Large amplitude oscillations in a trapped dissipative electron gas

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Abstract

A collisional trapped non-neutral plasma is described by an hydrodynamical model in one-dimensional geometry. For suitable initial conditions and velocity field, the Lagrangian variables method reduces the pressure dominated problem to a damped autonomous Pinney equation, representing a dissipative nonlinear oscillator with an inverse cubic force. An accurate approximate analytic solution derived from the Kuzmak-Luke perturbation theory is applied, allowing the assessment of the fully nonlinear dynamics. On the other hand, in the cold plasma case the Lagrangian variables approach allows the derivation of exact damped nonlinear oscillations. The conditions for the applicability of the hot, pressure dominated, or the cold gas assumptions are derived.

PACS numbers: 02.30.Hq, 52.25.Fp, 52.27.Jp, 52.35.Mw

Keywords: Trapped electron gas, arbitrary amplitude solution, Lagrangian variables, Pinney equation, collisional effects.

I. INTRODUCTION

The analysis of arbitrary amplitude structures in charged particle systems such as plasmas is a traditional research field [1]–[4] with an ongoing interest [5]–[7]. However, in most cases the previous literature regarding nonlinear solutions in fluid-plasma systems restrict to the approximation of a cold, collisionless system, which can be a drawback in view of realistic applications. The present work aims to remove these constraints, considering a trapped electron gas (a non-neutral plasma) possibly taking into account thermal effects and dissipation due to a collisional drag.

We will consider nonlinear structures derived by means of the Lagrangian coordinates method [1]–[3], applied to an hydrodynamical model with adiabatic equation of state, which is appropriate to fast processes where heat transfer does not take place. Lagrangian variables are recognized as an effective method in fluid problems and have been recently applied e.g. for the derivation of nonlinear waves in one-dimensional degenerate electron gases [8]. In the hydrodynamic model, the repulsive collective field due to the electron gas and the pressure term tend to produce expansion, while an external harmonic trap provides confinement. Two basic situations will be considered, according to the prevalence of thermal or Coulomb repulsion effects. Thanks to the more complete formulation of the original model equations, the precise conditions for the dominant effects can be evaluated in terms of physical parameters. This fills a gap in the literature, where e.g. the exact conditions for the cold plasma assumption are seldom evaluated.

Moreover, it will be shown that the presence of an external trap is a necessary condition for the existence of thermally dominated regime. For instance, such a possibility can not take place in an one-component plasma with fixed ionic background, since in this case the expansive rôle of the pressure, in equilibrium with the ions attraction, would be of the same order of the Coulomb repulsion (see Section III for more details). On the other hand, the external confinement allows the reduction of the complete problem to the solution of the Pinney equation [9], which is endemic in nonlinear physics. Pinney's equation applies to the stability analysis of beams in accelerators [10, 11], cosmologic models [12, 13], propagation of gravitational waves [14], rotating shallow water waves [15], Bose-Einstein condensates [16], the quantum Buneman instability [17] and many more.

Due to the presence of collisional drag, our version of the Pinney equation contains a damping term, so that it will become a dissipative Pinney equation [18], as apparent from Eq. (17) below. The damped Pinney equation is attracting much attention recently, in view of applications for dissipative quantum mechanics [19, 20], barotropic Friedman-Robertson-Walker universes with Chiellini damping [21], dissipative Milne-Pinney systems [22] and time-dependent non-commutative quantum mechanics [23]. In addition, although our treatment is restricted to systems with an one-dimensional (1D) geometry as a starting point, which is a frequently adopted choice [5, 24], it should be regarded as a general framework to be followed in more complex situations. For instance, it can be readily adapted to nonlinear spherical surface waves [25] or plasmas with a cylindrical symmetry [26], among other possibilities [27]. The 1D geometry is relevant for real trapped gases, such as in the two-stream instability in quasi-1D Bose-Einstein condensates [28], or Pierce diode plasmas [29, 30].

Although the present treatment has some similarity with bounded plasmas where rigid or virtual walls exist, demanding boundary conditions at the interfaces, here due to the gaseous nature, as well as due to the external confinement and electrostatic repulsion, the electron gas boundaries are self-consistently determined. For instance, in a bounded plasma one can have a velocity field $u(x,t)$ such that $u(\pm d,t) = 0$, with interfaces at $x = \pm d$. Such a situation has been treated in [24, 31, 32]. In our case, the starting model is exactly the same of [33] for a non-neutral plasma in a trap, except from the equation of state and because we also allow for damping. For a finite plasma, the boundary becomes defined in terms of the dynamics of the moving fluid, as described in [34]. Trapped clouds of identical charges have been treated in many contexts, like for the breathing mode of a quantum electron gas [35]. Experiments on such harmonically confined gases frequently use magneto-optical confinement techniques, tuned so as to obtain not only three dimensional but also quasi -two- or quasi-one-dimensional configurations [36]. We stress that in such systems by definition there is not the need for a positive feedback between “in” and “out”, as is the case of electronic devices as the Pierce diode.

This work is organized as follows. Section II introduces the basic set of hydrodynamic equations and the transformation to Lagrangian variables. Section III develops the arbitrary amplitude full solution when thermal effects are dominant and provides the precise applicability conditions of the solution in terms of the relevant physical parameters. Section IV performs the same job of Section III, but in the opposite case where Coulomb repulsion dominates thermal effects (cold plasma assumption). Section V shows sample applications of the results in the case of prevalent self-consistent fields, for a few initial conditions. Section VI is reserved to the conclusions.

II. BASIC MODEL AND LAGRANGIAN VARIABLES METHOD

In a slab geometry where the relevant physics develops in 1D space, the non-neutral plasma can be described by the following standard hydrodynamic equations,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\,u) = 0\,, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = -\frac{1}{m\,n}\frac{\partial P}{\partial x} - \frac{e\,E}{m} - \omega^2 x - \nu\,u\,, \tag{2}$$

$$\frac{\partial E}{\partial x} = -\frac{e\,n}{\varepsilon_0}\,. \tag{3}$$

By definiteness, the system is composed by electrons (charge $-e$, mass m) with a number density n, fluid velocity u and pressure P. Moreover, E is the internal electric field, ε_0 is the vacuum permittivity and a drag term with collision frequency ν is also included. Confinement is provided by an external harmonic field with angular frequency ω. As remarked in [33], the trapping potential can be provided by an homogeneous ionic background with number density n_i, in which case $\omega^2 = n_i e^2/(m\varepsilon_0)$, supposing rapid oscillations so that ions can be regarded as motionless. In this context E by definition is the electric field due to electrons only. Other popular particle confinement techniques are the radio-frequency Paul trap [37] and the Penning trap [38]. In view of the fast processes assumption, heat transport can be neglected. In this context, it can be assumed an adiabatic equation of state $P = n_0\kappa_B T(n/n_0)^\gamma$, where $\kappa_B T$ is a reference thermal energy and γ is the adiabatic index. For longitudinal waves it is adequate to chose $\gamma = 3$, corresponding to 1D compression. Weak collisionality is assumed, allowing wave propagation to remain essentially 1D.

In order to derive arbitrary-amplitude solutions for the system (1)-(3), we introduce Lagrangian coordinates (ξ, τ) given [1, 2] by

$$\xi = x - \int_0^\tau u(\xi, \tau')\,d\tau'\,, \quad \tau = t\,, \tag{4}$$

such that

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\,, \quad \frac{\partial}{\partial \xi} = \left(1 + \int_0^\tau \frac{\partial u(\xi,\tau')}{\partial \xi}\,d\tau'\right)\frac{\partial}{\partial x}\,. \tag{5}$$

The continuity equation (1) is then converted into

$$\frac{\partial}{\partial \tau}\left[\left(1 + \int_0^\tau \frac{\partial u(\xi,\tau')}{\partial \xi}\,d\tau'\right) n\right] = 0\,, \tag{6}$$

with solution

$$n = n(\xi,0)\left(1 + \int_0^\tau \frac{\partial u(\xi,\tau')}{\partial \xi}\, d\tau'\right)^{-1}\,, \tag{7}$$

where $n(\xi,0)$ is the electrons number density at $\tau = 0$.

The Gauss law (Eq. (3)) in transformed coordinates reads

$$\frac{\partial E}{\partial \xi} = -\frac{e}{\varepsilon_0}\, n(\xi,0)\,, \tag{8}$$

with solution

$$E = -\frac{e}{\varepsilon_0}\int n(\xi,0)\, d\xi + E_0(\tau)\,, \tag{9}$$

where $E_0(\tau)$ is at this stage an arbitrary function of the new time parameter. Physically $E_0(\tau)$ would be associated with an additional external field, besides the harmonic confinement, so we will set $E_0(\tau) = 0$ in the continuation.

The only remaining equation to be solved is the momentum transport equation (2), which becomes

$$\begin{aligned}\frac{\partial u}{\partial \tau} = &-\frac{3\,\kappa_B T}{2\,m}\left(1 + \int_0^\tau \frac{\partial u(\xi,\tau')}{\partial \xi}\, d\tau'\right)^{-1}\frac{\partial}{\partial \xi}\left[\left(\frac{n}{n_0}\right)^2\right] \\ &- \omega^2\left(\xi + \int_0^\tau u(\xi,\tau')\, d\tau'\right) + \frac{\omega_p^2}{n_0}\int n(\xi,0)\, d\xi - \nu\, u\,,\end{aligned} \tag{10}$$

where $\omega_p = [n_0 e^2/(m\varepsilon_0)]^{1/2}$ is the plasma frequency, for a reference number density n_0.

There are two manifestly repulsive contributions in Eq. (10). One of them is the pressure term proportional to $\kappa_B T$ and the another one is due to the electrons self-consistent field, proportional to ω_p^2/n_0. These repulsive effects are counterbalanced by the second term in the right-hand side of Eq. (10), due to the harmonic confinement. In the following, the solutions of Eq. (10) will be analyzed according to the strengths of the thermal and self-consistent field effects.

III. DOMINATING THERMAL EFFECTS

Equation (10) is too difficult to be analytically solved without further assumptions. As a working hypothesis, in this Section we consider sufficiently simple, linear velocity fields given by

$$u = \xi/T(\tau) + u_0(\tau)\,, \tag{11}$$

where $T(\tau)$ and $u_0(\tau)$ are functions to be determined. From Eq. (7), we get

$$n = n(\xi, 0)/\rho(\tau)\,, \tag{12}$$

where

$$\rho = \rho(\tau) = 1 + \int_0^\tau \frac{d\tau'}{T(\tau')}\,. \tag{13}$$

Inserting u from Eq. (11) and n from Eq. (12) into Eq. (10), the result is

$$\left(\ddot\rho + \nu\dot\rho + \omega^2\rho\right)\rho^3\xi + \left(\dot u_0 + \nu u_0 + \omega^2 \int_0^\tau u_0(\tau')\,d\tau'\right)\rho^3 = \\ -\frac{3\,\kappa_B T}{2\,m}\frac{\partial}{\partial\xi}\left[\left(\frac{n(\xi,0)}{n_0}\right)^2\right] + \frac{\omega_p^2\,\rho^3}{n_0}\int n(\xi,0)\,d\xi \tag{14}$$

where a dot denotes derivative with respect to τ. Deriving all terms in the Eq. (14) twice with respect to ξ and once with respect to τ gives $(\omega_p^2/n_0) \times\, d(\rho^3)/d\tau \times dn(\xi,0)/d\xi = 0$, a condition which is due to the electrons self-consistent repulsion term alone. Such a constraint can not be satisfied in nontrivial situations where neither $\rho(\tau)$ or $n(\xi,0)$ are constants. Hence, the only meaningful possibility occurs when the electrons repulsion can be neglected in comparison with the thermal effects.

Disregarding the electrons collective field in Eq. (14), for consistency still one needs to impose the pressure contribution as a linear function of ξ, in the same manner as the left-hand side of the equation is. By inspection, this requirement implies

$$n(\xi,0) = n_0\sqrt{1 + c_1\frac{\xi}{\xi_0} - \left(\frac{\xi}{\xi_0}\right)^2}\,, \quad |\xi| \le \xi_0\,, \tag{15}$$

where c_1 is a dimensionless numerical constant and $\xi_0 > 0$ is a reference position. Outside the bulk of the electron gas, where $|\xi| \ge \xi_0$, we set $n(\xi,0) = 0$. Since the more interesting physics takes place inside the electrons cloud, we will mainly discuss the problem for $|\xi| \le \xi_0$. Without loss of generality, in Eq. (15) it was chosen $n(0,0) = n_0$, which becomes the definition of n_0. In addition, for simplicity we shall consider only symmetric equilibrium densities, so that $c_1 \equiv 0$.

Taking into account Eq. (15), the term independent not depending on ξ gives

$$\dot u_0 + \nu u_0 + \omega^2\int_0^\tau u_0(\tau')\,d\tau' = 0\,. \tag{16}$$

It is apparent that u_0 will just execute linear transient oscillations. For simplicity, it will be set $u_0 = 0$ in what follows.

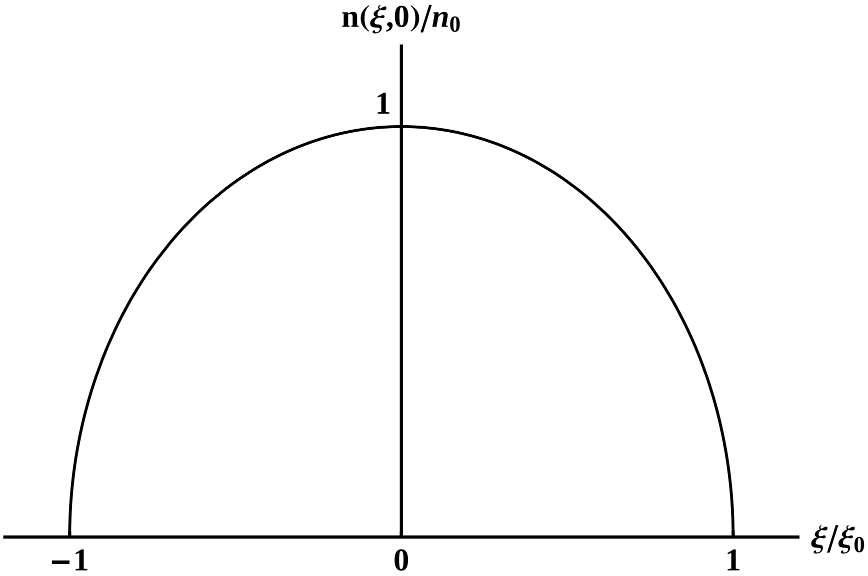

FIG. 1: Initial number density from Eq. (15) and $c_1 = 0$.

On the other hand, the term proportional to ξ in Eq. (14) provides

$$\ddot{\rho} + \nu\dot{\rho} + \omega^2\rho = \frac{3\kappa_B T}{m\xi_0^2}\,\frac{1}{\rho^3}\,. \tag{17}$$

Equation (17) is an autonomous damped, or dissipative Pinney equation [18]. The undamped case ($\nu = 0$) was solved by Pinney [9], including a time-dependent frequency $\omega = \omega(t)$, in terms of the linearly independent solutions of the associated Hill equation. Similar nonlinear equations were obtained in non-uniform non-neutral plasmas [33], from a moments method approach, without drag and for a pressure equation explicitly depending on the position.

By inspection it is possible to identify the equilibrium solution $\rho = \rho_{\rm eq}$ where $\dot{\rho} = \ddot{\rho} = 0$, given by

$$\rho_{\rm eq} = \left(\frac{3\kappa_B T}{m\omega^2\xi_0^2}\right)^{1/4}\,. \tag{18}$$

The existence of the (stable) equilibrium is due to the sign of the inverse cubic term, which in turn comes from the concavity of the number density in Eq. (15). Alternatively, one can write the conservative part of Eq. (17) in terms of a potential $V = V(\rho)$, defined by

$$V = \frac{m\omega^2\rho^2}{2} + \frac{3\kappa_B T}{2m\xi_0^2}\,\frac{1}{\rho^2}\,, \tag{19}$$

so that

$$\ddot{\rho} = -\frac{\partial V}{\partial\rho} - \nu\dot{\rho}\,. \tag{20}$$

Evidently, the repulsive term from Eq. (19) prevents collapse to the origin, as depicted in Fig. 2.

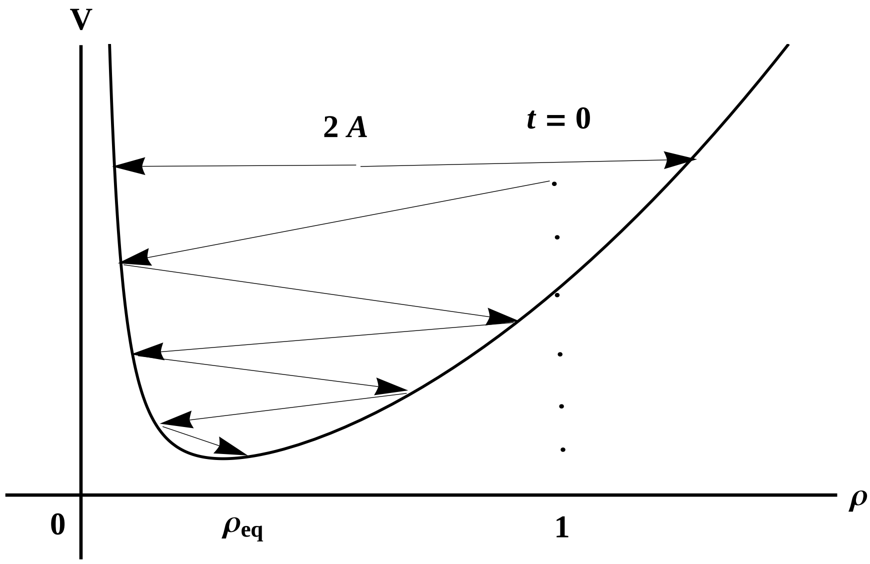

FIG. 2: Effective potential from Eq. (19), showing also a representative trajectory such that $\rho(0) = 1 > \rho_{\rm eq}$ and with initial amplitude $\approx A > 0$.

It happens that the damped Pinney equation (17) admits an accurate approximate solution, derived from Kuzmak-Luke perturbation theory [39, 40], which is an appropriate tool for weakly damped, nonlinear oscillator problems [41]. From Eq. (26) of [18], the solution reads

$$\rho^2 = \rho_{\rm eq}^2 + 2A^2 e^{-\nu\tau} + 2A\, e^{-\nu\tau/2} \left(\rho_{\rm eq}^2 + A^2 e^{-\nu\tau}\right)^{1/2} \cos\left(2\omega(\tau - \tau_0)\right) , \qquad (21)$$

where A, τ_0 are integration constants. As detailed in [18], the weak damping assumption should be valid for the accuracy of Eq. (21). In the undamped case ($\nu = 0$), Eq. (21) shows an exact oscillatory solutions in the interval $I = \{\rho > 0\,|\,\sqrt{\rho_{\rm eq}^2 + A^2} - |A| \le \rho \le \sqrt{\rho_{\rm eq}^2 + A^2} + |A|\}$, with the parameter $|A|$ playing the rôle of an initial amplitude, as depicted in Fig. 2. Moreover, since $\nu/\omega \ll 1$, during one oscillation period $\tau = \pi/\omega$ the quantity $|A|\exp(-\nu\tau/2)$ does not change very much and plays the rôle of a slowly varying time-dependent amplitude.

From Eq. (13) one has $\rho(0) = 1$, implying

$$\cos(2\omega\tau_0) = \frac{1 - \rho_{\rm eq}^2 - 2A^2}{2\,A\,(\rho_{\rm eq}^2 + A^2)^{1/2}} \, , \qquad (22)$$

which makes sense if and only if $A^2 \geq (1-\rho_{\rm eq}^2)^2/4$, or equivalently $\rho(0) = 1 \in I$, to avoid $\cos^2(2\omega\tau_0) > 1$. For any A satisfying the requirement, by construction the solution will remain regular and non-explosive, not producing multistream flow [1]. This is due to the repulsive inverse cubic term in the damped Pinney equation, which prevents $\rho \to 0$. Besides, taking into account Eq. (22), the parameter A is obviously related to $\dot\rho(0)$, but in an awkward algebraic way wish we refrain to exhibit.

As an example, we take realistic parameters for a trapped electron gas [42], namely $n_0 = 10^{10}\,{\rm m}^{-3}, \kappa_B T = 1\,{\rm eV}, \xi_0 = 5\,{\rm cm}, \omega = 5\,\omega_p = 25\,\nu$, together with $\omega_p = 5,64\,{\rm MHz}, \rho_{\rm eq} = 0.72$. The numerical simulation of Eq. (17) and the approximate solution from Eq. (21) yield almost identical results in this case, shown in Fig. 3.

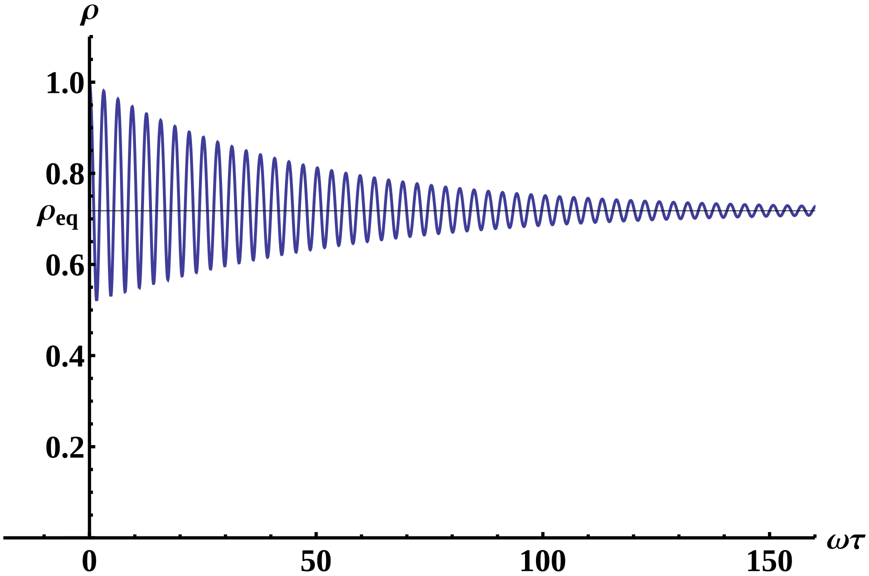

FIG. 3: Auxiliary function ρ as a function of time, from Eq. (21), approaching $\rho_{\rm eq} = 0.72$. Parameters: $n_0 = 10^{10}\,{\rm m}^{-3}, \kappa_B T = 1\,{\rm eV}, \xi_0 = 5\,{\rm cm}, \omega = 5\,\omega_p = 25\,\nu$. Initial conditions: $\rho(0) = 1, \dot\rho(0) = 0$. Correspondingly, $A = 0.24, \omega\tau_0 = 0.01$.

To complete the solution procedure, the electric field found from Eq. (9) and shown in Fig. 4 is

$$E = -\frac{n_0 e \xi_0}{2\varepsilon_0}\left(\frac{\xi}{\xi_0}\sqrt{1-\left(\frac{\xi}{\xi_0}\right)^2} + \arcsin\left(\frac{\xi}{\xi_0}\right)\right) , \tag{23}$$

which is also symmetric with respect to the origin $\xi = 0$.

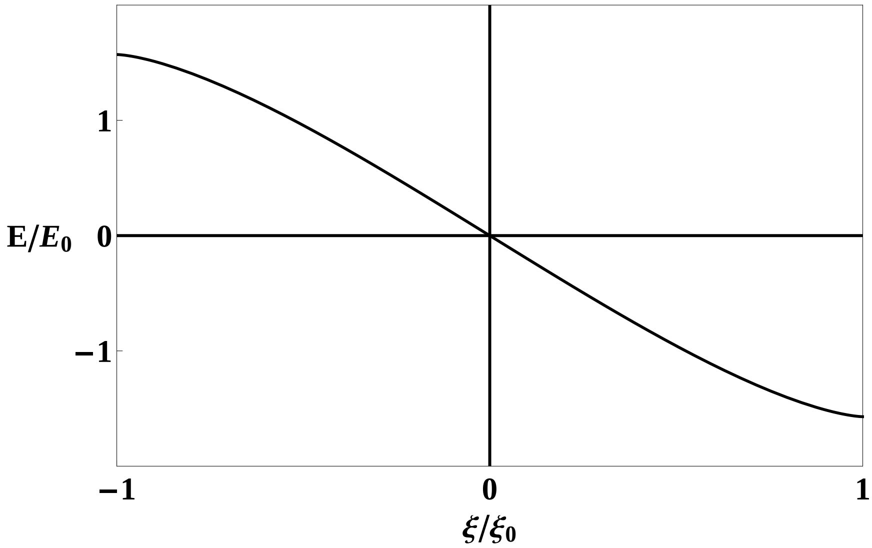

FIG. 4: Electric field from Eq. (23), normalized to $E_0 = m\,\omega_p^2\,\xi_0/(2\,e)$.

To summarize, an accurate nonlinear solution of the full hydrodynamic problem was found in terms of Lagrangian variables, with the number density given by Eq. (12) where $n(\xi, 0)$ is shown in Eq. (15), which also defines the initial condition $n(x, 0)$ since $\rho(0) = 1$ by definition (see Eq. (13)), the velocity field given by Eq. (11) with $1/T = \dot{\rho}$ and the electric field in Eq. (23). In terms of physical coordinates, one has from Eq. (4) that $x = \xi/\rho, t = \tau$, which also imply the damped nonlinear oscillations of the electron gas cloud boundaries. Namely, $|\xi| \leq \xi_0$ maps to $|x| \leq \xi_0/\rho$. In brief, the whole procedure is reducible to the dissipative Pinney equation (17), with the approximate solution (21). However, it is necessary to have a more detailed account on the validity conditions of the solution, regarding weak collisionality and thermal effects prevalence. These issue are discussed in the immediate continuation.

A. Weak damping condition

In the context of the dissipative Pinney equation (17), the approximate solution (21) holds for $\nu \ll \omega$ and not necessarily for $\nu \ll \omega_p$, which turns out to be a more stringent constraint. However, a strongly collisional plasmas would barely remains 1D. Hence we need $\nu \ll \omega_p$. It should be noticed that the undamped case is a particular case of the more general treatment. For real applications it is necessary to measure the strength of the drag force, in terms of suitable physical mechanisms, setting $\nu = 0$ whenever possible. Besides, as apparent from Eq. (40) below, damping plays a regularizing rôle to avoid a collapsing dynamics.

Hence, it is useful to reproduce at least a few explicit expressions of the damping rate. For instance, it can be originated from electron-neutral collisions. In this case one has the estimate

$$\nu = n_N < \sigma\, u > \approx n_N\, \pi a_0^2\, u_T \ll \omega_p\,, \tag{24}$$

where the symbol $<>$ denotes average, n_N is the neutrals number density, $\sigma \approx \pi a_0^2$ the electron-neutrals collisions cross section, a_0 the Bohr radius and $u_T = \sqrt{\kappa_B T/m}$ the electrons thermal velocity.

Also notice that considering the electron gas as a whole, the (elastic) electron-electron collisions can not dissipate its momentum. However, another possibility considers an electron-ion gas, where the electrons momentum could be dissipated into the ion species due to electron-ion collisions. For fast processes where the average ions velocity u_i can be neglected to a first approximation, a drag term reduces to $-\nu(u - u_i) \approx -\nu u$, with the damping rate given by the Landau frequency ν_{ei} of electron-ion collisions [43]. In this case, the weak collisionality holds for

$$\nu = \nu_{ei} = \frac{2\,\omega_p}{3}\,\frac{\ln\Lambda}{\Lambda} \ll \omega_p\,, \quad \Lambda = \frac{4\pi n_0 \lambda_D^3}{3}\,, \quad \lambda_D = \frac{u_T}{\omega_p}\,. \tag{25}$$

However, in such an alternative scheme from the start one needs to explicitly take into account the ion background, which would slightly modify some of the analytic results in this Section.

B. Pressure dominance: validity conditions

We can now analytically evaluate the pressure dominance condition. Specifically, from comparison of the repulsive terms in Eq. (14) and using Eq. (15), it amounts to $3\kappa_B T\xi/(m\xi_0^2) \gg -eE/m$, where the electric field is given by Eq. (23). A short algebra then yields

$$\frac{3\kappa_B T}{m\omega_p^2\xi_0^2} \gg \rho^3\, f(\xi/\xi_0)\,, \quad f(\xi/\xi_0) = \frac{1}{2}\left(\sqrt{1 - \left(\frac{\xi}{\xi_0}\right)^2} + \frac{\arcsin(\xi/\xi_0)}{\xi/\xi_0}\right) \approx 1\,. \tag{26}$$

The last estimate happens because $f(\xi/\xi_0)$ does not change appreciably from unity in the interval $|\xi| \leq \xi_0$, as seen in Fig. 5.

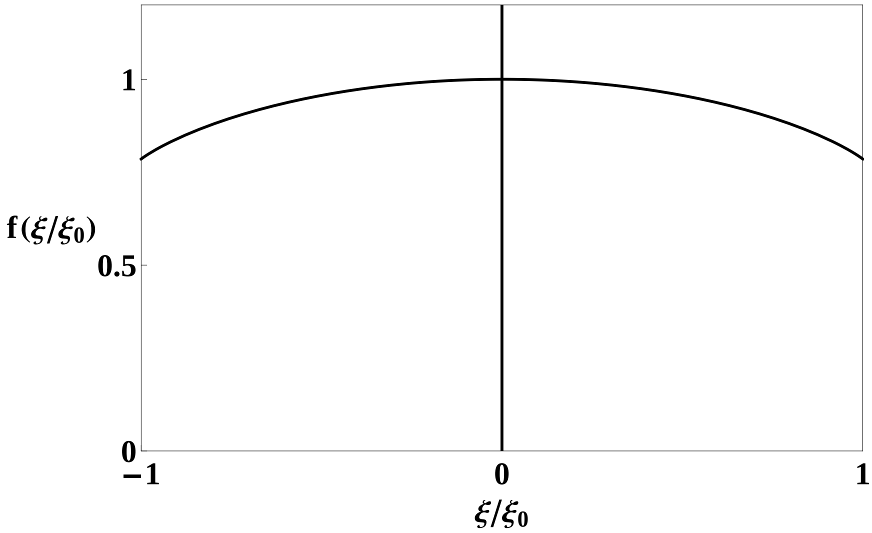

FIG. 5: Function $f(\xi/\xi_0)$ from Eq. (26).

Finally, the most stringent constraint from the inequality in Eq. (26) is for the maximum value $\rho = \rho_{\rm max} \approx \sqrt{\rho_{\rm eq}^2 + A^2} + |A|$, a return point obtained from the perturbative solution (21) where damping was neglected, for the sake of the estimate. Hence, the pressure dominance assumption holds for

$$\frac{3\kappa_B T}{m\omega_p^2\xi_0^2} \gg \left(\sqrt{\rho_{\rm eq}^2 + A^2} + |A|\right)^3 \geq \left[\frac{1}{2}\left(1 + \rho_{\rm eq}^2 + |1 - \rho_{\rm eq}^2|\right)\right]^3 , \tag{27}$$

where the last inequality is due to the internal condition reported below Eq. (22). In terms of the physical parameters, the inequality (27) unveils two subclasses, as follows.

1. *Thermal dominated equilibrium*

The thermal dominated equilibrium case corresponds to

$$\rho_{\rm eq}^2 \geq 1 \quad \Rightarrow \quad \frac{\omega^4}{\omega_p^4} \gg \frac{3\kappa_B T}{m\xi_0^2\omega^2} \geq 1\,. \tag{28}$$

2. *Harmonic confinement dominated equilibrium*

Although in this Section the main repulsive influence in the electrons momentum equation is always due to the pressure term, it can happens that the external force is so strong that $\rho_{\rm eq}^2 \leq 1$, which we refer to as the harmonic confinement dominated equilibrium case. It corresponds to

$$\rho_{\rm eq}^2 \leq 1 \quad \Rightarrow \quad 1 \geq \frac{3\kappa_B T}{m\xi_0^2\omega^2} \gg \frac{\omega_p^2}{\omega^2}\,. \tag{29}$$

It must be observed that for all values of $\rho_{\rm eq}$ one needs $\omega^2 \gg \omega_p^2$. This is because the pressure term in the hydrodynamic equations is compensated by the confinement term, which is proportional to ω^2. Hence, to disregard the Coulomb repulsion (proportional to ω_p^2) in comparison to the thermal effects, necessarily $\omega^2 \gg \omega_p^2$. The example in Fig. 3 fits the harmonic confinement dominated scenario. For the corresponding parameters, the plasma is almost ideal ($\Lambda \approx 10^7$) and a suitable damping mechanism would be collisions with neutrals.

IV. NEGLIGIBLE THERMAL EFFECTS

When the Coulomb repulsion dominates the pressure effects, one can drop the $\sim \kappa_B T$ term in Eq. (10). In this situation, the undamped problem was solved in [1], with an ionic background but without external confinement. Our aim is the treatment of the $\nu \neq 0$ case with harmonic trap, which was not fully performed before, to the best of our knowledge.

Ignoring thermal effects and differentiating all terms in Eq. (10) with respect to τ we get

$$\frac{\partial u}{\partial \tau^2} + \nu\frac{\partial u}{\partial \tau} + \omega^2 u = 0\,, \tag{30}$$

whose general solution is

$$u = e^{-\nu\tau/2}\Big(u(\xi,0)\,\cos(\Omega\tau) + \omega\, X(\xi)\,\sin(\Omega\tau)\Big)\,, \quad \Omega = \sqrt{\omega^2 - \nu^2/4}\,, \tag{31}$$

where $X(\xi)$ is at this stage an arbitrary function of the indicated argument and dimensions of a length. We will consider only the more interesting case, where the damping is weak so that $\Omega^2 > 0$.

Inserting the velocity field from Eq. (31) back into Eq. (10) gives

$$X(\xi) = \frac{1}{\Omega\omega}\left(-\omega^2\xi + \frac{\omega_p^2}{n_0}\int n(\xi,0)\,d\xi - \frac{\nu u(\xi,0)}{2}\right)\,. \tag{32}$$

Then, inserting $X(\xi)$ from Eq. (32) into Eq. (7), the result is

$$n = n(\xi,0)\left[1 + \frac{e^{-\nu\tau/2}}{\Omega}\,\frac{\partial u(\xi,0)}{\partial\xi}\,\sin(\Omega\tau) + 2\left(\frac{\omega_p^2\, n(\xi,0)}{\omega^2\, n_0} - 1\right)\psi(\tau)\right]^{-1}\,, \tag{33}$$

where

$$\psi(\tau) = \frac{1}{2}\left(1 - e^{-\nu\tau/2}\,\cos(\Omega\tau) - \frac{\nu\, e^{-\nu\tau/2}}{2\,\Omega}\,\sin(\Omega\tau)\right)\,, \tag{34}$$

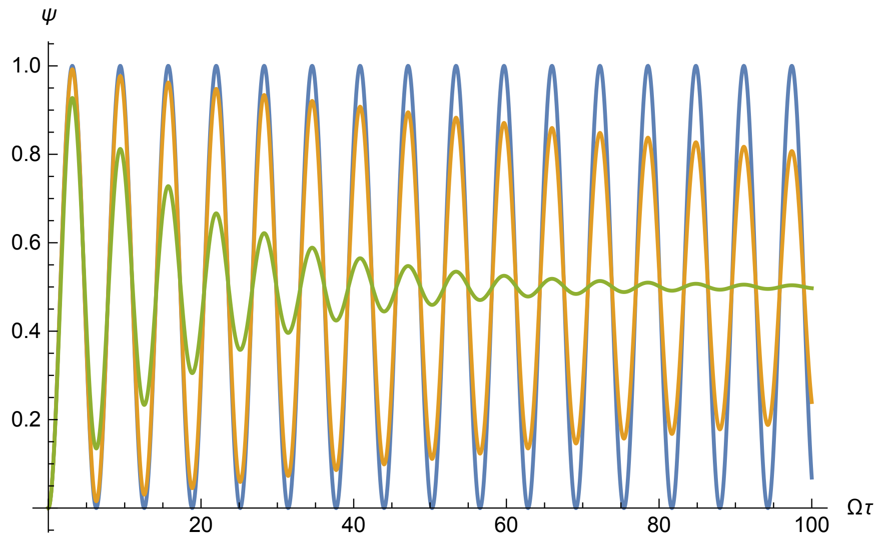

FIG. 6: Function ψ from Eq. (34) for different damping strengths. Upper curve, blue: $\nu/\Omega = 0$; mid curve, orange: $\nu/\Omega = 1/100$; lower curve, green: $\nu/\Omega = 1/10$. One has $\psi \to 1/2$ as $\tau \to \infty$, except in the undamped case.

an oscillatory function frequently appearing in what follows.

The electric field follows from Eq. (9), with the choice $E_0(\tau) = 0$. To finalize the arbitrary amplitude solution, the original spatial coordinate if found from Eq. (4) and reads

$$x = \xi + \frac{e^{-\nu\tau/2}}{\Omega}\, u(\xi,0)\sin(\Omega\tau) + 2\left(-\xi + \frac{\omega_p^2}{\omega^2\, n_0}\int n(\xi,0)\,d\xi\right)\psi(\tau)\,. \tag{35}$$

The results generalize those of chapter 3 of [1] and reproduce them in the dissipation-free ($\nu = 0$) and balanced ($\omega = \omega_p$) case. Interestingly, combining Eqs. (9) and (35), one concludes that asymptotically the electric and harmonic forces balance, or $eE = -m\,\omega^2\,x$ as $\tau \to \infty$, as expected.

Although the initial conditions $n(\xi,0), u(\xi,0)$ remain rather general, a constraint arises from the requirement of a positive definite number density, as follows.

A. Admissible initial conditions in the cold case

Unlike the thermal dominated case, corresponding to regular solutions by inspection, in the cold case there is the need to determine in which circumstances explosive solutions can take place. The mathematical conditions to avoid the associated multistream flow are described e.g. in page 37 of [1], which we strictly follow here. Supposing $n(\xi,0) \geq 0$

everywhere, from Eq. (33) one has $n(\xi,\tau)\geq 0$ for all time provided

$$\frac{\omega_p^2\, n(\xi,0)}{\omega^2\, n_0} + F(\xi,\tau) \geq 0\,, \tag{36}$$

where

$$F(\xi,\tau) = e^{-\nu\tau/2}\,\left(a\,\sin(\Omega\tau) + b\,\cos(\Omega\tau)\right)\,, \tag{37}$$

in terms of

$$a = \frac{1}{\Omega}\,\frac{\partial u(\xi,0)}{\partial\xi} + \frac{\nu}{2\,\Omega}\left(1 - \frac{\omega_p^2\, n(\xi,0)}{\omega^2\, n_0}\right)\,, \qquad b = 1 - \frac{\omega_p^2\, n(\xi,0)}{\omega^2\, n_0}\,. \tag{38}$$

It is easy to verify that in terms of time the minimum value of $F(\xi,\tau)$ occurs at $\tau=\tau_*$ such that

$$\Omega\tau_* = \arctan\left(\frac{\Omega\, a - \nu\, b/2}{\Omega\, b + \nu\, a/2}\right) + \pi = \pi\,, \tag{39}$$

where the last equality holds for $\partial u(\xi,0)/\partial\xi = 0$, which for simplicity we assume, to avoid too cumbersome expressions. Evaluating then Eq. (36) at $\tau=\tau_*=\pi/\Omega$ and isolating $n(\xi,0)$, we derive

$$\frac{n(\xi,0)}{n_0} \geq \frac{\omega^2}{\omega_p^2}\,\frac{1}{1+\exp[\pi\nu/(2\,\Omega)]}\,, \tag{40}$$

which is the required constraint on the initial number density, in the case of an uniform initial velocity field $u(\xi,0)$. If the inequality (40) is violated, one has that eventually $n(\xi,\tau)$ becomes negative in certain regions. Moreover, inversion of the Lagrangian variable transform in Eq. (35) becomes multivalued in this case. From Eq. (40), it is apparent the striking influence of the balance factor ω^2/ω_p^2, besides the regularizing rôle of damping, which allows smaller values of $n(\xi,0)$.

The undamped ($\nu=0$) case can be easily evaluated from Eq. (36), without restriction on initial velocity fields, with the result

$$\frac{n(\xi,0)}{n_0} \geq \frac{\omega^2}{2\,\omega_p^2}\left[1 + \frac{1}{\omega^2}\left(\frac{\partial u(\xi,0)}{\partial\xi}\right)^2\right]\,. \tag{41}$$

In brief, Eqs. (40) and (41) provide meaningful necessary conditions for regular solutions in the cold case.

V. SAMPLE APPLICATIONS IN THE COLD CASE

Unlike the hot, or thermally dominated case, there is no strong restriction on the functional form of the initial condition $n(\xi,0)$ and on the general form of the velocity field, when pressure effects can be disregarded. This allows the explicit construction of an infinite class of solutions, provided the mild constraints of Section IVA are obeyed. As an illustration, we consider two initial conditions: a localized, homogeneous electron gas and an an initial condition describing a bunching of electrons.

A. Homogeneous initial condition

For the sake of illustration, suppose an uniform electron gas at $\tau = 0$ restricted to $|\xi| < \xi_0$, with the following initial conditions,

$$n(\xi, 0) = \begin{cases} n_0\,, & |\xi| < \xi_0\,; \\ 0\,, & |\xi| > \xi_0\,, \end{cases} \tag{42}$$

together with $u(\xi, 0) = 0$.

Using Eqs. (8), (31) and (33), the complete solution in the bulk of the electron cloud ($|\xi| < \xi_0$) is found as

$$n = n_0 \left[1 + 2\left(\frac{\omega_p^2}{\omega^2} - 1\right)\,\psi(\tau)\right]^{-1}\,, \tag{43}$$

$$u = \exp(-\nu\tau/2)\sin(\Omega\,\tau)\,(\omega_p^2 - \omega^2)\,\xi/\Omega\,, \quad E = -n_0\,e\,\xi/\varepsilon_0\,, \tag{44}$$

where $\psi(\tau)$ is defined in Eq. (34).

The Lagrangian coordinate follows from Eq. (35) yielding (inside the electron's bulk)

$$\xi = x\left[1 + 2\left(\frac{\omega_p^2}{\omega^2} - 1\right)\,\psi(\tau)\right]^{-1}\,. \tag{45}$$

Therefore the domain of the electrons cloud will be given by $|x| \leq \xi_0\,\left[1 + 2\left(\omega_p^2/\omega^2 - 1\right)\,\psi(\tau)\right]$, asymptotically tending to $|x| \leq \omega_p^2\,\xi_0/\omega^2$ since $\psi(\tau) \to 1/2$ as $\tau \to \infty$. In addition, $n \to n_0\,\omega^2/\omega_p^2$ in the long time limit. As expected, a stronger trapping $\sim \omega^2$ produces a more localized solution, while a bigger electronic density $\sim \omega_p^2$ yields the inverse effect. Notice that the solution becomes stationary when $\omega = \omega_p$, in which case the initial condition represents an equilibrium state.

The number density in Eq. (43) is positive definite and well behaved for all time provided

$$\frac{\omega_p^2}{\omega^2} > \frac{1}{1 + \exp[\pi\,\nu/(2\,\Omega)]}\,, \tag{46}$$

in agreement with Eq. (40). The strict inequality sign in Eq. (46) is necessary to avoid explosive solutions associated to wave breaking, where the density and the gradient of the velocity field blow up to infinity in a finite time. Finally, since the electrons cloud is always spatially homogeneous, temperature effects are automatically zero.

B. Gaussian initial condition

Suppose an infinite system with an initially Gaussian concentration of electrons,

$$n(\xi,0) = \frac{n_0\,\omega^2}{2\,\omega_p^2}\,\left(1 + \Delta\,\exp(-\xi^2/\xi_0^2)\right)\,, \quad u(\xi,0) = 0\,, \tag{47}$$

where Δ and ξ_0 are positive parameters. The constraint (40) is manifestly satisfied. Working out Eq. (33), the exact number density is expressed as

$$n(\xi,\tau) = \frac{n_0\,\omega^2}{2\,\omega_p^2}\,\frac{\left[1 + \Delta\exp(-\xi^2/\xi_0^2)\right]}{\left[1 + \left(-1 + \Delta\exp(-\xi^2/\xi_0^2)\right)\psi(\tau)\right]}\,, \tag{48}$$

in terms of the same function $\psi(\tau)$ defined in Eq. (34). Asymptotically, one has $n \to n_0\omega^2/\omega_p^2$.

Carrying on the necessary steps, the velocity and electric fields become

$$u = \frac{\omega^2\,\xi_0}{2\,\Omega}\,\exp(-\nu\,\tau/2)\,\sin(\Omega\,\tau)\left[-\frac{\xi}{\xi_0} + \frac{\sqrt{\pi}\,\Delta}{2}\,\mathrm{Erf}\Big(\frac{\xi}{\xi_0}\Big)\right], \tag{49}$$

$$E = -\frac{m\,\omega^2\,\xi_0}{2\,e}\Big[\frac{\xi}{\xi_0} + \frac{\sqrt{\pi}\,\Delta}{2}\,\mathrm{Erf}\Big(\frac{\xi}{\xi_0}\Big)\Big]\,. \tag{50}$$

Remembering that $\tau = t$, the remaining step of the integration rests on the inversion of the Lagrangian coordinate transformation, which from Eq. (35) turns out to be

$$\frac{x}{\xi_0} = \frac{\xi}{\xi_0} + \left[-\frac{\xi}{\xi_0} + \frac{\sqrt{\pi}\,\Delta}{2}\,\mathrm{Erf}\left(\frac{\xi}{\xi_0}\right)\right]\,\psi(\tau)\,, \tag{51}$$

where $\mathrm{Erf}(s) = (2/\sqrt{\pi})\int_0^s \exp(-s'^2)\,ds'$ denotes the Error function in terms of a generic argument s.

Although Eq. (51) represents a higher transcendental equation, it can be easily solved numerically to obtain ξ as a function of (x,t), provided all pertinent parameters are furnished. Due to $0 \le \psi(\tau) < 1$, it can be shown that the solution is unique. An example of such a procedure is shown in Fig. 8.

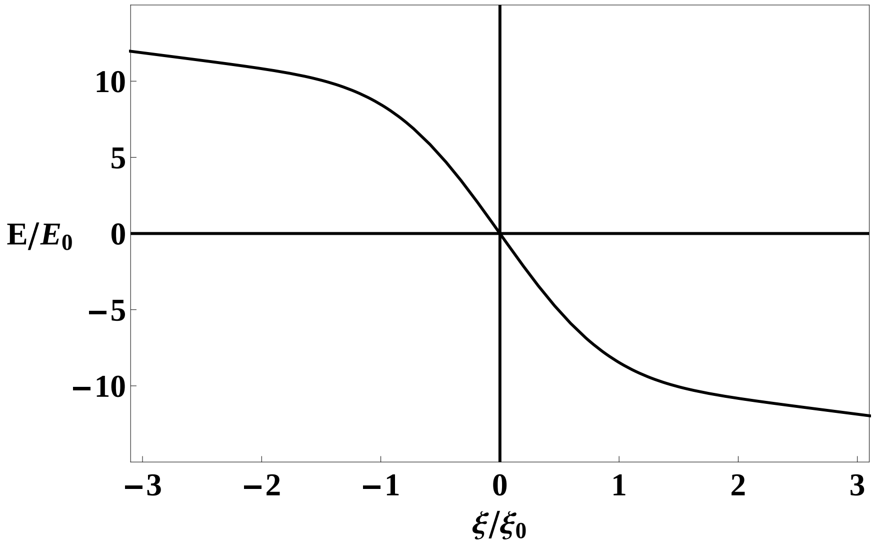

FIG. 7: Electric field from Eq. (50), with $\Delta = 10$, normalized to $E_0 = m\,\omega^2 \xi_0/(2\,e)$.

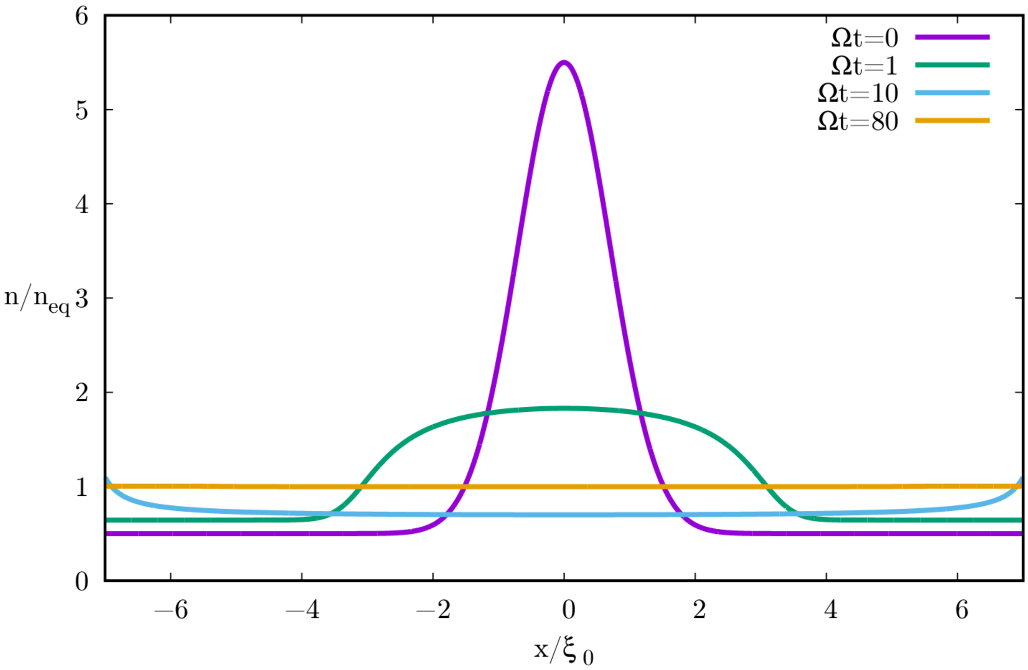

FIG. 8: Number density from Eq. (48), in the laboratory frame, normalized to the asymptotic value $n_{\rm eq} = (\omega^2/\omega_p^2)\, n_0$, for $\Delta = 10$, $\nu/\Omega = 1/10$ and different times.

Using Eq. (10) it can be shown that the cold electron gas assumption is satisfied provided

$$\frac{\kappa_B\, T}{m} \ll \frac{\omega_p^4\, \xi_0^2}{\omega^2\, \Delta\, (1+\Delta)}\,. \tag{52}$$

In particular, if there is no initial bunching ($\Delta \approx 0$), the electron gas becomes everywhere homogeneous (see Eq. (48)) and pressure effects are automatically negligible, as manifest from Eq. (52).

VI. CONCLUSION

In this work, new arbitrary amplitude structures were derived for trapped charged particle systems, in terms of the Lagrangian variables method. Two classes of solutions have been identified. One of them is entirely new and valid for thermally dominated systems, becoming essentially reducible to the Pinney equation, a traditional ordinary differential equation in nonlinear Physics. Moreover, in the case of non-ideal non-neutral plasma, dissipation can be also handled by perturbation theory provided the damping is weak. The second class of solutions, applicable to Coulomb repulsion dominated systems, significantly generalizes the known results in the literature, now including collisional effects and an external harmonic trap. The conditions for the validity of the solutions have been fully determined in terms of the physical parameters, which helps the experimental verification of the prescribed dynamics.

The spirit of this work can be generalized in several directions, allowing for more complex geometries [44], relativistic effects [45, 46] and harmonic traps with a time-dependent frequency or external forcing. For instance, the dissipative Pinney equation has well-known accurate perturbative solutions in the weakly damped non-autonomous case [18]. In addition, the situation where thermal and Coulomb repulsion effects are of the same order can in principle be handled to some extent, using a linearization procedure of the pressure term [1]. These possibilities are under investigation and will be reported elsewhere.

Acknowledgments: F. H. and L. G. F. S. acknowledge the support by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

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