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ESTIMATION OF THE SAVING RATE FUNCTION IN THE SPATIAL SOLOW-SWAN MODEL BY PINNs

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Introduction

The spatial Solow-Swan model [1] is central to analyzing the impact of location on economic growth. The deep learning technique known as Physics-Informed Neural Networks (PINNs) has been widely used to solve Partial Differential Equations (PDEs). This technique has proven to be highly effective in addressing inverse problems, even with limited data sets [2]. Previous studies have addressed inverse problems related to the production function of the model [3], but no one has investigated its saving rate function. In this work, we aim to estimate the saving rate function by solving the inverse problem in the spatial Solow-Swan model using PINNs.

Objectives

The main goal of this work is to develop a PINNs based method to solve the inverse problem of estimating the spatially dependent saving rate function, assuming the spatial Solow-Swan model.

Methodology

Setup of the Inverse Problem

- **Data:** Observations of the capital stock (K)

$$\mathcal{D} = \{(t_i, x_i; \hat{K}_i)\}_{i=1}^{n_d} \quad \text{with} \quad \hat{K}_i = K(x_i, t_i) \tag{1}$$

- **Spatial Solow-Swan model**

$$K_t = s(x) K^{\phi} L^{1-\phi}(x) - \delta K + d K_{xx},\ x \in (0, l),\ t > 0 \tag{2a}$$

$$K(0, x) = K_0(x),\ x \in \Omega = [0, l],\ t = 0 \tag{2b}$$

$$K_x = 0,\ x \in \partial\Omega = \{0, l\},\ t > 0 \tag{2c}$$

- $s(x) \in (0, 1)$ is the unknown saving rate function, and given $\delta \in (0, 1)$ the capital depreciation rate, and $d > 0$ the capital diffusion coefficient

PINNs Method

1. ANNs architectures
   - Neural network (MLP): $\tilde{s}(x) = N_s(x; \theta_s)$
   - Neural network (MLP): $\tilde{K}(t, x) = N_K(t, x; \theta_K)$
2. Loss function

$$\mathcal{L}(\theta_K, \theta_s) = \omega_{\text{PDE}} \mathcal{L}_{\text{PDE}} + \omega_{\text{IC}} \mathcal{L}_{\text{IC}} + \omega_{\text{BC}} \mathcal{L}_{\text{BC}} + \omega_{\text{data}} \mathcal{L}_{\text{data}}$$

   - PDE loss

$$\mathcal{L}_{PDE} = \frac{1}{n_{\text{PDE}}} \sum_{i=1}^{n_{\text{PDE}}} |\mathcal{R}(t_i, x_i)|^2$$

   - PDE residual

$$\mathcal{R}(t, x) = \partial_t \tilde{K}(t, x) - \text{s(x)} \tilde{K}^{\phi}(t, x) L^{1-\phi}(x) + \delta \tilde{K}(t, x) - d\, \partial_{xx} \tilde{K}(t, x)$$

3. Training procedure

$$\min_{\theta_K, \theta_s} \mathcal{L}(\tilde{K}, \tilde{s})$$

   - Both networks ($\mathcal{N}_K$ and $\mathcal{N}_s$) trained simultaneously using the Adam optimizer
   - Training points (t_i, x_i) uniformly random samples in the domain

Results

- **Case study:** FEM numerical manufactured solution
  - Parameters: $l = 10$, $T = 10$, $\phi = 1/3$, $\delta = 0.05$, $d = 0.25$
  - Functions: $L(x) = 1 + 0.3x^2 [1 - \cos(4\pi x/l)]$, $K_0(x) = \cos^4(\pi x/(2l) - \pi/2)$, $s(x) = 0.2 \cos^4(\pi x/(2l))$

- **MLPs architecture**

We selected the capital network architecture by testing different numbers of neurons (n_n) and training samples (n_s) with 2 layers, choosing the configuration that best approximates $K(x, t)$ when solving the direct problem with PINNs. Similarly, for the saving rate function, we explored architectures to find the one that best approximates $s(x)$.

$n_n \backslash n_s$	25^2	50^2	75^2	100^2
50	26.473	21.085	17.068	19.127
60	18.447	16.031	15.925	17.718
70	19.788	14.888	17.838	18.745

Table 1: number of epochs for different network sizes (n_n, n_s), until reaching the stopping criterion with loss tolerance 10^{-5}.

- **Solution of $K(x, t)$ and $s(x)$ by PINNs**
  - MLP Architecture: $\mathcal{N}_K : 2{-}70 \times 2{-}1$, $\mathcal{N}_s : 1{-}40 \times 1{-}1$
  - Learning rate: $lr = 10^{-3}$
  - Activation functions: $N_K(t, x; \theta_K)$: Tanh, Softplus, $N_s(x; \theta_s)$: Tanh, Sigmoid

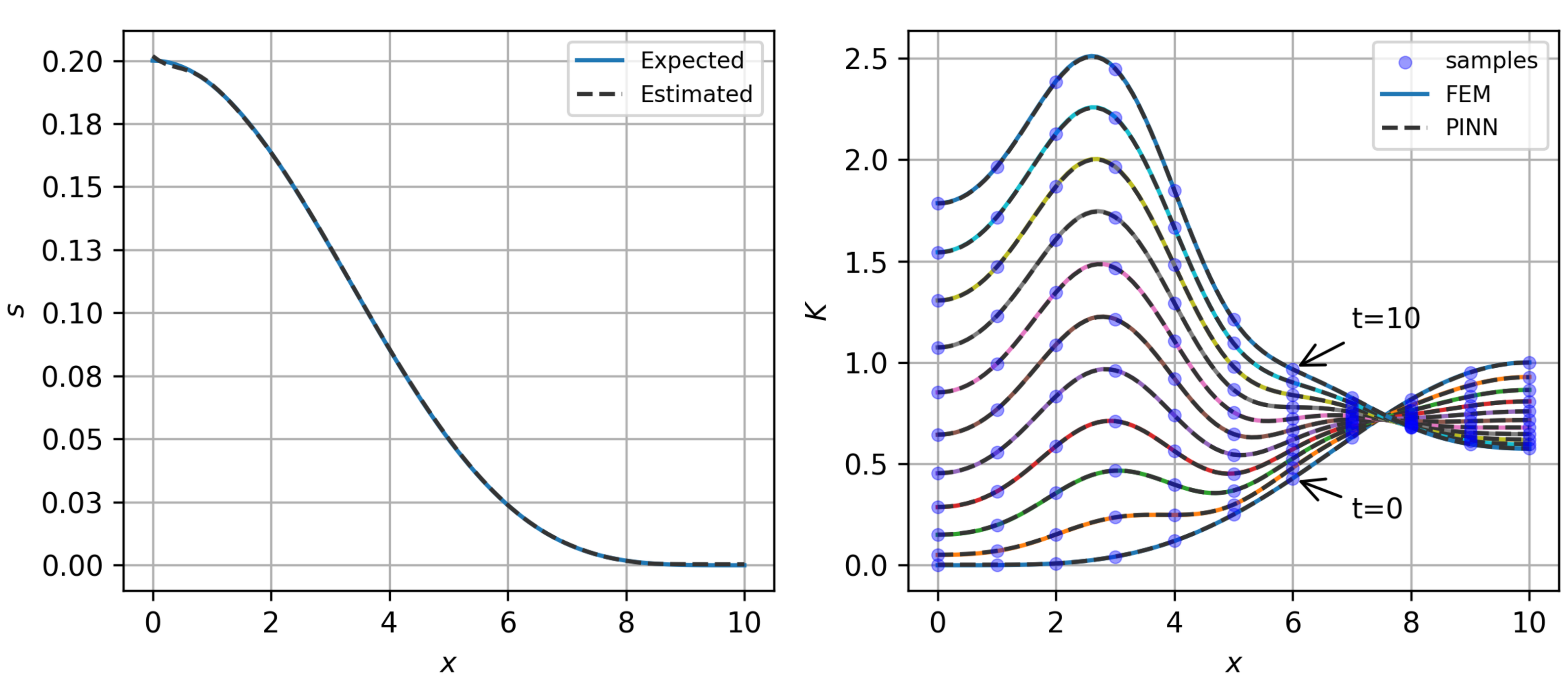

Figure 1: PINN-estimated saving rate $\tilde{s}(x)$ and capital distribution $\tilde{K}(x, t)$ versus finite element solutions.

Conclusion

The results sugest that PINNs can effectively estimate the saving rate function in the spatial Solow-Swan model. The estimated function closely matches the expected solution. Future work will perform a stability analysis with respect to noise in the data, extend this approach to two-dimensional scenarios, and apply it to real data available from IBGE.

References

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