Mini-course at IMPA on area, curvature and radius estimates for constant mean curvature surfaces

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Lectures based on references appearing in last slide. Slides of talks appear on web page `http://professor.ufrgs.br/alvaro/pages/30-s̄cbm`.

- Lecture 1: Background material, statements of the main results.
- Lecture 2: Area estimates for embedded 3-periodic H-surfaces.
- Lecture 3: H-surfaces in homogeneous 3-manifolds.
- **Lecture 4: Reduction of Hopf uniqueness theorem to area estimates.**
- Lecture 5: Area estimates for H-spheres in Hopf uniqueness theorem.

The Representation Theorem

Suppose Σ is a simply connected Riemann surface with conformal parameter z, X is a metric Lie group, and $H \in \mathbb{R}$.
Let $g\colon \Sigma \to \overline{\mathbb{C}}$ be a solution of the complex elliptic PDE

$$g_{z\overline{z}} = \frac{R_q}{R}(g)\, g_z g_{\overline{z}} + \left(\frac{R_{\overline{q}}}{R} - \frac{\overline{R_q}}{\overline{R}} \right)(g)\, |g_z|^2, \tag{1}$$

such that $g_z \neq 0$ everywhere, and such that the H-potential R of X does not vanish on $g(\Sigma)$ (for instance, this happens if X admits a compact H-surface). Then, there exists an immersed H-surface $f\colon \Sigma \to X$, unique up to left translations, whose Gauss map is g.
Conversely, if $g\colon \Sigma \to \overline{\mathbb{C}}$ is the Gauss map of an immersed H-surface $f\colon \Sigma \to X$ in a metric Lie group X, and the H-potential R of X does not vanish on $g(\Sigma)$, then g satisfies the equation (1), and moreover $g_z \neq 0$ holds everywhere.

Existence and Uniqueness Theorem (Meeks-Mira-Pérez-Ros, 2017).

Let M be a homogeneous 3-manifold. Then, any two **spheres** in M of the same absolute constant mean curvature differ, as sets, by an isometry of M. Moreover, if X is the universal covering space of M, it holds:

- If X is not diffeomorphic to $\mathbb{R}^3$, then, for every $H \in \mathbb{R}$, there exists a sphere of constant mean curvature H in M.
- If X is diffeomorphic to $\mathbb{R}^3$, then the values $H \in \mathbb{R}$ for which there exists a sphere of constant mean curvature H in M are exactly those with $|H| > \mathrm{Ch}(X)/2 = H(X)$.

Properties Theorem (Meeks-Mira-Pérez-Ros, 2017).

Let S be an H-sphere in M and let $\widetilde{S}$ to be a lift of S to X. Then:

- If X is a product $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $H = 0$, then
  - S is totally geodesic, stable and has nullity 1 for its Jacobi operator.
  - S represents a nontrivial element in the second homology group of M.
- Otherwise, S has index 1 and nullity 3 for its Jacobi operator and the immersion of S into M extends as the boundary of an isometric immersion $F: B \to M$ of a Riemannian 3-ball B which is mean convex.
- There is a point $p_S \in M$ such that every isometry of M that fixes p_S also leaves invariant S.

M a homogeneous 3-manifold and X is its universal covering.

There are two cases to consider:

Case 1. X is isometric to a metric Lie group.

Case 2. X is isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$.

Here, we will treat uniquely Case 1, and remark that Case 2 was proved by R. Souam. Also, Case 2 follows from Case 1 using the *sister surfaces correspondence* introduced by B. Daniel (2007).

Main steps of the proof assuming X is a metric Lie group.

- The left invariant Gauss map of any index-1 H-sphere is a orientation preserving diffeomorphism. (Step 4)
- Let $H \in \mathbb{R}$ be given and let S_1, S_2 be two H-spheres such that the Gauss map of S_1 is a diffeomorphism. Then, S_2 is a left translation of S_1. (Step 5)
- Every H-sphere in X has index 1. (Step 9)
  (This proves that any two spheres in X with the same constant mean curvature differ by a left translation.)

To prove the theorem in M:

We need the definition of a special point for H-spheres in X, called the **center of symmetry** or **center of mass**.

Let X be a simply connected metric Lie group of dimension 3 and let $\mathcal{M}(X)$ denote the space, *up to left translations* of index 1 immersed H-spheres in X.

Step 1.

There exists some $\Sigma \in \mathcal{M}(X)$ that is embedded. Furthermore, Σ is the boundary of a mean convex ball B_Σ in X, and the constant mean curvature of Σ can be chosen arbitrarily large.
Proof: Take a solution to the isoperimetric problem for a small volume.

Step 2.

Such a Σ has nullity three.

- Let F a nonzero right-invariant vector field of X. Then, F is not everywhere tangent to Σ.
- If η is an unitary vector field orienting Σ, $\langle F, \eta \rangle$ is a nonzero Jacobi function of Σ.
- Hence, the linear map from the 3-dimensional vector space of right-invariant vector fields to the space of Jacobi functions of Σ is injective, hence the nullity of Σ is at least three.
- However, Cheng (1974) proved that the dimension of the kernel of any Schrodinger operator of an index-1 Riemannian 2-sphere cannot be greater than 3, proving Step 2.

Corollary (Step 3)

$\mathcal{M}(X)$ is an analytic 1-manifold locally parameterized by the mean curvature values of its elements.

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Idea of the proof

Use Implicit Function Theorem and Step 2.

Step 4.

The left invariant Gauss map of any index-1 H-sphere is an orientation preserving diffeomorphism.

Proof.

Let S_H be an index-1 H-sphere in X, which we view as a conformal immersion $f\colon \overline{\mathbb{C}} \to X$. Let $g\colon S_H \equiv \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be its stereographically projected left invariant Gauss map.

Claim

g is a diffeomorphism.

Proof that $g\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a diffeomorphism.

Since $\overline{\mathbb{C}}$ is simply connected, it suffices to show it is a local diffeomorphism.

By contradiction, assume there is a point $z_0 \in S_H$ such that dg_{z_0} is not surjective. The Representation Theorem gives that $g_z \neq 0$ everywhere (and that the H-potential for X does not vanish), hence there is a conformal parameter $z = x + iy$ near z_0 so that $g_x(z_0) = 0$ and $g_y(z_0) \neq 0$.

Consider the second order ODE given by specializing the Representation PDE to functions that depend uniquely on the real variable y, i.e.,

$$\widehat{g}_{yy} = \frac{R_q}{R}(\widehat{g})(\widehat{g}_y)^2 + \left(\frac{R_{\overline{q}}}{R} - \frac{\overline{R_q}}{\overline{R}}\right)(\widehat{g})|\widehat{g}_y|^2,$$

and let $\widehat{g} = \widehat{g}(y)$ be its (unique) solution with initial conditions $\widehat{g}(0) = g(0)$, $\widehat{g}_y(0) = g_y(0)$.

Proof that $g\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a diffeomorphism.

From the Representation Theorem, there exists a conformally immersed H-surface $\widehat{f}$ in X with left invariant Gauss map $\widehat{g}$, and such that $f(z_0) = \widehat{f}(z_0)$.

Since g and $\widehat{g}$ satisfy the same initial data, it follows that the immersed surfaces f, $\widehat{f}$ have at least a second order contact. However, $\widehat{f}$ is invariant under the flow of a right invariant vector field F of X. Defining the function $u = \langle \eta, F \rangle$, it follows that u is not identically zero.

The contradiction:

- Since u is a Jacobi function, $u(0) = 0$ and $u_z(0) = 0$, the nodal set $u^{-1}(0)$ of u around 0 consists of $n \geq 2$ analytic arcs that intersect transversely at 0.
- On the other hand, since S_H has index 1, again Cheng (1974) proves that $u^{-1}(0)$ is an analytic simple closed curve in S_H.

Proof that $g \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ preserves the orientation.

The Representation Theorem gives that $g_z \neq 0$ everywhere, hence g is nowhere antiholomorphic and hence cannot reverse the orientation around any point.

Step 5.

Let $H \in \mathbb{R}$ be given and let S_1, S_2 be two H-spheres such that the Gauss map g_1 of S_1 is a diffeomorphism. Then, S_2 is a left translation of S_1.

Idea of the proof.

- Since g_1 is a diffeomorphism, a function $L\colon \overline{\mathbb{C}} \to \mathbb{C}$ be defined implicitly in terms of g_1 and of the H-potential R of X (which does not vanish) by

$$L(g_1(z)) = -\frac{\overline{g_1}_z}{R(g_1(z)){g_1}_z}.$$

- Given a conformally immersed H-surface $f\colon \Sigma \to X$ with left invariant Gauss map g, define the complex quadratic differential

$$Q_H(dz)^2 = \left(L(g) g_z^2 + \frac{1}{R(g)} g_z \overline{g}_z \right) (dz)^2.$$

Step 5.

Let $H \in \mathbb{R}$ be given and let S_1, S_2 be two H-spheres such that the Gauss map g_1 of S_1 is a diffeomorphism. Then, S_2 is a left translation of S_1.

Idea of the proof (cont.)

Q_H (defined in terms of the Gauss map of S_1) satisfies:

- If Q_H does not vanish identically for f, then Q_H has only isolated zeros of negative index.
- $Q_H \equiv 0$ for f if and only if f is, up to left translation, an open piece of S_1.

Corollary.

Each element of $\mathcal{M}(X)$ is uniquely defined by its mean curvature.

Remarks.

- At this point of the proof, we still need to show that **any** H-sphere in X has index 1.
- It suffices to show that for every *possible* H there is an index 1 H-sphere in X.
- Let $H(X) = \inf\{\|H\|_\infty(\Sigma) \mid \Sigma \text{ is a compact surface immersed in } X\}$.
- $H(X) = 2\mathrm{Ch}(X)$ and there are no H-surfaces in X with $|H| < H(X)$ and equality can only happen when $X = \mathrm{SU}(2)$ and $H = 0$.

Existence Claim

- If X is diffeomorphic to $\mathbb{R}^3$ then for any $|H| > H(X)$ there exists an H-sphere in X of index 1.
- If $X = \mathrm{SU}(2)$, for any $H \in \mathbb{R}$ there exists an H-sphere in X with index 1.

Notation.

Let $\mathcal{C}$ be a connected component of $\mathcal{M}(X)$. Then $\mathcal{C}$ is parametrized by an open (possibly unbounded) interval $I_{\mathcal{C}} \subset \mathbb{R}$, defined by the mean curvature of its elements.

Proof of Existence Claim

- For $|H|$ large enough, there exists a sphere as claimed by Step 1.
- Let $\mathcal{C}$ be a connected component of $\mathcal{M}(X)$ containing a positive number. The next arguments are to show that
  - $I_{\mathcal{C}} = (H(X), +\infty)$, if X is diffeomorphic to $\mathbb{R}^3$;
  - $I_{\mathcal{C}} = \mathbb{R}$, if $X = \mathrm{SU}(2)$.

Step 6 (Curvature estimates).

Given any $H_1 > 0$ there exists a constant $C = C(H_1)$ such that for every $H \in \mathbb{R}$ satisfying $|H| \leq H_1$ and for every index-1 H-sphere Σ, the norm of the second fundamental form of Σ is uniformly bounded by C.

Step 7 (Area Estimates).

- If X is isomorphic to $\mathrm{SU}(2)$, then the areas of spheres in $\mathcal{M}(X)$ are uniformly bounded.
- If X is not isomorphic to $\mathrm{SU}(2)$, then for any $\delta > 0$ the areas of H-spheres in $\mathcal{M}(X)$ with $|H| \in [H(X) + \delta, \infty)$ are uniformly bounded.

Consequence

- If X is isomorphic to $\mathrm{SU}(2)$, then $I_{\mathcal{C}}$ is closed (then $I_{\mathcal{C}} = \mathbb{R}$);
- Otherwise, $I_{\mathcal{C}}$ is closed in $[H(X) + \delta, \infty)$ for any $\delta > 0$, hence $I_{\mathcal{C}} \supset (H(X), \infty)$, then $I_{\mathcal{C}} = (H(X), \infty)$.

Proof of Step 6 (Curvature Estimates).

By contradiction, we fix $H_1 > 0$ and assume that there exists a sequence of index-1 spheres $\{\Sigma_n\}_{n\in\mathbb{N}}$, each Σ_n having constant mean curvature $H_n \in [-H_1, H_1]$ and admitting points $p_n \in \Sigma_n$ such that the norm of the second fundamental form of Σ_n at p_n is $\lambda_n = \|A_{\Sigma_n}\|(p_n) \geq n$.

- Assume that $\lambda_n = \sup_{\Sigma_n} \|A_{\Sigma_n}\|$ and, after a left translation, that $e = p_n$, $\forall n \in \mathbb{N}$.
- Let $X_n = \lambda_n X$ be the homogeneous manifold obtained by the rescaling of the metric of X by λ_n.
- Let $S_n = \lambda_n \Sigma_n$, then S_n is an immersed (H_n/λ_n)-sphere in X_n of index 1.
- The norm of the second fundamental form of each S_n is such that

$$\|A_{S_n}\| \leq 1, \quad \|A_{S_n}\|(e) = 1.$$

Proof of Step 6 (Curvature Estimates).

- When $n \to \infty$, X_n converges to the Euclidean space $\mathbb{R}^3$ with its flat metric.
- A subsequence of $\{S_n\}_{n\in\mathbb{N}}$ converges to a minimal surface S in $\mathbb{R}^3$, passing through the origin $\vec{0}$, with second fundamental form satisfying $\|A_S\| \leq 1$, $\|A_S\|(\vec{0}) = 1$.
- The group structure of X_n also converges to the abelian group structure of $\mathbb{R}^3$, hence the respective left invariant Gauss maps $G_n\colon S_n \to \mathbb{S}^2$ converge to the Gauss map $G\colon S \to \mathbb{S}^2$.
- S has strictly negative Gaussian curvature at $\vec{0}$, hence, G is orientation reversing in a neighborhood of S around $\vec{0}$. However, for all $n \in \mathbb{N}$, G_n is an orientation preserving diffeomorphism, which gives a contradiction.

Up to Area estimates, the description Theorem holds when M is isometric to a simply connected metric Lie group.

Lemma (center of symmetry)

Given an H-sphere Σ in X, there exists a uniquely defined point $p_\Sigma \in X$ such that any isometry of X that fixes p_Σ must also leave invariant Σ. We call this point p_Σ the **center of symmetry** of Σ. Furthermore, if two spheres of the same constant mean curvature have the same center of symmetry, then they coincide.

Apply the center of symmetry to prove the Theorem in M

- Suppose M is a homogenous 3-manifold with universal covering $\pi\colon X \to M$, where X is isometric to a metric Lie group.
- Let Σ_1, Σ_2 be any two immersed spheres in M with the same absolute constant mean curvature $H \geq 0$.
- Let $\widetilde{\Sigma}_1$, $\widetilde{\Sigma}_2$ be the respective lifts of Σ_1, Σ_2 to X with respective centers of symmetry p_1, p_2.
- Take q_1, q_2, the projections of p_1, p_2 to M and let $\varphi\colon M \to M$ be an isometry of M which maps q_1 to q_2.
- Let $\widetilde{\varphi}\colon X \to X$ the isometry of X such that $\pi \circ \widetilde{\varphi} = \varphi \circ \pi$, with $\widetilde{\varphi}(p_1) = p_2$.
- $\widetilde{\varphi}(\widetilde{\Sigma}_1)$ and $\widetilde{\Sigma}_2$ are two immersed spheres with the same constant mean curvature and the same center of mass in X, thus, they coincide.
- In particular, $\pi(\widetilde{\varphi}(\widetilde{\Sigma}_1)) = \Sigma_2$, which implies that $\varphi(\Sigma_1) = \Sigma_2$.