Ion-acoustic cnoidal waves in a quantum plasma

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Abstract

Nonlinear ion-acoustic cnoidal wave structures are studied in an unmagnetized quantum plasma. Using the reductive perturbation method, a Korteweg-de Vries equation is derived for appropriate boundary conditions and nonlinear periodic wave solutions are obtained. The corresponding analytical solution and numerical plots of the ion-acoustic cnoidal waves and solitons in the phase plane are presented using the Sagdeev pseudo-potential approach. The variations in the nonlinear potential of the ion-acoustic cnoidal waves are studied at different values of quantum parameter H_e which is the ratio of electron plasmon energy to electron Fermi energy defined for degenerate electrons. It is found that both compressive and rarefactive ion-acoustic cnoidal wave structures are formed depending on the value of the quantum parameter. The dependence of the wavelength and frequency on nonlinear wave amplitude is also presented.

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I. INTRODUCTION

The study of nonlinear wave propagation in quantum plasmas has gain importance due its application in understanding the particle or energy transport phenomenon on short scale lengths i.e., in micro and nano scale electronic devices and in dense compact stars [1–7]. Typically, the quantum effects in plasmas become important when the Fermi temperature, which is related to the particles density, becomes equal or greater than the system's thermal temperature or the inter-particle distance becomes smaller or of the same order of the particle's de Broglie thermal wavelength. In order to study the dynamics in quantum plasmas, the quantum hydrodynamic (QHD) model is frequently used [1, 2, 6]. The QHD model consists of a set of equations describing the transport of charge, momentum and energy in a quantum charged particle system interacting through a self consistent electromagnetic field. In QHD model, the quantum effects appear through the quantum statistical (Fermi) pressure and the Bohm potential (due to quantum diffraction or tunneling effects). The QHD is useful to study collective effects on short scale lengths and has its limitation for systems that are large compared to the Fermi Debye lengths of the species in the system. Quantum ion-acoustic waves were investigated by Haas *et al.* [1, 2] using the QHD model. They derived a Korteweg-de Vries (KdV) equation in the weakly nonlinear amplitude wave limit for studying the propagation of ion-acoustic solitons in a quantum plasma. It is reported that the compressive or rarefactive soliton solution depends on a quantum parameter (H_e) defined for degenerate electrons, which is the ratio of the electron plasmon energy to the Fermi energy. In the fully nonlinear regime, the existence of periodic traveling wave patterns were reported for ion-acoustic waves in quantum plasmas. The arbitrary amplitude ion-acoustic solitary waves using the QHD model in electron-ion quantum plasmas with the Sagdeev potential approach has also been investigated [8]. Nonlinear electrostatic wave structures such as solitons, envelope and shocks have been studied in quantum electron-ion (EI) plasmas [9] but no one has reported the propagation of cnoidal wave structures in quantum plasmas. The purpose of the present work is to investigate the formation of ion-acoustic cnoidal waves in a quantum EI plasma using the well known QHD model.

In classical plasmas, a lot of research work has been done in studying nonlinear ion-acoustic wave structures such as solitons, cnoidal waves and envelopes. Solitons are single pulse structures which are formed due to the balance between nonlinearity and dispersion effects in the system [10] and they consist of isolated hump or dip like wave profile with no rapid oscillations inside the packet. Envelope structure contains both fast and slow oscillations, obtained when nonlinearity balances the wave group dispersion effects. The envelope is a localized modulated wave packet whose dynamics is governed by the nonlinear Schrödinger (NLS) equation [11, 12]. The periodic (cnoidal) wave is the exact nonlinear periodic wave solution of the KdV equation with appropriate boundary conditions. These solutions of the KdV equation are also termed as cnoidal waves because they are written in terms of Jacobian elliptic-function cn. In general, the nonlinear periodic waves are expressed in terms of Jacobian elliptic-functions such as dn, sn or cn, and the nonlinear dn waves are believed to be generated in the defocusing region of the ionospheric plasmas [13–16]. The ion-acoustic soliton and double layer structures are observed in auroral and magnetospheric plasmas and also nonlinear periodic wave signals appear frequently in these observations [17]. The periodic signals are also observed at the edge of the tokamak plasma, which can be described by cnoidal waves [18]. Kono *et al.* [19] studied the higher order contributions in the reductive perturbation theory for the nonlinear ion-acoustic wave propagation under the periodic boundary condition. The nonlinear periodic wave solution for small amplitude Langmuir waves in electron-ion plasmas was studied by Schamel [20]. Jovanovic and Shukla [21] presented a solution in the form of a cnoidal wave provided the minimum value of the electrostatic potential remain finite in studying coherent electric field structures. Prudskikh [22] studied the ion-acoustic nonlinear periodic waves in dusty plasmas. The ion-acoustic cnoidal wave and the associated nonlinear ion flux in dusty plasmas was studied by Jain *et al.* [23]. They derived the coupled evolution equations for the first and second order potentials for ion-acoustic waves in unmagnetized dusty plasmas using reductive perturbation method with appropriate boundary conditions. Kaladze *at al.* [24] investigated acoustic cnoidal waves and solitons in unmagnetized pair-ion (PI) plasmas consisting of the same mass ion species with different temperatures. They reported the formation of both compressive and rarefactive cnoidal wave structures in PI plasmas which depends on the temperature ratio of PI species. Recently, Kaladze and Mahmood [25] studied the effect of positrons density and nonthermal parameter kappa on the propagation of the ion-acoustic cnoidal waves in electron-positron-ion plasmas. **Saha and Chatterjee studied electron acoustic periodic and solitary wave solutions in unmagnetized [26] and magnetized [27, 28] quantum plasmas. They derived a KdV equation using the reductive**

perturbation method and investigated the associated nonlinear structures using bifurcation theory.

The manuscript is organized in the following way. In the next section, the model and set of dynamic equations for studying nonlinear ion-acoustic waves in unmagnetized quantum plasmas is presented. Using the reductive perturbation method, the KdV equation is also derived with appropriate boundary conditions. In Section III, the stationary wave solution is obtained for ion-acoustic cnoidal waves using the Sagdeev potential approach. In Section IV, the numerical analysis and plots are presented for degenerate plasma cases at different plasma densities chosen from literature and the conclusion is drawn in the final Section V.

II. BASIC MODEL AND DERIVATION OF KORTEWEG-DE VRIES EQUATION

In order to study the electrostatic nonlinear periodic (cnoidal) waves in unmagnetized electron-ion (EI) quantum plasmas, we will derive a KdV equation using the reductive perturbation method. A KdV equation for quantum ion-acoustic waves has already been derived by Haas *et al.* [2] with emphasis on localized solutions obtained under decaying boundary conditions. For cnoidal waves, periodic boundary conditions are more appropriate, hence we will derive again the KdV equation for quantum ion-acoustic waves. The set of dynamic equations for ion-acoustic wave using QHD model is described as follows.

The ion continuity and momentum equations for ion fluid are given by

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \tag{1}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{e}{m_i}\frac{\partial \phi}{\partial x}. \tag{2}$$

The dynamic equation for the inertialess electron quantum fluid is described by

$$0 = e\frac{\partial \phi}{\partial x} - \frac{1}{n_e}\frac{\partial p_e}{\partial x} + \frac{\hbar^2}{2m_e}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{n_e}}\frac{\partial^2}{\partial x^2}\sqrt{n_e}\right). \tag{3}$$

The Poisson equation is written as

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0}(n_e - n_i), \tag{4}$$

where ϕ is the electrostatic potential. The ion fluid density and velocity are represented by n_i and u_i respectively, while n_e is the electron fluid density. Also, m_e and m_i are the electron and ion masses, $-e$ is the electronic charge, ε_0 and $\hbar$ are the dielectric and scaled Planck's constants. In equilibrium, we have $n_{e0} = n_{i0} = n_0$ (say). Here p_e is the electron pressure and $p_e(n_e)$ is obtained from the equation of state for the electron fluid. The electrons are assumed to obey the equation of state pertaining to one-dimensional zero-temperature Fermi gas [2, 3], which is $p_e = m_e v_{Fe}^2 n_e^3/3n_0^2$ where n_0 is the equilibrium plasma density. Here v_{Fe} is the Fermi velocity of electron, connected to the Fermi temperature by $m_e v_{Fe}^2/2 = k_B T_{Fe}$ and k_B is the Boltzmann constant. The last term on the right hand side of the momentum equations for electrons quantum fluid is the quantum force, which arises due to the quantum Bohm potential and gives quantum diffraction or quantum tunneling effects due to the wave-like nature of the charged particles. The quantum effects due to ions are ignored in the model as they have large inertia in comparison with the electrons.

In order to find the nonlinear ion-acoustic periodic waves in a quantum plasma, the set of nonlinear dynamic equations are written in a normalized form as follows,

$$\frac{\partial \tilde{n}_i}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}(\tilde{n}_i \tilde{u}_i) = 0, \tag{5}$$

$$\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_i \frac{\partial}{\partial \tilde{x}} \tilde{u}_i = -\frac{\partial \Phi}{\partial \tilde{x}}, \tag{6}$$

$$0 = \frac{\partial \Phi}{\partial \tilde{x}} - \frac{1}{2}\frac{\partial \tilde{n}_e^2}{\partial \tilde{x}} + \frac{H_e^2}{2}\frac{\partial}{\partial \tilde{x}}\left(\frac{1}{\sqrt{\tilde{n}_e}}\frac{\partial^2}{\partial \tilde{x}^2}\sqrt{\tilde{n}_e}\right), \tag{7}$$

$$\frac{\partial^2 \Phi}{\partial \tilde{x}^2} = \tilde{n}_e - \tilde{n}_i. \tag{8}$$

The normalization of space, time, ion velocity and electrostatic potential is defined by $\tilde{x} \to \omega_{pi} x/c_s$, $\tilde{t} \to \omega_{pi} t$, $\tilde{u}_i \to u_i/c_s$ and $\Phi = e\phi/2k_B T_{Fe}$ respectively, where the ion plasma frequency and ion-acoustic speed are $\omega_{pi} = (n_0 e^2/\varepsilon_0 m_i)^{1/2}$ and $c_s = (2k_B T_{Fe}/m_i)^{1/2}$ respectively, and the non-dimensional quantum parameter for electrons is defined as $H_e = \hbar\omega_{pe}/2k_B T_{Fe}$ i.e., the ratio of electron plasmon energy to the Fermi energy, here $\omega_{pe} = (n_0 e^2/\varepsilon_0 m_e)^{1/2}$ is the electron plasma frequency. The normalization of electron and ion fluid density is defined as $\tilde{n}_j = n_j/n_0$ $(j = e, i)$. In the following, for simplicity we will not use the tilde sign.

In order to find an nonlinear evolution equation, a stretching of independent variables x, t is defined as follows [10],

$$\xi = \varepsilon^{1/2}(x - V_0 t), \quad \tau = \varepsilon^{3/2} t,$$

where ε is a small parameter and V_0 is the phase velocity of the wave to be determined later on. The perturbed quantities can be expanded in the powers of ε,

$$
\begin{aligned}
n_{i1} &= 1 + \varepsilon n_{i1} + \varepsilon^2 n_{i2} + ..., \\
n_{e1} &= 1 + \varepsilon n_{e1} + \varepsilon^2 n_{e2} + ..., \\
u_i &= \varepsilon u_{i1} + \varepsilon^2 u_{i2} + ..., \\
\Phi &= \varepsilon \Phi_1 + \varepsilon^2 \Phi_2 + ...
\end{aligned} \tag{9}
$$

Moreover, $\partial/\partial x = \varepsilon^{1/2}\, \partial/\partial \xi$ and $\partial/\partial t = \varepsilon^{3/2} \partial/\partial \tau - V_0 \varepsilon^{1/2} \partial/\partial \xi$.

From ion continuity and momentum equations the lowest order ($\sim \varepsilon^{3/2}$) terms gives

$$-V_0 \frac{\partial n_{i1}}{\partial \xi} + \frac{\partial u_{i1}}{\partial \xi} = 0, \tag{10}$$

$$-V_0 \frac{\partial u_{i1}}{\partial \xi} + \frac{\partial \Phi_1}{\partial \xi} = 0, \tag{11}$$

$$\frac{\partial \Phi_1}{\partial \xi} - \frac{\partial n_{e1}}{\partial \xi} = 0. \tag{12}$$

The lowest order ($\sim \varepsilon$) term of Poisson equation gives

$$n_{i1} = n_{e1}. \tag{13}$$

Multiplying Eq. (10) by V_0 and adding with Eq. (11), we have

$$\frac{\partial n_{i1}}{\partial \xi} = \frac{1}{V_0^2} \frac{\partial \Phi_1}{\partial \xi}. \tag{14}$$

Using Eqs. (12), (13) and (14), we have

$$V_0 = \pm 1. \tag{15}$$

which is the normalized phase velocity of the ion-acoustic wave. From now on, we set $V_0 = 1$ without loss of generality.

Now integrating Eqs. (11), (12), (14) and using (13) we have

$$n_{i1} = n_{e1} = \Phi_1 + c_1(\tau) \tag{16}$$

and

$$u_{i1} = \Phi_1 + c_2(\tau), \tag{17}$$

where $c_1(\tau)$ and $c_2(\tau)$ are at this point arbitrary functions of τ only.

Now collecting the next higher order terms from ion dynamic equations, we have

$$-\frac{\partial n_{i2}}{\partial \xi} + \frac{\partial u_{i2}}{\partial \xi} + \frac{\partial n_{i1}}{\partial \tau} + \frac{\partial}{\partial \xi}(n_{i1}u_{i1}) = 0, \tag{18}$$

$$-\frac{\partial u_{i2}}{\partial \xi} + \frac{\partial \Phi_2}{\partial \xi} + \frac{\partial u_{i1}}{\partial \tau} + u_{i1}\frac{\partial u_{i1}}{\partial \xi} = 0, \tag{19}$$

$$\frac{\partial \Phi_2}{\partial \xi} - \frac{\partial n_{e2}}{\partial \xi} - \frac{1}{2}\frac{\partial}{\partial \xi}n_{e1}^2 + \frac{H_e^2}{4}\frac{\partial^3 n_{e1}}{\partial \xi^3} = 0. \tag{20}$$

The next higher order ($\sim \varepsilon^2$) term of Poisson equation gives

$$\frac{\partial^2 \Phi_1}{\partial \xi^2} = n_{e2} - n_{i2}. \tag{21}$$

Adding Eqs. (18) and (19), we obtain

$$\frac{\partial n_{i2}}{\partial \xi} = \frac{\partial n_{i1}}{\partial \tau} + \frac{\partial}{\partial \xi}(n_{i1}u_{i1}) + \frac{\partial u_{i1}}{\partial \tau} + u_{i1}\frac{\partial}{\partial \xi}u_{i1} + \frac{\partial \Phi_2}{\partial \xi}, \tag{22}$$

and from Eq. (20) we have

$$\frac{\partial n_{e2}}{\partial \xi} = \frac{\partial \Phi_2}{\partial \xi} - \frac{1}{2}\frac{\partial}{\partial \xi}n_{e1}^2 + \frac{H_e^2}{4}\frac{\partial^3 n_{e1}}{\partial \xi^3}. \tag{23}$$

Using n_{i1}, u_{i1} and n_{e1} from Eqs. (16) and (17) in Eqs. (22) and (23), we obtain

$$\frac{\partial n_{i2}}{\partial \xi} = 2\frac{\partial \Phi_1}{\partial \tau} + 3\Phi_1\frac{\partial \Phi_1}{\partial \xi} + (2c_2 + c_1)\frac{\partial \Phi_1}{\partial \xi} + \frac{\partial \Phi_2}{\partial \xi} \tag{24}$$

and

$$\frac{\partial n_{e2}}{\partial \xi} = -\Phi_1\frac{\partial \Phi_1}{\partial \xi} - c_2\frac{\partial \Phi_1}{\partial \xi} + \frac{H_e^2}{4}\frac{\partial^3 \Phi_1}{\partial \xi^3} + \frac{\partial \Phi_2}{\partial \xi}. \tag{25}$$

Applying periodic boundary conditions we get $\partial c_1/\partial \tau = \partial c_2/\partial \tau = 0$, so that the functions c_1 and c_2 become independent of both ξ and τ and are from now on constants.

Differentiating Eq. (21) and using Eqs. (24) and (25), and after some simplifications, we have the KdV equation for the nonlinear dynamics of ion-acoustic waves in a quantum plasma as follows,

$$\frac{\partial \Phi}{\partial \tau} + 2\Phi\frac{\partial \Phi}{\partial \xi} + c_0\frac{\partial \Phi}{\partial \xi} + D\frac{\partial^3 \Phi}{\partial \xi^3} = 0, \tag{26}$$

where

$$\begin{aligned} c_0 &= c_1 + c_2, \\ D &= \frac{1}{2}\left(1 - \frac{H_e^2}{4}\right). \end{aligned} \tag{27}$$

Here Φ_1 has been replaced by Φ.

In the above KdV equation (26), the term containing an arbitrary constant c_0 can be removed with a Galilean transformation. Hence we can set $c_0 = 0$ without loss of generality. It can be noticed easily from dispersive coefficient D that the cnoidal wave solution exist only when $H_e \neq 2$, so that the dispersive coefficient does not disappear to balance the nonlinearity.

III. NONLINEAR PERIODIC WAVE SOLUTIONS

In order to find the steady state cnoidal and solitary waves solutions of the KdV Eq. (26) for quantum ion-acoustic waves, we follow the same procedure as already done in Refs.[24, 25]. Assume a solution $\Phi(\eta)$, where $\eta = \xi - u\tau$ and u is the velocity of the nonlinear structure moving with the frame. Therefore, Eq. (26) can be written as

$$D\frac{d^3\Phi}{d\eta^3} + \frac{d}{d\eta}\left(\Phi^2 - u\Phi\right) = 0. \tag{28}$$

As said, the arbitrary constant c_0 has been ignored, since it gives just a shift in the velocity of the nonlinear structure.

After integration of Eq. (28), we get the equation of a conservative nonlinear oscillator i.e.,

$$\frac{d^2\Phi}{d\eta^2} = -\frac{dW}{d\Phi}, \tag{29}$$

where its potential energy $W = W(\Phi)$ is defined as

$$W(\Phi) = \frac{1}{3D}\Phi^3 - \frac{u}{2D}\Phi^2 + \rho_0\Phi. \tag{30}$$

Here ρ_0 is an integration constant. The potential function $W(\Phi)$ has two points of extremum for Φ i.e. $\Phi = \tilde{\Phi}_{1,2}$ defined by $\partial W/\partial\Phi = 0$, which are given by

$$\tilde{\Phi}_{1,2} = \frac{u}{2} \pm \sqrt{\frac{u^2}{4} - D\rho_0}. \tag{31}$$

Thus, there are two equilibrium states. One of them defines a saddle point while the other one represents a center point i.e., the bottom of the potential well [24]. Moreover $u^2/4 - D\rho_0 > 0$ must hold for real values. The zero's of the potential energy (30) i.e., $\Phi = z_1, z_2, z_3$ are given as follows,

$$z_1 = 0\,, \quad z_{2,3} = \frac{3}{2}\left(\frac{u}{2} \pm \sqrt{\frac{u^2}{4} - \frac{4}{3}D\rho_0}\right). \tag{32}$$

To get the shape of the real potential well $u^2/4 - 4D\rho_0/3 > 0$ must holds. Note that the potential well having the center for $\Phi > 0$ (positive) defines the compressive cnoidal waves and solitons, while in the case of $\Phi < 0$ (negative) the potential well defines the rarefactive cnoidal waves and solitons [24]. The shape of the potential well strongly depends on the sign of dispersive coefficient D (see Eq. (30)). As described in Eq. (27), the value of the dispersive coefficient D in the KdV equation is positive for electron quantum parameter $H_e < 2$ case, while it becomes negative for $H_e > 2$. Therefore, the formation of compressive or rarefactive ion-acoustic nonlinear structure depends on the value of the quantum parameter H_e. The amplitude of the nonlinear structures is defined by the width of the potential well, which is the length between the last zero of the potential well and the saddle point (see Eqs. (31) and (32)).

The energy first integral associated to (29) is

$$\frac{1}{2}\left(\frac{d\Phi}{d\eta}\right)^2 + W(\Phi) = \frac{E_0^2}{2}, \tag{33}$$

where E_0^2 is the integration constant (assumed positive definite in order to access cnoidal wave solutions).

Using Eq. (30) in (33), we have

$$\left(\frac{d\Phi}{d\eta}\right)^2 = E_0^2 - \frac{2}{3D}\Phi^3 + \frac{u}{D}\Phi^2 - 2\rho_0\Phi. \tag{34}$$

Let us consider the initial conditions $\Phi(0) = \Phi_0$ and $d\Phi(0)/d\eta = 0$. Then we can define

$$E_0^2 = \frac{2}{3D}\Phi_0^3 - \frac{u}{D}\Phi_0^2 + 2\rho_0\Phi_0. \tag{35}$$

Using Eq. (35) in Eq. (34) and after factorization, we have

$$\left(\frac{d\Phi}{d\eta}\right)^2 = \frac{2}{3D}(\Phi_0 - \Phi)(\Phi - \Phi_1)(\Phi - \Phi_2), \tag{36}$$

where

$$\Phi_{1,2} = \frac{3}{2}\left[\frac{u}{2} - \frac{\Phi_0}{3} \pm \sqrt{\frac{1}{3}(b_1 - \Phi_0)(\Phi_0 - b_2)}\right], \tag{37}$$

and

$$b_{1,2} = \frac{u}{2} \pm 2\sqrt{\frac{u^2}{4} - D\rho_0}. \tag{38}$$

In addition, the following inequalities should be kept: $b_2 \leq \Phi_0 \leq b_1$ or $b_1 \leq \Phi_0 \leq b_2$. From Eqs. (34)-(36), we have the following relation,

$$u = \frac{2}{3}(\Phi_0 + \Phi_1 + \Phi_2). \tag{39}$$

The periodic (cnoidal) wave solution of Eq. (36) is given [29] by

$$\Phi(\eta) = \Phi_1 + (\Phi_0 - \Phi_1)\mathrm{cn}^2(R\eta, s), \tag{40}$$

where cn is the Jacobian elliptic function, s is the modulus defined as

$$s^2 = \frac{(\Phi_0 - \Phi_1)}{(\Phi_0 - \Phi_2)} < 1, \tag{41}$$

and the quantity $R = \sqrt{\frac{1}{6D}(\Phi_0 - \Phi_2)}$.

The amplitude A of the cnoidal wave is defined from Eq. (40) as follows,

$$A = \Phi_0 - \Phi_1. \tag{42}$$

As it is seen from the solution (40) at $\eta = 0$, we have the initial condition $\Phi(0) = \Phi_0$. In addition, the real numbers Φ_i $(i = 0, 1, 2)$ are ordered as $\Phi_0 > \Phi_1 \geq \Phi_2$ and $\Phi_1 \leq \Phi \leq \Phi_0$ for $D > 0$.

The modulus $0 < s < 1$ is a measure of the nonlinearity of the wave. The case $s << 1$ corresponds to the weakly nonlinear oscillations near the bottom of the potential well and the elliptic functions are close to trigonometric ones. At $s \to 0$, the expression (40) passes to solution of linear equations [29].

The wavelength λ of the cnoidal waves is defined as

$$\lambda = 4\sqrt{\frac{3D}{2(\Phi_0 - \Phi_2)}}K(s), \tag{43}$$

where $K(s)$ is the complete elliptic integral of the first kind and the corresponding frequency is $f = v_1/\lambda$ (where v_1 is the velocity of the cnoidal waves). The velocity v_1 of the cnoidal waves in the laboratory frame is equal to $v_1 = V_0 + u$, where the expression for the frame velocity u is given by,

$$u = \frac{2}{3}\frac{(\Phi_0 - \Phi_1)}{s^2}(2 - s^2) + 2\Phi_2, \tag{44}$$

which has been obtained using the expression of the modulus s described by Eq. (41).

The mean value of Φ can be expressed as

$$\bar{\Phi} = \frac{1}{\lambda}\int_0^{\lambda}\Phi(\eta)d\eta = \Phi_2 + (\Phi_0 - \Phi_2)\frac{E(s)}{K(s)}, \tag{45}$$

where $E(s)$ is the complete integral of the second kind.

The limiting case of the soliton i.e. $s = 1$, can be obtained at $\Phi_0 \approx b_1$ or $\Phi_0 \approx b_2$ (see Eq. (38)), so that

$$\Phi_1 \approx \Phi_2 = \frac{u}{2} \mp \sqrt{\frac{u^2}{4} - D\rho_0}. \tag{46}$$

Further, we take into account,

$$K(s) \approx \frac{1}{2}\ln\left(\frac{16}{1-s^2}\right) \to \infty\,, \quad \mathrm{cn}\,\mathrm{z} \to \frac{1}{\mathrm{ch}\,\mathrm{z}}, \tag{47}$$

which imply that the wavelength of the cnoidal waves defined in (43) tends to infinity and the solution (40) passes to a soliton-like shape [29] i.e.,

$$\Phi(\eta) = \Phi_1 + \frac{A}{\mathrm{ch}^2\left(\sqrt{\frac{1}{6D}(\Phi_0 - \Phi_1)}\eta\right)}, \tag{48}$$

where $\triangle = \sqrt{6\,|D/A|}$ is the width of the soliton and A is its amplitude, defined by Eq. (42). As it follows from Eq. (44), the propagation velocity of the solitons becomes

$$u = 2\Phi_1 + \frac{2}{3}A. \tag{49}$$

From Eq. (48), we see that Φ_1 defines the potential at $\eta \to \pm\infty$.

Thus, at large values of s i.e., $s \to 1$, $A = const$ the periodic wave asymptotically approaches to the sequence of solitons having the amplitude A (relatively to the level $\Phi = \bar{\Phi} = \Phi_1$). By the order of magnitude, the distance between them is equal to

$$\lambda = \triangle\left|\ln(1-s^2)\right|, \tag{50}$$

where $\triangle$ is the width of the soliton already defined above.

IV. NUMERICAL ANALYSIS

The numerical plots of the nonlinear wave potential and the phase plane plots of the cnoidal wave structures and solitons are shown in Figs.1-3 at different densities for a degenerate electron plasma cases, such as astrophysical plasmas, laser plasmas and ultra-cold plasmas. For a completely degenerate electron plasma, the electron Fermi energy and density are related as $k_B T_{Fe} = \hbar^2(3\pi^2 n_0)^{2/3}/2m_e$ and the Fermi temperature of degenerate electrons (T_{Fe}) should be much greater than the thermal temperature T of the system i.e., $T \ll T_{Fe}$. The quantum parameter for electrons is related to density as $H_e \sim n_0^{-1/6}$, which shows that in a completely degenerate electron plasma case, the value of the quantum parameter decreases with the increase in the plasma density. So the quantum diffraction effects tend to be less relevant in dense plasmas. In case of astrophysical plasma conditions i.e., $n_0 = 10^{36} m^{-3}$, the quantum parameter for degenerate electrons comes out to be $H_e = 0.05$ and the condition for thermal temperature becomes $T << 10^9 K$, while for laser plasmas we have $n_0 = 10^{32} m^{-3}$ then $H_e = 0.24$ and $T << 10^7 K$. Further, for ultra-cold plasmas, we have $n_0 = 2.7 \times 10^{26} m^{-3}$ for which $H_e = 2.002$ and $T << 1800K$ [30, 31].

The formation of ion-acoustic compressive (rarefactive) nonlinear structures depends on the value of the quantum parameter for electrons i.e., $H_e < 2$ ($H_e > 2$). It is also noticed that the velocity of the nonlinear structure is positive i.e., $u > 0$ for compressive cnoidal waves and solitons with quantum parameter for degenerate electrons lies in the range $0 \leq H_e < 2$. For the $H_e > 2$ case, rarefactive ion-acoustic cnoidal waves and solitons structures are formed and its solution exist only when the velocity of the nonlinear structure is negative $u < 0$ i.e., it moves in the backward direction.

It can be seen from Figs. 1a and 2a that the Sagdeev potential $W(\Phi)$ are formed for $\Phi > 0$ for the electron quantum parameter values $H_e = 0.05$ and 0.24. The corresponding compressive ion-acoustic cnoidal wave (solid curve) and solitons (dotted curve) structures are shown in the phase plane plots of Fig. 1b and 2b respectively. The cnoidal wave structure (solid bounded curve) is formed inside the separatrix (dotted curve) which represent a soliton structure as shown in the figures. The Sagdeev potential plot $W(\Phi)$ is formed with $\Phi < 0$ for the electron quantum parameter value $H_e = 2.002$ (> 2) as shown in the Fig. 3a. The corresponding rarefactive ion-acoustic cnoidal wave (solid bounded curve) and soliton (dotted curve) structures in the phase plane plot are shown in the Fig. 3b. Rarefactive nonlinear ion-acoustic cnoidal wave (solid bounded curve) structures are also formed inside the separatrix (dotted curve), which represent the soliton.

The plots of the compressive ion-acoustic cnoidal wave from Eq. (40) and solitons from Eq. (48) for $H_e < 2$ case are shown in the Figs. 4a and 4b respectively. It can be seen from the Fig. 4a that there is a little decrease in the

Fig.1

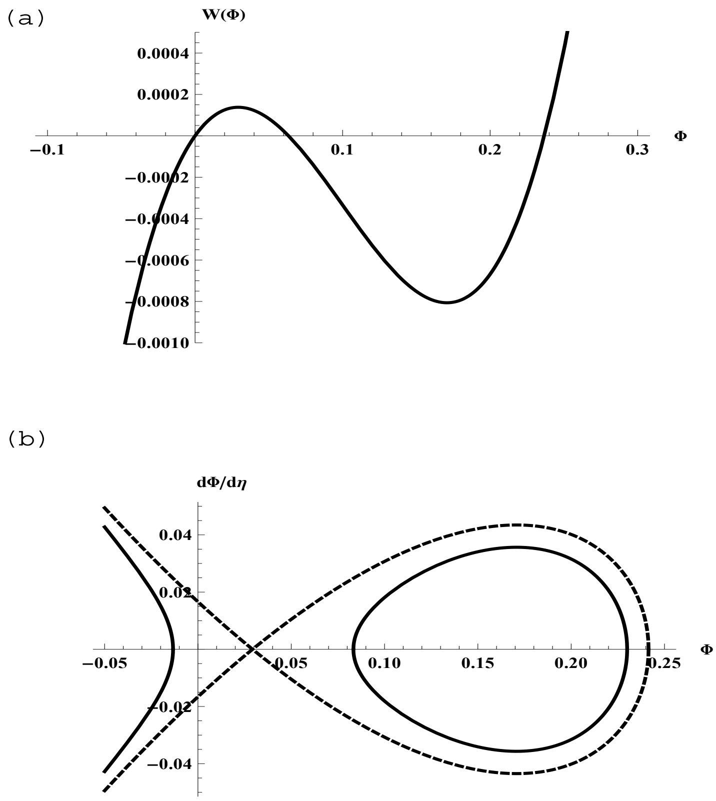

FIG. 1: (a) The nonlinear potential $W(\Phi)$ is plotted for $\rho_0 = 0.01$, $u = 0.2$ and $H_e = 0.05$. (b) The phase plane plots of the compressive ion-acoustic cnoidal wave (solid bounded curve) and soliton (dotted curve) are shown for the same numerical values as in Fig. 1a. Dimensionless variables are used.

wavelength (frequency) of the compressive ion-acoustic cnoidal wave case with the increase in the quantum parameter of the degenerate electrons. The little increase in the width of the compressive ion-acoustic solitons with the increase in the value of quantum parameter is shown in the Fig. 4b. Similarly, the plots for the rarefactive ion-acoustic cnoidal wave and soliton structures for $H_e > 2$ case are shown in the Figs. 5a and 5b respectively. It can be seen from the Fig. 5a that wavelength (frequency) of the rarefactive ion-acoustic cnoidal wave increases and its amplitude decreases significantly with the little increase in the value of quantum parameter for $H_e > 2$ case. Also, the decrease in the wave

Fig.2

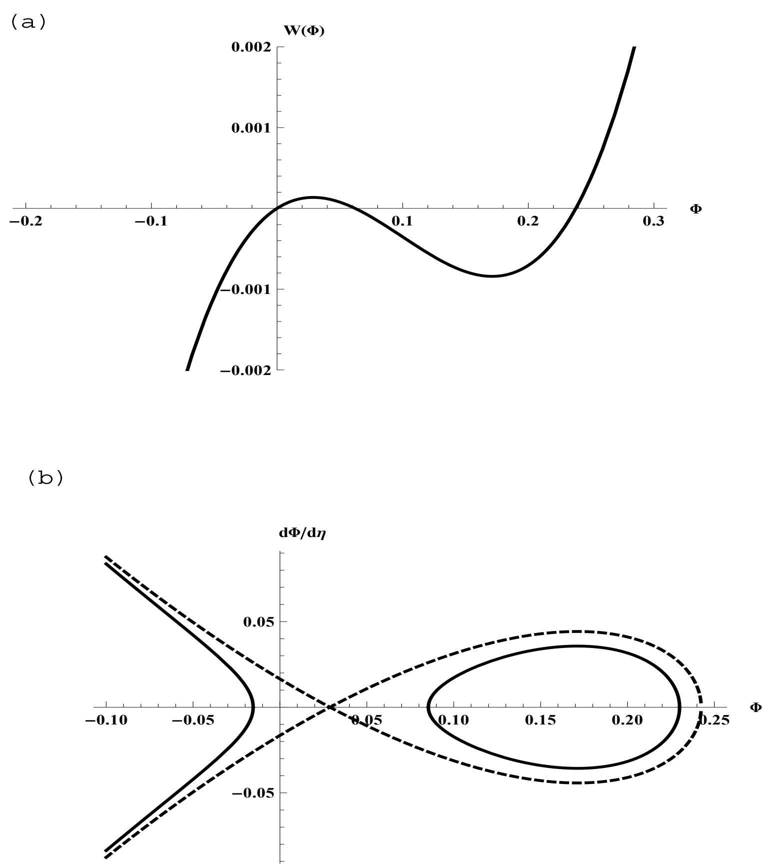

FIG. 2: (a) The nonlinear potential $W(\Phi)$ is plotted for $\rho_0 = 0.01$, $u = 0.2$ and $H_e = 0.24$ (b) The phase plane plots of the compressive ion-acoustic cnoidal wave (solid bounded curve) and soliton (dotted curve) are shown for the same numerical values as in Fig. 2a. Dimensionless variables are used.

amplitude as well as increase in the width of the rarefactive ion-acoustic soliton with the increase in the value of quantum parameter is shown in Fig. 5b.

The variations of the wavelength and frequency with wave amplitude of the compressive ion-acoustic cnoidal waves at different quantum parameters are shown in Fig. **6a** and **6b**, respectively. It can be seen from the figures that the wavelength increases, while the frequency decreases with the increase in the amplitude. Also, the decrease in the wavelength and increase in the wave frequency is found with the increase in the electron quantum parameter for compressive ($H_e < 2$) ion-acoustic cnoidal waves case. However, this decrease in the wavelength and increase in the wave frequency with the wave amplitude for the ion-

Fig.3

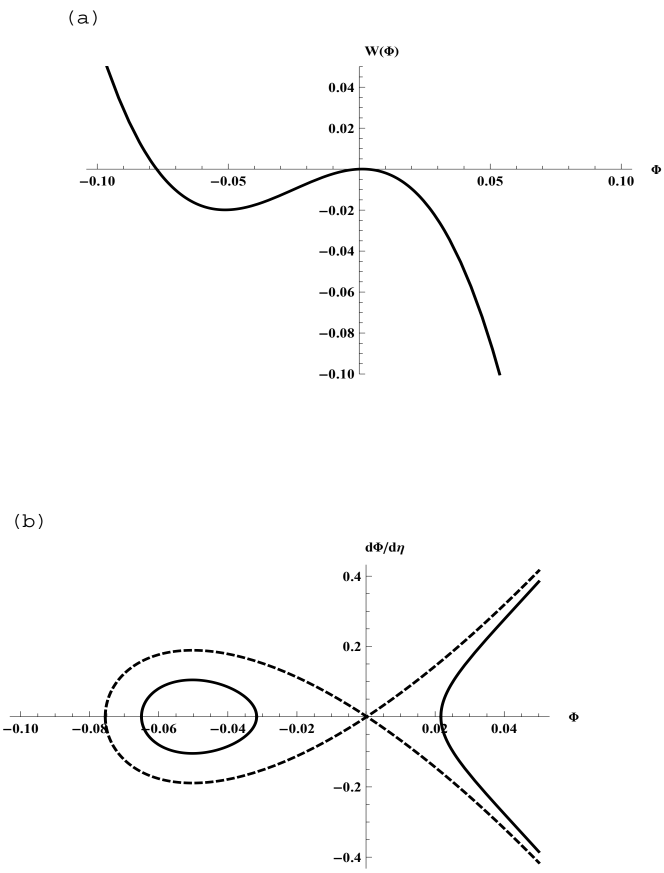

FIG. 3: (a) The nonlinear potential $W(\Phi)$ is plotted for $\rho_0 = 0.01$, $u = -0.05$ and $H_e = 2.002$ (b) The phase plane plots of the rarefactive ion-acoustic cnoidal wave (solid bounded curve) and soliton (dotted curve) are shown for the same numerical values as in Fig. 3a. Dimensionless variables are used.

acoustic compressive cnoidal waves case seems to be very small at the chosen degenerate plasma densities as indicated in Figs. **6a** and **6b**. The dependence of the wavelength and frequency on the wave amplitude in case of rarefactive ion-acoustic cnoidal waves case i.e., at quantum parameters with values $H_e > 2$ are shown in Figs. **7a** and **7b**, respectively. The wavelength increases while the frequency decreases with the increase in the amplitude of the rarefactive ion-acoustic cnoidal waves in quantum plasmas. On the other hand, the wavelength is found to be increased, while frequency decreases for rarefactive ion-acoustic cnoidal waves case with the increase in the value of the quantum parameter for degenerate electrons as shown in the Figs. **7a** and **7b**.

Fig.4

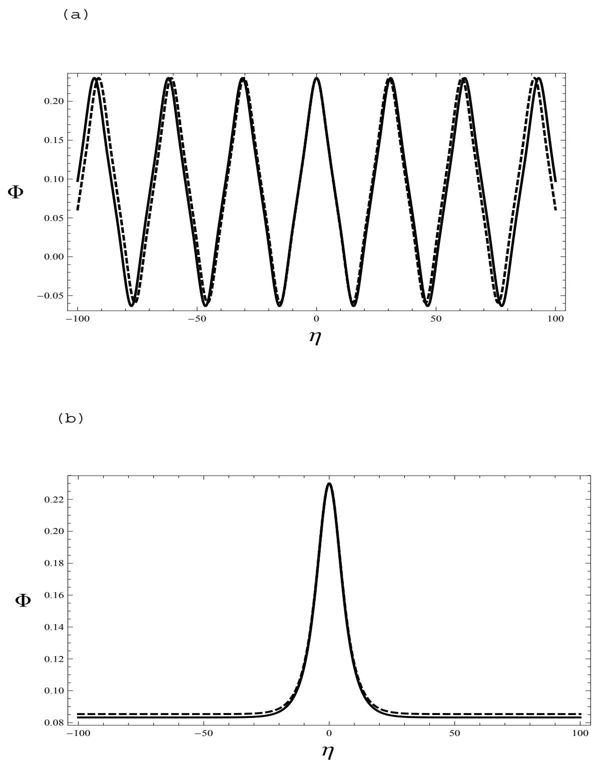

FIG. 4: **(a) The plots of compressive ion-acoustic cnoidal waves (periodic wave oscillations) from Eq. (40) and (b) solitons (single pulse) from Eq. (48) are shown for the $H_e < 2$ case i.e., with $H_e = 0.05$ (solid) and $H_e = 0.24$ (dotted) respectively. Dimensionless variables are used.**

V. CONCLUSION

To conclude, we have studied for the first time the ion-acoustic cnoidal waves and solitons in an unmagnetized quantum plasma. The KdV equation for ion-acoustic waves in a quantum plasma was obtained using the reductive perturbation method with periodic wave boundary conditions, appropriate to study cnoidal waves. It is found that both compressive and rarefactive nonlinear ion-acoustic cnoidal wave structures are formed in such a degenerate plasma, which depends on the quantum parameter i.e., $H_e \gtrless 2$. The dependence of wave frequency and wavelengths on the nonlinear ion-acoustic wave amplitude is also investigated

Fig.5

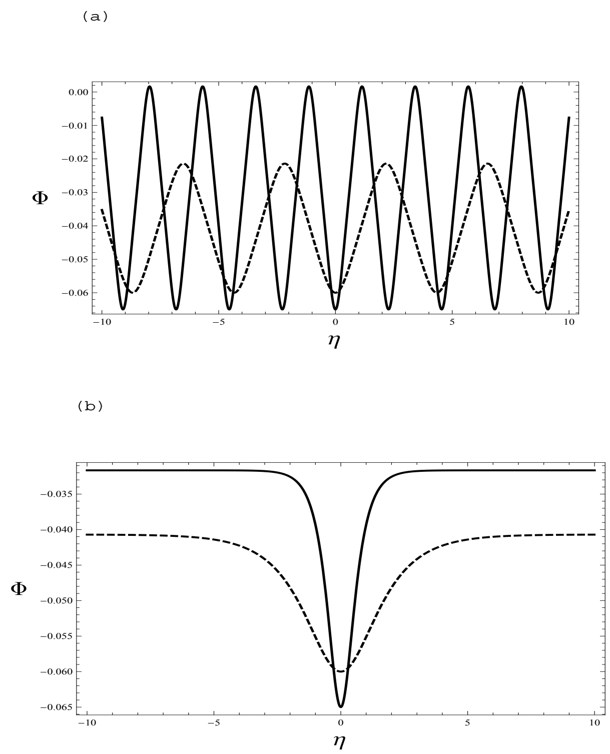

FIG. 5: **(a) The plots of compressive ion-acoustic cnoidal waves (periodic wave oscillations) from Eq. (40) and (b) solitons (single pulse) from Eq. (48) are shown for the** $H_e > 2$ **case i.e., with** $H_e = 2.002$ **(solid) and** $H_e = 2.01$ **(dotted) respectively. Dimensionless variables are used.**

at different values of quantum parameters with the degenerate plasma densities exist in astrophysical and laboratory plasmas. It is found that the dependencies of wavelength and frequency on wave amplitude at different quantum parameters for electrons behave differently for compressive and rarefactive ion-acoustic cnoidal wave cases. The results are useful to understand how nonlinear wave propagates in quantum plasmas.

Fig.6

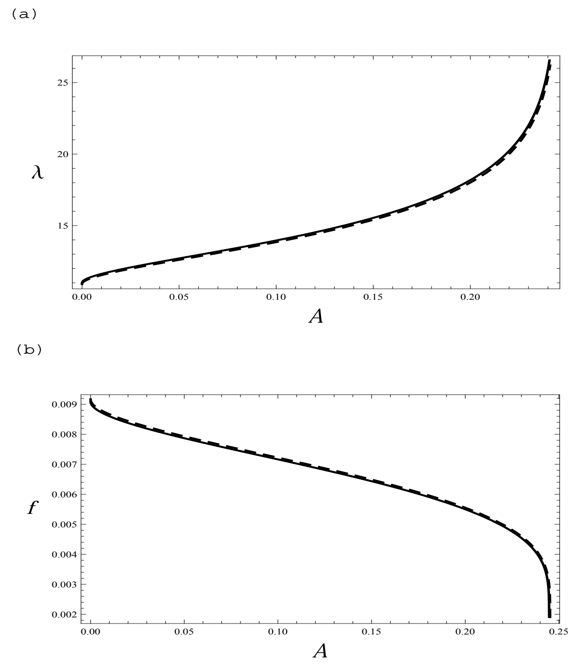

FIG. 6: (a) The dependence of wavelength λ on compressive ion-acoustic cnoidal wave amplitude A is shown for degenerate electron quantum parameter $H_e = 0.05$ (solid curve) and $H_e = 0.24$ (dotted curve) with $v_1 = 0.1$. (b) The dependence of frequency f on compressive ion-acoustic cnoidal wave amplitude A with same parameters as in Fig. 4a. Dimensionless variables are used.

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Fig.7

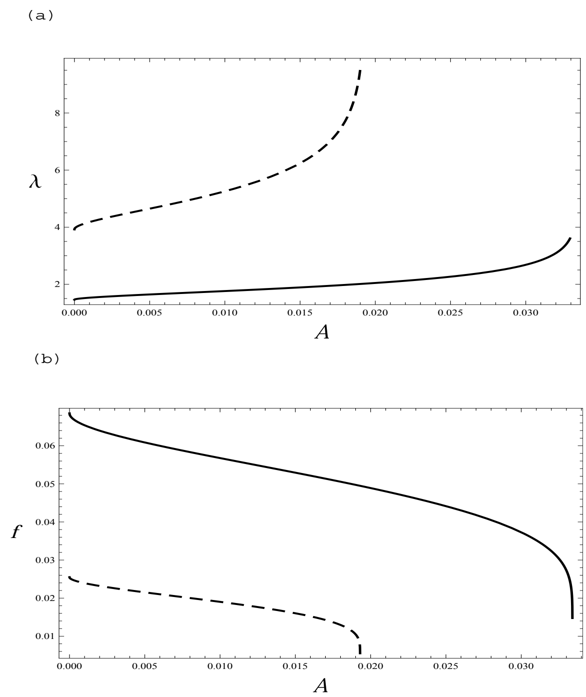

FIG. 7: (a) The dependence of wavelength λ on rarefactive ion-acoustic cnoidal wave amplitude A is shown for degenerate electron quantum parameter $H_e = 2.002$ (solid curve) and $H_e = 2.01$ (dotted curve). (b) The dependence of frequency f on rarefactive ion-acoustic cnoidal wave amplitude A with the same parameters as in Fig. 6a. Dimensionless variables are used.

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