[a(t)s(t) + 2\dot{s}(t)\big]\dot{z} + c(t).$$

However, the reduced LDE takes a more simple form if one introduces the 'Wronskian variable'

$$w = \dot{q}s(t) - q\dot{s}(t) = s^2(t)\dot{z} \tag{9}$$

which is also a first order invariant of $\mathscr{L}(\varepsilon)$. Indeed, it becomes

$$\dot{w} + a(t)w + c(t)s(t) = 0.$$

Applying formula (8) to that first order LDE, one obtains directly the first integral

$$I(t,q,\dot{q}) := \Big(\dot{q}s(t) - q\dot{s}(t)\Big)\,\mathrm{e}^{\int a(t)\mathrm{d}t} + \int c(t)s(t)\,\mathrm{e}^{\int a(t)\mathrm{d}t}\,\mathrm{d}t = C. \tag{10}$$

It is by itself an invariant of $\mathscr{L}(\varepsilon)$. In the homogeneous case where $c(t) = 0$, Eq. (10) is nothing else but Abel's identity in which the exponential factor compensates exactly the amplitude damping of both $q(t)$ and $s(t)$. Substituting $s^2(t)\dot{z}$ for w in (10) yields an expression of $\dot{z}$ as a function of t which can easily be integrated to give the general solution $q(t)$ of (1), *videlicet*

$$q(t) = s(t)\left\{\int \frac{1}{s^2(t)}\left[C - \int c(t)s(t)\,\mathrm{e}^{\int a(t)\mathrm{d}t}\,\mathrm{d}t\right]\mathrm{e}^{-\int a(t)\mathrm{d}t}\,\mathrm{d}t + C'\right\},$$

where C' is another constant of integration.

Since the solution space of the homogeneous equation (2) is a two dimensional vector space, the whole group associated with the linearity symmetry is actually the two-parameter symmetry group

$$\mathscr{L}(\varepsilon_1,\varepsilon_2)\colon (t,q) \longrightarrow (t, q + \varepsilon_1 s_1(t) + \varepsilon_2 s_2(t)), \tag{11}$$

where $s_1(t)$ and $s_2(t)$ are two independent solutions of (2). The knowledge of $s_1(t)$ and $s_2(t)$ enables an algebraic resolution of (1) which constitutes an alternative to the usual method of variation of the parameters. Indeed, they induce respectively two first integrals $I_1(t,q,\dot{q}) = C_1$ and $I_2(t,q,\dot{q}) = C_2$ as given by formula (10). They form a Cramer's system in q and $\dot{q}$ from which $q(t)$ can be extracted.

We end this subsection with the couples of first integrals generated by the linearity symmetry in the two most relevant examples encountered in physics, at the undergraduate level.

2.2.1 Example 1: the harmonic oscillator

Let us consider the equation of motion of the harmonic oscillator

$$\ddot{q} + \omega_0^2 q = 0,$$

where the natural frequency ω_0 is a constant. Here, the formal first integral (10) is the Wronskian itself:

$$I(t,q,\dot{q}) = \dot{q}s(t) - q\dot{s}(t).$$

Since two independent solutions of the homogeneous equation (1) are $s_1(t) = \cos(\omega_0 t)$ and $s_2(t) = \sin(\omega_0 t)$, the linearity symmetry generates the two independent first integrals

$$I_1 = \dot{q}\cos(\omega_0 t) + \omega_0 q \sin(\omega_0 t) \quad \text{and} \quad I_2 = \dot{q}\sin(\omega_0 t) - \omega_0 q\cos(\omega_0 t).$$

2.2.2 Example 2: the driven damped harmonic oscillator

Let us now move on to the equation of motion of a damped harmonic oscillator driven by a sinusoidal excitation force:

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q - F\cos(\omega_\mathrm{e} t) = 0,$$

where the natural and excitation frequencies, ω_0 and ω_e, are constants as well as the dissipation rate γ and the characteristic force F. The formal first integral is

$$I(t,q,\dot{q}) = \Big(\dot{q}s(t) - q\dot{s}(t)\Big)\,\mathrm{e}^{2\gamma t} - F\int \cos(\omega_\mathrm{e} t)s(t)\,\mathrm{e}^{2\gamma t}\,\mathrm{d}t.$$

Suppose that we are in the underdamped regime in which case $s_1(t) = \cos(\omega t)\,\mathrm{e}^{-\gamma t}$ and $s_2(t) = \sin(\omega t)\,\mathrm{e}^{-\gamma t}$ are two real independent solutions of the homogeneous equation, where $\omega = (\omega_0^2 - \gamma^2)^{1/2}$. The corresponding first integrals are

$$I_1 = \left[\dot{q}\cos(\omega t) + q(\gamma\cos(\omega t) + \omega\sin(\omega t)) - \frac{F\sin((\omega+\omega_\mathrm{e})t + \beta_+)}{2\sqrt{(\omega+\omega_\mathrm{e})^2+\gamma^2}} - \frac{F\sin((\omega-\omega_\mathrm{e})t + \beta_-)}{2\sqrt{(\omega-\omega_\mathrm{e})^2+\gamma^2}}\right]\mathrm{e}^{\gamma t},$$

$$I_2 = \left[\dot{q}\sin(\omega t) + q(\gamma\sin(\omega t) - \omega\cos(\omega t)) + \frac{F\cos((\omega+\omega_\mathrm{e})t + \beta_+)}{2\sqrt{(\omega+\omega_\mathrm{e})^2+\gamma^2}} + \frac{F\cos((\omega-\omega_\mathrm{e})t + \beta_-)}{2\sqrt{(\omega-\omega_\mathrm{e})^2+\gamma^2}}\right]\mathrm{e}^{\gamma t},$$

where one has introduced the angles

$$\beta_\pm = \arctan\left(\frac{\gamma}{\omega \pm \omega_\mathrm{e}}\right).$$

3 Linearity as a Noether point symmetry

The equation of motion (1) is known [28, 29] to be equivalent to the Euler-Lagrange equation derived from the Lagrangian

$$L(t,q,\dot{q}) = \left(\frac{1}{2}\,\dot{q}^2 - \frac{1}{2}\,b(t)q^2 - c(t)q\right)\mathrm{e}^{\int a(t)\mathrm{d}t}\,. \tag{12}$$

A first manner of constructing this Lagrangian is to perform the point transformation $q \to Q = \exp(\int a(t)\mathrm{d}t/2)q$ in Eq. (1). It maps the initial problem onto a problem of a non dissipative oscillator with amplitude Q governed by the equation

$$\ddot{Q} + \left[b(t) - \frac{1}{2}\,\dot{a}(t) - \frac{1}{4}\,a^2(t)\right]Q + c(t)\,\mathrm{e}^{\frac{1}{2}\int a(t)\mathrm{d}t} = 0$$

derivable from the standard Lagrangian

$$L = \frac{1}{2}\,\dot{Q}^2 - \frac{1}{2}\left[b(t) - \frac{1}{2}\,\dot{a}(t) - \frac{1}{4}\,a^2(t)\right]Q^2 - c(t)\,\mathrm{e}^{\frac{1}{2}\int a(t)\mathrm{d}t}\,Q. \tag{13}$$

Then, performing the inverse transformation $Q \to q$ in (13) provides

$$L = \left(\frac{1}{2}\,\dot{q}^2 - \frac{1}{2}\,b(t)q^2 - c(t)q\right)\mathrm{e}^{\int a(t)\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{4}\,a(t)q^2\,\mathrm{e}^{\int a(t)\mathrm{d}t}\right)$$

and one has thereby obtained (12) up to a removable total derivative. However, this result may be derived without any guess, only by the application of the linearity symmetry in conjunction with the theory of Jacobi multipliers [4, 30, 31] which generalize Euler's ones. It is well-known since the works of Lie [32] that if a second order differential equation

$$\ddot{q} - F(t,q,\dot{q}) = 0 \tag{14}$$

possesses two independent Lie point symmetries

$$(t,q) \longrightarrow (t + \varepsilon\,\xi_i(t,q), q + \varepsilon\,\eta_i(t,q)) \qquad (i = 1,2)$$

then the quantity

$$M(t,q,\dot q)=\begin{vmatrix}1 & \dot q & F\\ \xi_1 & \eta_1 & \dot\eta_1-\dot q\dot\xi_1\\ \xi_2 & \eta_2 & \dot\eta_2-\dot q\dot\xi_2\end{vmatrix}^{-1} \tag{15}$$

is such that

$$\frac{\partial}{\partial \dot q}(MF)+\frac{\partial}{\partial q}(M\dot q)+\frac{\partial M}{\partial t}=0.$$

It is a so-called Jacobi last multiplier of Eq. (14), see also Refs. [4] (chap. 15, § 5, theorem 32) and [31]. What is important for our purpose is that, beyond their profound signification, Jacobi last multipliers bring solutions to the inverse Lagrange problem in the case of a single coordinate. Indeed, the existence of a last multiplier M brings a Lagrangian L, constrained by [31]

$$\frac{\partial^2 L}{\partial \dot q^2}=M,$$

whose Euler-Lagrange equation amounts to Eq. (14). Using the linearity symmetries generated by $s_1(t)$ and $s_2(t)$, Eq. (1) admits the Jacobi last multiplier

$$M=\begin{vmatrix}s_1(t) & \dot s_1(t)\\ s_2(t) & \dot s_2(t)\end{vmatrix}^{-1}=\frac{1}{s_1(t)\dot s_2(t)-\dot s_1(t)s_2(t)}=K\,\mathrm{e}^{\int a(t)\mathrm{d}t},$$

thanks to Abel's identity. Here, K is a nonzero constant without any signification depending on the choices of $s_1(t)$ and $s_2(t)$. It will be set to 1. Interestingly enough, the explicit forms of $s_1(t)$ and $s_2(t)$ are unnecessary to infer the last multiplier, the knowledge of their existence suffices. Then, the Lagrangian has *a priori* the form

$$L(t,q,\dot q)=\left(\frac{1}{2}\,\dot q^2+f_1(t,q)\dot q+f_0(t,q)\right)\mathrm{e}^{\int a(t)\mathrm{d}t}.$$

However, in a Lagrangian, one can always remove a term linear in $\dot q$ by a gauge transformation, that is, by adding the total derivative of a suitable function of (t,q). Thus, without lost in generality, one can make the gauge choice $f_1(t,q)=0$ yielding the Euler-Lagrange equation

$$\mathsf{E}(L)=\frac{\partial L}{\partial q}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q}=-\left(\ddot q+a(t)\dot q-\frac{\partial f_0}{\partial q}(t,q)\right)\mathrm{e}^{\int a(t)\mathrm{d}t}.$$

It is equivalent to (1) if

$$f_0(t,q)=-\frac{1}{2}\,b(t)q^2-c(t)q$$

and one has re-obtained the Lagrangian[3] (12). Under the transformation (5), it becomes

$$L(t,q+\varepsilon s(t),\dot q+\varepsilon\dot s(t))=L(t,q,\dot q)+\varepsilon\Big(\dot q\dot s(t)-\big[b(t)q+c(t)\big]s(t)\Big)\mathrm{e}^{\int a(t)\mathrm{d}t}+\mathrm{O}(\varepsilon^2).$$

[3] We point out that, whatever the potential $V(q,t)$ be, a dynamic equation $\ddot q+a(t)\dot q+\partial_q V(q,t)$ is always deducible from the Caldirola-Kanai Lagrangian

$$L=\left(\frac{1}{2}\,\dot q^2-V(q,t)\right)\mathrm{e}^{\int a(t)\mathrm{d}t}.$$

This is because $M=\mathrm{e}^{\int a(t)\mathrm{d}t}$ is actually an universal last multiplier of the dynamic equation. However, it is *a priori* not a consequence of any symmetry (one can find in Ref. [33] the list of potentials for which the equation of motion admits a point symmetry). Interestingly, for linear equations, the existence of the last multiplier ceases to be 'accidental'.

Hence, using the fact that $s(t)$ is a solution of (2), one has

$$\delta L = L(t, q + \varepsilon s(t), \dot{q} + \varepsilon \dot{s}(t)) - L(t, q, \dot{q}) = \varepsilon \frac{\mathrm{d}f}{\mathrm{d}t} + \mathrm{O}(\varepsilon^2), \tag{16}$$

where

$$f(t, q) = q\dot{s}(t)\,\mathrm{e}^{\int a(t)\mathrm{d}t} - \int c(t)s(t)\,\mathrm{e}^{\int a(t)\mathrm{d}t}\,\mathrm{d}t.$$

Since L merely undergoes a gauge transformation under (5), the later is a Noether point symmetry. On the other hand, one has independently on the form of L:

$$\begin{aligned} L(t, q + \varepsilon s(t), \dot{q} + \varepsilon \dot{s}(t)) &= L(t, q, \dot{q}) + \varepsilon \left[s(t)\frac{\partial L}{\partial q} + \dot{s}(t)\frac{\partial L}{\partial \dot{q}} \right] + \mathrm{O}(\varepsilon^2) \\ &= L(t, q, \dot{q}) + \varepsilon \left[s(t)\mathsf{E}(L) + \frac{\mathrm{d}}{\mathrm{d}t}\left(s(t)\frac{\partial L}{\partial \dot{q}} \right) \right] + \mathrm{O}(\varepsilon^2). \end{aligned} \tag{17}$$

Equations (16) and (17) give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[s(t)\frac{\partial L}{\partial \dot{q}} - f(t, q) \right] = -s(t)\mathsf{E}(L).$$

Along the solutions $q(t)$, the right-hand side vanishes so the symmetry generates the conservation of the expression in brackets, namely the Noether invariant which coincide obviously with Lie's one (10).

4 Higher-order linear differential equations

4.1 Linearity symmetry of third-order linear differential equations

One can wonder about the extension of the linearity symmetry and the associated Wronskian-type conservation law to higher-order LDEs. For this purpose and for the sake of illustration, we now consider a third order LDE

$$\Delta_3(t, q, \dot{q}, \ddot{q}, \dddot{q}) = \dddot{q} + a(t)\ddot{q} + b(t)\dot{q} + c(t)q + d(t) = 0. \tag{18}$$

Let $s_1(t)$ be a nonzero solution of the homogeneous equation

$$\Delta_{3\mathrm{h}}(t, q, \dot{q}, \ddot{q}, \dddot{q}) = \dddot{q} + a(t)\ddot{q} + b(t)\dot{q} + c(t)q = 0. \tag{19}$$

Yet again, the transformation

$$\mathscr{L}_1(\varepsilon): \quad (t, q) \to (t, q + \varepsilon s_1(t))$$

is a symmetry of Δ_3. Introducing the canonical variable $z_1 = q/s_1(t)$, Eq. (18) is reduced to a second order LDE

$$\Delta_3'(t, \dot{z}_1, \ddot{z}_1, \dddot{z}_1) = \Delta_3(t, q, \dot{q}, \ddot{q}, \dddot{q}) = 0$$

whose dependent variable is the invariant

$$\dot{z}_1 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{q}{s_1(t)} \right) = \frac{1}{s_1^2(t)} \begin{vmatrix} s_1(t) & q \\ \dot{s}_1(t) & \dot{q} \end{vmatrix}$$

of $\mathscr{L}_1(\varepsilon)$. Now, let $s_2(t)$ be another independent solution of (19). By construction, Δ_3' inherits the invariance under the symmetry group

$$\mathscr{L}_2(\varepsilon): \quad (t, q) \to (t, q + \varepsilon s_2(t)).$$

The action of $\mathscr{L}_2(\varepsilon)$ on $\dot{z}_1$ is simply

$$\dot{z}_1 \longrightarrow \frac{1}{s_1^2(t)} \begin{vmatrix} s_1(t) & q+\varepsilon s_2(t) \\ \dot{s}_1(t) & \dot{q}+\varepsilon \dot{s}_2(t) \end{vmatrix} = \dot{z}_1 + \frac{\varepsilon}{s_1^2(t)} \begin{vmatrix} s_1(t) & s_2(t) \\ \dot{s}_1(t) & \dot{s}_2(t) \end{vmatrix}.$$

Hence, $\mathscr{L}_2(\varepsilon)$ merely translates by ε the variable

$$z_2 = \frac{s_1^2(t)\dot{z}_1}{\begin{vmatrix} s_1(t) & s_2(t) \\ \dot{s}_1(t) & \dot{s}_2(t) \end{vmatrix}} = \frac{\begin{vmatrix} s_1(t) & q \\ \dot{s}_1(t) & \dot{q} \end{vmatrix}}{\begin{vmatrix} s_1(t) & s_2(t) \\ \dot{s}_1(t) & \dot{s}_2(t) \end{vmatrix}}$$

and one obtains a first order LDE in

$$\dot{z}_2 = \frac{s_1(t)w}{\begin{vmatrix} s_1(t) & s_2(t) \\ \dot{s}_1(t) & \dot{s}_2(t) \end{vmatrix}^2} \qquad \text{where} \qquad w = \begin{vmatrix} s_1(t) & s_2(t) & q \\ \dot{s}_1(t) & \dot{s}_2(t) & \dot{q} \\ \ddot{s}_1(t) & \ddot{s}_2(t) & \ddot{q} \end{vmatrix},$$

or in the Wronsrkian variable w itself. It may be deduced directly by deriving w. Since each row is the derivative of the preceding, one has

$$\dot{w} = \begin{vmatrix} s_1(t) & s_2(t) & q \\ \dot{s}_1(t) & \dot{s}_2(t) & \dot{q} \\ \dddot{s}_1(t) & \dddot{s}_2(t) & \dddot{q} \end{vmatrix} = -a(t)w - d(t) \begin{vmatrix} s_1(t) & s_2(t) \\ \dot{s}_1(t) & \dot{s}_2(t) \end{vmatrix}.$$

It is easily integrated to yield the first integral

$$I_3(t,q,\dot{q},\ddot{q}) := \begin{vmatrix} s_1(t) & s_2(t) & q \\ \dot{s}_1(t) & \dot{s}_2(t) & \dot{q} \\ \ddot{s}_1(t) & \ddot{s}_2(t) & \ddot{q} \end{vmatrix} e^{\int a(t)\mathrm{d}t} + \int d(t) \begin{vmatrix} s_1(t) & s_2(t) \\ \dot{s}_1(t) & \dot{s}_2(t) \end{vmatrix} e^{\int a(t)\mathrm{d}t}\mathrm{d}t = C_3 \qquad (20)$$

which, like $\dot{z}_2$ or w, is a simultaneous Lie invariant of $\mathscr{L}_1(\varepsilon)$ and $\mathscr{L}_2(\varepsilon)$. The last equality may be integrated in z_2, then in z_1 to provide the general expression of $q(t)$. However, things becomes easier when one considers the whole linearity symmetry group through the introduction of a third independent solution $s_3(t)$ of (19). Then, permuting cyclically the indices $1, 2, 3$ in (20) yields two other first integrals $I_1 = C_1$ and $I_2 = C_2$. The three first integrals define a Cramer system in $q, \dot{q}, \ddot{q}$ from which $q(t)$ can be extracted.

4.2 Linearity symmetries of third-order Lagrangians

We now consider a Lagrangian depending also on the acceleration, so that $L = L(t, q, \dot{q}, \ddot{q})$. In this case [34, 35] the Euler-Lagrange equation reads

$$\mathsf{E}(L) := \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{q}} = 0\,.$$

It is certainly linear if the Lagrangian has the form

$$L(t,q,\dot{q},\ddot{q}) = \frac{1}{2}\,A(t)\,\ddot{q}^2 + \frac{1}{2}\,B(t)\,\dot{q}^2 + \frac{1}{2}\,C(t)\,q^2 - D(t)\,q, \qquad (21)$$

where A, B, C and D are arbitrary functions of time. Notice that further monomials in $q\ddot{q}, \dot{q}\ddot{q}, q\dot{q}, \ddot{q}, \dot{q}$ would be superfluous. Indeed, they can be accommodated in the above picture by repeating Leibniz' product rule as many time as necessary and taking into account that total derivatives do not contribute to the Euler-Lagrange equations. For instance, one has

$$E(t)q\ddot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(E(t)q\dot{q}\Big) - \dot{E}(t)q\dot{q} - E(t)\dot{q}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\Big(E(t)q\dot{q} - \frac{1}{2}\,\dot{E}(t)q^2\Big) + \frac{1}{2}\,\ddot{E}(t)q^2 - E(t)\dot{q}^2,$$

and the relevant terms of the right-hand side are seen to fit with (21). The Euler-Lagrange equation for (21) reads

$$A(t)\ddddot{q} + 2\dot{A}(t)\dddot{q} + [\ddot{A}(t) - B(t)]\ddot{q} - \dot{B}(t)\dot{q} + C(t)q - D(t) = 0.$$

Notice that not all fourth-order LDE fulfils the above equation, so that the variational principle imposes some restriction. We are not aware of a suitable Lagrangian for the general (time-dependent) fourth-order LDE. In addition, observe that for $A(t) = 0$ one goes directly from a fourth-order to a second-order equation in this case.

Now, let $s(t)$ be a nonzero solution of the associated homogeneous equation:

$$A(t)\ddddot{q} + 2\dot{A}(t)\dddot{q} + [\ddot{A}(t) - B(t)]\ddot{q} - \dot{B}(t)\dot{q} + C(t)q = 0.$$

Using this property, one verifies that the transformation $q \to q + \varepsilon s(t)$ is a Noether point symmetry of L since

$$\delta L = L(t, q + \varepsilon s(t), \dot{q} + \varepsilon\dot{s}(t), \ddot{q} + \varepsilon\ddot{s}(t)) - L(t, q, \dot{q}, \ddot{q}) = \varepsilon\,\frac{\mathrm{d}f}{\mathrm{d}t} + \mathrm{O}(\varepsilon^2), \tag{22}$$

with

$$f(t, q, \dot{q}) = A(t)\ddot{s}(t)\dot{q} - A(t)\dddot{s}(t)q - \dot{A}(t)\ddot{s}(t)q + B(t)\dot{s}(t)q - \int D(t)s(t)\mathrm{d}t\,.$$

However, independently of the form of L, the action of the transformation is generally

$$\begin{aligned}\delta L &= \varepsilon\Big[s(t)\frac{\partial L}{\partial q} + \dot{s}(t)\frac{\partial L}{\partial \dot{q}} + \ddot{s}(t)\frac{\partial L}{\partial \ddot{q}}\Big] + \mathrm{O}(\varepsilon^2)\\ &= \varepsilon\Big[s(t)\mathsf{E}(L) + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{s}(t)\frac{\partial L}{\partial \ddot{q}} - s(t)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}} + s(t)\frac{\partial L}{\partial \dot{q}}\Big)\Big] + \mathrm{O}(\varepsilon^2).\end{aligned} \tag{23}$$

Then, one concludes from (22) and (23) the Noether invariant

$$\begin{aligned}I(t, q, \dot{q}, \ddot{q}, \dddot{q}) &= \dot{s}(t)\frac{\partial L}{\partial \ddot{q}} - s(t)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}} + s(t)\frac{\partial L}{\partial \dot{q}} - f(t, q, \dot{q})\\ &= A(t)\big(\dot{s}(t)\ddot{q} - \ddot{s}(t)\dot{q} + \dddot{s}(t)q - s(t)\dddot{q}\big) + \dot{A}(t)\big(\ddot{s}(t)q - s(t)\ddot{q}\big)\\ &\quad + B(t)\big(s(t)\dot{q} - \dot{s}(t)q\big) + \int D(t)s(t)\mathrm{d}t.\end{aligned} \tag{24}$$

5 General higher-order linear differential equations

5.1 Lie symmetry approach

The reasoning about the Lie linearity symmetry remains evidently valid for an LDE of any order

$$\Delta_n(t, q, q^{(1)}, \ldots, q^{(n)}) = q^{(n)} + a_{n-1}(t)q^{(n-1)} + \cdots + a_1(t)q^{(1)} + a_0(t)q + c(t) = 0, \tag{25}$$

where $q^{(k)}$ designates the k-th derivative of q. Let $s_1(t), \ldots, s_{n-1}(t)$ be independent solutions of the homogeneous equation

$$\Delta_{n\mathrm{h}}(t, q, q^{(1)}, \ldots, q^{(n)}) = q^{(n)} + a_{n-1}(t)q^{(n-1)} + \cdots + a_1(t)q^{(1)} + a_0(t)q = 0. \tag{26}$$

Using successively the $n-1$ linearity symmetries, one decreases the order of the LDE one by one until a first-order LDE. At each step, the intermediate LDE of order $n-k$ is expressed in terms of the derivative of the canonical variable

$$z_k = \frac{w_k}{\mathscr{D}_k(t)}$$

where one has introduced the kth-order Wronskian variable

$$w_k = \begin{vmatrix} s_1(t) & s_2(t) & \dots & s_{k-1}(t) & q \\ \dot{s}_1(t) & \dot{s}_2(t) & \dots & \dot{s}_{k-1}(t) & \dot{q} \\ \vdots & \vdots & & \vdots & \vdots \\ s_1^{(k-1)}(t) & s_2^{(k-1)}(t) & \dots & s_{k-1}^{(k-1)}(t) & q^{(k-1)} \end{vmatrix}$$

and where $\mathscr{D}_k(t)$ is the Wronskian obtained from w_k by substituting q for $s_k(t)$ along the last column. Differentiating w_n row by row produces the first order LDE

$$\dot{w}_n + a_{n-1}(t)w_n + c(t)\mathscr{D}_{n-1}(t) = 0. \tag{27}$$

Once integrated, it yields the first integral

$$I(t, q, q^{(1)}, \dots, q^{(n-1)}) = w_n \, \mathrm{e}^{\int a_{n-1}(t)\mathrm{d}t} + \int c(t)\mathscr{D}_{n-1}(t)\, \mathrm{e}^{\int a_{n-1}(t)\mathrm{d}t} = C.$$

Solving (27) for w_n as in paragraph 2.1 gives $z_n(t)$. Then, thanks to the relation

$$\boxed{\dot{z}_k = \frac{\mathscr{D}_{k-1}(t)\mathscr{D}_{k+1}(t)}{\mathscr{D}_k(t)^2}\, z_{k+1}},$$

$n-1$ successive quadratures allow us to obtain $z_1(t)$ and to deduce the general solution $q(t)$. Alternatively, if one knows a last independent solution $s_n(t)$ of (26), one can construct a Cramer system of n first integrals from which one extracts $q(t)$.

5.2 Noether symmetry approach

An immediate generalisation of the Lagrangian (12) to the n-th order with linear Euler-Lagrange equation

$$\mathsf{E}(L) = \sum_{k=0}^{n} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L}{\partial q^{(k)}}$$

is *a priori* of the form

$$L = \frac{1}{2} \sum_{i,j=0}^{n} a_{ij}(t)\, q^{(i)} q^{(j)} + \sum_{i=0}^{n} b_i(t) q^{(i)}.$$

However, it is a simple task to show, by induction on the non-negative integer j, that any term of the form $A(t)q^{(i)}$ can be decomposed as a sum of a term $B(t)q$ and a total derivative. In the same manner, any term of the form $A(t)q^{(i)}q^{(i+j)}$ can be decomposed as a sum of quadratic terms $B_k(t)(q^{(k)})^2$ plus a total derivative. All the total derivatives in L can be gauged out and it is sufficient to restrict ourself to a Lagrangian

$$L = \frac{1}{2} \sum_{k=0}^{n} \alpha_k(t)(q^{(k)})^2 - \beta(t)q.$$

The Euler-Lagrange equation of order $2n$ is

$$\mathsf{E}(L) = \sum_{k=0}^{n} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big(\alpha_k(t) q^{(k)}\Big) - \beta(t) = 0.$$

Let $s(t)$ be a solution of the homogeneous equation, i.e. a function such that

$$\sum_{k=0}^{n} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big(\alpha_k(t) s^{(k)}(t)\Big) = \alpha_0 s(t) + \sum_{k=1}^{n} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big(\alpha_k(t) s^{(k)}(t)\Big) = 0. \tag{28}$$

The transformation $q \to q + \varepsilon s(t)$ affects the derivatives according to $q^{(k)} \to q^{(k)} + \varepsilon s^{(k)}(t)$. To the first order in ε, it produces the following variation of the Lagrangian:

$$\delta L = \varepsilon \mathsf{X}^{[n]}(L) = \varepsilon\Big[q\,\alpha_0(t) s(t) + \sum_{k=1}^{n} q^{(k)} \alpha_k(t) s^{(k)}(t) - \beta(t) s(t)\Big].$$

Exploiting Eq. (28), the variation reads

$$\delta L = \varepsilon\Big[-\sum_{k=1}^{n} (-1)^k q \frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big(\alpha_k(t) s^{(k)}(t)\Big) + \sum_{k=1}^{n} q^{(k)} \alpha_k(t) s^{(k)}(t) - \beta(t) s(t)\Big].$$

Then, using the identity

$$u(t) v^{(k)}(t) = (-1)^k u^{(k)}(t) v(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=0}^{k-1} (-1)^j u^{(j)}(t) v^{(k-j-1)}(t)\right), \tag{29}$$

one obtains

$$\delta L = \varepsilon \frac{\mathrm{d}f}{\mathrm{d}t} \tag{30}$$

with

$$f(t, q, q^{(1)}, \ldots, q^{(n-1)}) = -\sum_{k=1}^{n} (-1)^k \sum_{j=0}^{k-1} (-1)^j q^{(j)} \frac{\mathrm{d}^{k-j-1}}{\mathrm{d}t^{k-j-1}}\Big(\alpha_k(t) s^{(k)}(t)\Big) - \int \beta(t) s(t) \mathrm{d}t.$$

Hence, the transformation is definitely a Noether symmetry of L. On the other hand, one has

$$\delta L = \varepsilon \sum_{k=0}^{n} s^{(k)}(t) \frac{\partial L}{\partial q^{(k)}} = \varepsilon\left\{s(t)\mathsf{E}(L) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{k=1}^{n} \sum_{j=0}^{k-1} (-1)^j s^{(k-j-1)}(t) \frac{\mathrm{d}^j}{\mathrm{d}t^j}\Big(\alpha_k(t) q^{(k)}(t)\Big)\right]\right\},$$

where use has been made of (29). This expression together with (30) yield the first integral of order $2n-1$

$$I = \sum_{k=1}^{n} \sum_{j=0}^{k-1} (-1)^j \left[s^{(k-j-1)}(t) \frac{\mathrm{d}^j}{\mathrm{d}t^j}\Big(\alpha_k(t) q^{(k)}(t)\Big) + (-1)^k q^{(j)} \frac{\mathrm{d}^{k-j-1}}{\mathrm{d}t^{k-j-1}}\Big(\alpha_k(t) s^{(k)}(t)\Big)\right] + \int \beta(t) s(t) \mathrm{d}t.$$

6 Conclusions

In this work, we started with a brief review of the role of the linearity symmetry for linear second-order ordinary differential equations. The corresponding Lie symmetry group canonical variables allow the quadrature of the system and the identification of a Wronskian-like first-order Lie invariant. Later the variational approach is pursued, where the well-known Kanai-Caldirola Lagrangian is recognized as a consequence of Jacobi last multiplier associated to two independent linearity symmetries. The linearity symmetry is shown to be a Noether symmetry, in accordance with previous findings [15, 16]. Finally, extension to linear ordinary differential equations of degree $2n$, where $n \geq 2$, is obtained, where the linearity symmetry is shown to be a Noether symmetry. The corresponding first-integrals are in this context higher-order generalizations of the traditional Wronskian-like quantities. It is hoped that the present review and generalized results will disseminate the invariance principles associated to linearity and Wronskian-like-conservation laws.

Acknowledgments: F. H. acknowledges the support by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq). R. L. is grateful to the whole Statistical Physics Group and to D. Malterre for regular stimulating discussions.

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