Mini-course at IMPA on area, curvature and radius estimates for constant mean curvature surfaces

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Lectures based on references appearing in last slide. Slides of talks appear on web page `http://professor.ufrgs.br/alvaro/pages/30-s̄cbm`.

- Lecture 1: Background material, statements of the main results.
- Lecture 2: Area estimates for embedded 3-periodic H-surfaces.
- **Lecture 3: H-surfaces in homogeneous 3-manifolds.**
- Lecture 4: Reduction of Hopf uniqueness theorem to area estimates.
- Lecture 5: Area estimates for H-spheres in Hopf uniqueness theorem.

Homogeneous manifolds

Definition

A Riemannian manifold M is called *homogeneous* if for any given points p, $q \in M$ there exists an isometry $\varphi\colon M \to M$ such that $\varphi(p) = q$.

Examples

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$\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$

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Metric Lie groups

Metric Lie groups

Definition

- A *Lie group* Z is a smooth manifold with a group structure, whose group operation $*\colon Z \times Z \to Z$ satisfies that $(x, y) \mapsto x^{-1} * y$ is a smooth map.
- Let $y \in Z$. The respective *left* and *right* translations by y are the maps defined by

$$\begin{array}{rcl} l_y\colon Z & \to & Z \\ x & \mapsto & y * x, \end{array} \qquad \begin{array}{rcl} r_y\colon Z & \to & Z \\ x & \mapsto & x * y. \end{array}$$

- A Riemannian metric on a Lie group Z is called *left invariant* if the translations l_x are isometries for all $x \in Z$. A Lie group equipped with a left invariant metric is called a metric Lie group.

Every metric Lie group is a homogeneous manifold:

Proof: If X is a metric Lie group and $x, y \in X$, then $l_{yx^{-1}}$ is an isometry of X which maps x to y. □

Examples

The euclidean space.

$(\mathbb{R}^3, +, ds^2_{\text{flat}})$

The hyperbolic space.

$\mathbb{H}^3$ can be seen as the group of similarities of $\mathbb{R}^2$, i.e., as a set $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\} = \mathbb{R}^3_+$ and it acts on $\mathbb{R}^2$ via

$$\phi_{(x,y,z)}: \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (p_1, p_2) & \mapsto & z(p_1, p_2) + (x, y) \end{array}$$

The group structure is that of composition of maps and is given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + z_1 x_2, y_1 + z_1 y_2, z_1 z_2)$$

Examples

The three sphere.

If identify $\mathbb{R}^4$ with the quaternion group $(x, y, z, w) \equiv x + iy + jz + kw$ and endow it with the quaterion operation defined by

$$ij = k, \quad jk = i, \quad ki = j, \quad i^2 = j^2 = k^2 = -1,$$

then one can identify the unit sphere $\mathbb{S}^3 = \{(x,\ y,\ z,\ w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$ as the subgroup of quaternions of length 1, also denoted $\mathrm{SU}(2)$.

Remark.

The underlying group $\mathrm{SU}(2)$ admits a three-parameter family of non-isometric left invariant metrics. When the dimension of the isometry group of $\mathrm{SU}(2)$ with one of such metrics g has dimension 4, we say that $(\mathrm{SU}(2), g)$ is a *Berger sphere*. When the isometry group of $(\mathrm{SU}(2), \widetilde{g})$ has dimension 6, we obtain $\mathbb{S}^3(\kappa)$, the simply connected homogeneous manifold with constant sectional curvature $\kappa > 0$.

More properties on Lie groups

- A vector field E on a Lie group Z is called *left invariant* if it is invariant under left translations, i.e., if $d(l_x)_y(E(y)) = E(l_x(y))$ for all $x,\ y \in Z$.
- If $E_1,\ E_2$ are two left invariant vector fields on Z, then their Lie bracket $[E_1,\ E_2]$ is also left invariant.
- The set of all left invariant vector fields on Z is called the *Lie algebra* of Z.
- A left invariant vector field is uniquely determined by its value at some point. Hence, there is a canonical identification between the Lie algebra of Z and its tangent space at the identity element.
- A vector field F on a Lie group Z is *right invariant* if it is invariant under right translations.
- The flow of a right invariant vector field F on Z is by left translations. In particular, if $X = (Z, \langle\ ,\ \rangle)$ is a metric Lie group, any right invariant vector field is a Killing field.

Simply connected homogeneous 3-manifolds and metric Lie groups

Remark:

- Since $\mathbb{S}^2$ does not admit a everywhere nonzero tangent vector field, there is no Lie group diffeomorphic to $\mathbb{S}^2$. In particular, $\mathbb{S}^2(\kappa)$ is not isometric to any metric Lie group;
- Moreover, since the second homotopy group of any Lie group is trivial, $\mathbb{S}^2(\kappa) \times \mathbb{R}$ is also not isometric to a metric Lie group.

Theorem.

A simply connected homogeneous 3-manifold Y is isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$, where $\mathbb{S}^2(\kappa)$ is a sphere of constant curvature $\kappa > 0$, or Y is isometric to a **metric Lie group**.

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Sketch of the Proof.

Let $I(Y)$ denote the isometry group of Y and let $p \in Y$ be any point. Let $I_p(Y) = \{\varphi \in I(Y) \mid \varphi(p) = p\}$. Then:

- $I_p(Y)$ is isomorphic to a subgroup of the orthogonal group $O(3)$;
- There are three possible dimensions for such subgroup: $0, 1$ or 3;
- There are three possible dimensions for $I(Y)$: $3, 4$ or 6.

Sketch of the Proof (cont.)

- If $\dim(I(Y)) = 6$, then Y is isometric to a space of constant sectional curvature, $\mathbb{S}^3(\kappa > 0)$, $\mathbb{R}^3$ or $\mathbb{H}^3(\kappa < 0)$, all of which are metric Lie groups;
- If $\dim(I(Y)) = 4$, then Y is isometric to a Riemannian bundle $\mathbb{E}(\kappa, \tau)$ over a complete, simply connected surface of constant curvature $\kappa \in \mathbb{R}$ and bundle curvature $\tau \in \mathbb{R}$. Each of these spaces has the structure of some metric Lie group except for the case of $\mathbb{E}(\kappa, 0)$, $\kappa > 0$, which is isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$.
- If $\dim(I(Y)) = 3$:
    - Let $I_0(Y)$ be the connected component of $I(Y)$ through the identity and $S = \{\varphi \in I_0(Y) \mid \varphi(p) = p\}$;
    - Since Y is simply connected, then S is trivial and Y is diffeomorphic to $I_0(Y)$;
    - Endowing Y with the Lie group structure of $I_0(Y)$, we have that the point p plays the role of the identity element of Y and the original metric of Y is left invariant. □

Question.

What are the simply connected metric Lie groups of dimension 3?

Definition.

A (connected) Lie group G is called *unimodular* if for every element x in its Lie algebra $\mathfrak{g}$, the endomorphism $\mathrm{ad}_x \colon \mathfrak{g} \to \mathfrak{g}$, $ad_x(y) = [x, y]$ has trace zero.

Remark.

This is equivalent to the property that the left invariant Haar measure on G is also right invariant. (In less fancy words, this means that right translations preserve the volume)

- Unimodular Lie groups;
- Non-unimodular Lie groups.

Unimodular Lie groups

Let G be a 3-dimensional unimodular Lie group. Then, for any $e \in \mathfrak{g}$

$$\mathrm{trace}(X \in \mathfrak{g} \mapsto [e, X] \in \mathfrak{g}) = 0.$$

From this, it is possible to find a basis $\{E_1, E_2, E_3\}$ and constants c_1, c_2, c_3, (among which at most one is negative) and such that

$$[E_2, E_3] = c_1 E_1, \quad [E_3, E_1] = c_2 E_2, \quad [E_1, E_2] = c_3 E_3.$$

Definition

The vectors $\{E_1, E_2, E_3\}$ form the *cannonical basis* of G and the constants $\{c_1, c_2, c_3\}$ are the called *structure constants* of G.

Simply connected unimodular metric Lie groups of dimension three

Signs of c_1, c_2, c_3	$\dim \mathrm{I}(X) = 3$	$\dim \mathrm{I}(X) = 4$	$\dim \mathrm{I}(X) = 6$
$+, +, +$	$\mathrm{SU}(2)$	$\mathbb{S}^3_{\text{Berger}} = \mathbb{E}(\kappa > 0, \tau)$	$\mathbb{S}^3(\kappa)$
$+, +, -$	$\widetilde{\mathrm{SL}}(2, \mathbb{R})$	$\mathbb{E}(\kappa < 0, \tau)$	$\emptyset$
$+, +, 0$	$\widetilde{\mathrm{E}}(2)$	$\emptyset$	$(\widetilde{\mathrm{E}}(2), \text{flat})$
$+, -, 0$	Sol_3	$\emptyset$	$\emptyset$
$+, 0, 0$	$\emptyset$	$\mathrm{Nil}_3 = \mathbb{E}(0, \tau)$	$\emptyset$
$0, 0, 0$	$\emptyset$	$\emptyset$	$\mathbb{R}^3$

Non-unimodular metric Lie groups

Semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$:

The set

$\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$

The group structure

- Let $A \in M_2(\mathbb{R})$ be a fixed 2×2 real matrix;
- For $z \in \mathbb{R}$, let $e^{Az} : \mathbb{R}^2 \to \mathbb{R}^2$ be the exponential map, written as

$$e^{Az} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}.$$

Then, one can define a group operation $*$ on $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$:

$$(\mathbf{p}_1, z_1) * (\mathbf{p}_2, z_2) = (\mathbf{p}_1 + e^{Az_1}\mathbf{p}_2, z_1 + z_2),$$

and define (the group) $\mathbb{R}^2 \rtimes_A \mathbb{R} = (\mathbb{R}^2 \times \mathbb{R}, *)$.

Semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$:

The Metric

- The coordinate vectors ∂_x, ∂_y, ∂_z are orthogonal at $e = (0,0,0)$;
- Extend $\{\partial_x(e), \partial_y(e), \partial_z(e)\}$ to a left invariant orthogonal frame

$$E_1 = a_{11}(z)\partial_x + a_{21}(z)\partial_y, \quad E_2 = a_{12}(z)\partial_x + a_{22}(z)\partial_y, \quad E_3 = \partial_z,$$

  producing a left invariant metric

$$ds^2 = Q_{11}(z)dx^2 + Q_{22}(z)dy^2 + dz^2 + Q_{12}(z)(dx \otimes dy + dy \otimes dx).$$

- $$Q_{11}(z) = e^{-2z\mathrm{trace}(A)}\left[a_{21}(z)^2 + a_{22}(z)^2\right],$$
  $$Q_{22}(z) = e^{-2z\mathrm{trace}(A)}\left[a_{11}(z)^2 + a_{12}(z)^2\right],$$
  $$Q_{12}(z) = -e^{-2z\mathrm{trace}(A)}\left[a_{11}(z)a_{21}(z) + a_{12}(z)a_{22}(z)\right].$$
- We denote $\mathbb{R}^2 \rtimes_A \mathbb{R} = (\mathbb{R}^3, *, ds^2)$ and say that ds^2 is the **cannonical left invariant metric** of $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

Examples

A	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\mathbb{R}^2 \rtimes_A \mathbb{R}$	$\mathbb{R}^3$ (abelian, flat)	$\mathbb{H}^3$ (const curv -1)	$\mathbb{H}^2 \times \mathbb{R}$
A	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\dots$
$\mathbb{R}^2 \rtimes_A \mathbb{R}$	Sol_3 (solvable)	Nil_3 (nilpotent)	$\dots$ (others)

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Theorem.

Every simply connected Lie group of dimension three is isomorphic to $\mathrm{SU}(2)$ (topology of $\mathbb{S}^3$), $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ or to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some $A \in M_2(\mathbb{R})$.

Classification theorem.

Let Y be a simply connected homogeneous manifold of dimension three. Then one of the following happens:

- Y is isometric to the Riemannian product $\mathbb{S}^2(\kappa) \times \mathbb{R}$ for some $\kappa > 0$;
- Y is isometric to a metric Lie group. In this case, the possibilities are:
  - Y is unimodular. Then either
    - Y is isometric to $SU(2)$ endowed with a left invariant metric;
    - Y is isometric to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ endowed with a left invariant metric;
    - Y is isometric to $\mathbb{R}^2 \rtimes_A \mathbb{R}$, with $\mathrm{trace}(A) = 0$.
  - Y is nonunimodular. Then Y is isometric to $\mathbb{R}^2 \rtimes_A \mathbb{R}$, with $\mathrm{trace}(A) \neq 0$. Moreover, after an homothety and an isometry, one may assume that $A = \begin{pmatrix} 1+a & -(1-a)b \\ (1+a)b & 1-a \end{pmatrix}$ for $a,\ b \geq 0$.

The left invariant Gauss map

Definition.

Given an oriented immersed surface $f\colon \Sigma \to X$ with unit normal vector field $N\colon \Sigma \to TX$, we define the *left invariant Gauss map* of the immersed surface to be the map $G\colon \Sigma \to \mathbb{S}^2 \subset T_eX$ given by

$$G(p) = d(l_{p^{-1}})_p(N_p).$$

How to obtain a CMC surface from its Gauss map?

Important case: $\mathbb{R}^3$.

We have the Weierstrass representation: From a Riemann surface Σ, a meromorphic function g on Σ and a holomorphic one-form ϕ on Σ, one can obtain a **minimal** (branched) conformal immersion of Σ in $\mathbb{R}^3$ such that (in some sense) g is its Gauss map.

Our case

If X is a simply connected metric Lie group of dimension 3 and Σ is a simply connected immersed H-surface in X with Gauss map G, one may consider the stereographic projection of G to $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $g\colon \Sigma \to \overline{\mathbb{C}}$.

Key fact.

Endowing Σ with a conformal coordinate z, then g satisfies an elliptic partial differential equation.

How to obtain a CMC surface from its Gauss map?

Let X be a three-dimensional simply connected metric Lie group.

Definition

If X is nonunimodular (and assume it is $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric, where $A = \begin{pmatrix} 1+a & -(1-a)b \\ (1+a)b & 1-a \end{pmatrix}$). Given $H \in \mathbb{R}$, we define the *H-potential* of X to be the map $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by

$$R(q) = H\left(1+|q|^2\right)^2 - (1-|q|^4) - a\left(q^2 - \overline{q}^2\right) - ib\left(2|q|^2 - a\left(q^2 + \overline{q}^2\right)\right).$$

How to obtain a CMC surface from its Gauss map?

Let X be a three-dimensional simply connected metric Lie group.

Definition

If X is a <u>unimodular</u> metric Lie group with structure constants c_1, c_2, c_3, given $H \in \mathbb{R}$, we define the *H-potential* of X as the map $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by

$$R(q) = H\left(1+|q|^2\right)^2 - \frac{i}{2}\left(\mu_2|1+q^2|^2 + \mu_1|1-q^2|^2 + 4\mu_3|q|^2\right),$$

where
$\mu_1 = \frac{1}{2}(-c_1+c_2+c_3), \quad \mu_2 = \frac{1}{2}(c_1-c_2+c_3), \quad \mu_3 = \frac{1}{2}(c_1+c_2-c_3).$

How to obtain a CMC surface from its Gauss map?

Representation Theorem.

Suppose Σ is a simply connected Riemann surface with conformal parameter z, X is a metric Lie group, and $H \in \mathbb{R}$.
Let $g\colon \Sigma \to \overline{\mathbb{C}}$ be a solution of the complex elliptic PDE

$$g_{z\bar{z}} = \frac{R_q}{R}(g)\, g_z g_{\bar{z}} + \left(\frac{R_{\bar{q}}}{R} - \frac{\overline{R_q}}{\overline{R}} \right)(g)\, |g_z|^2,$$

such that $g_z \neq 0$ everywhere, and such that the H-potential R of X does not vanish on $g(\Sigma)$. Then, there exists an immersed H-surface $f\colon \Sigma \to X$, unique up to left translations, whose Gauss map is g.

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Conversely, if $g\colon \Sigma \to \overline{\mathbb{C}}$ is the Gauss map of an immersed H-surface $f\colon \Sigma \to X$ in a metric Lie group X, and the H-potential R of X does not vanish on $g(\Sigma)$, then g satisfies the equation above, and moreover $g_z \neq 0$ holds everywhere.

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$$R(q) = H\left(1+|q|^2\right)^2 - (1-|q|^4) - a\left(q^2-\overline{q}^2\right) - ib\left(2|q|^2 - a\left(q^2+\overline{q}^2\right)\right).$$

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Non-zero H-potential Lemma

Let X be a metric Lie group and $H \in \mathbb{R}$. Then, the H-potential for X is everywhere nonzero if and only if:

- X is isomorphic to $\mathrm{SU}(2)$, or
- X is not isomorphic to $\mathrm{SU}(2)$, is unimodular and $H \neq 0$, or
- X is nonunimodular with D-invariant $D \leq 1$ and $|H| > 1$, or
- X is nonunimodular with D-invariant $D > 1$ and $|H| \neq 1$.

Proposition

Assume that there exists a compact H-surface Σ in X. Then, the H-potential of X is everywhere nonzero.

Questions about H-surfaces.

Given *any* ambient space M:

- Are there compact minimal surfaces in M?
- What are the possible topologies for such surfaces?
- If there are more than one, how are they related?
- What are the values of $H \in \mathbb{R}$ for which there exist a compact H-surface in M?
- ...

Answers in $\mathbb{R}^3$.

There are **no** compact minimal surfaces in $\mathbb{R}^3$:

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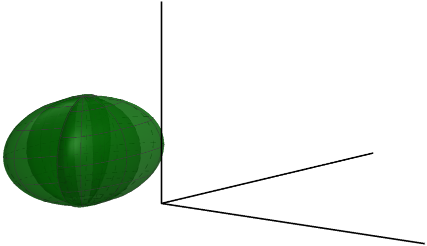

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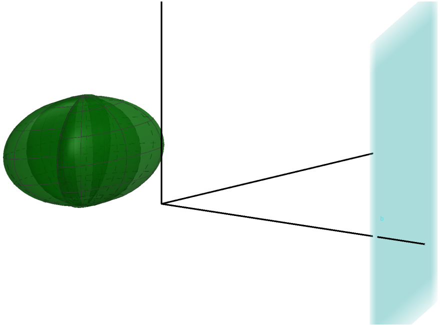

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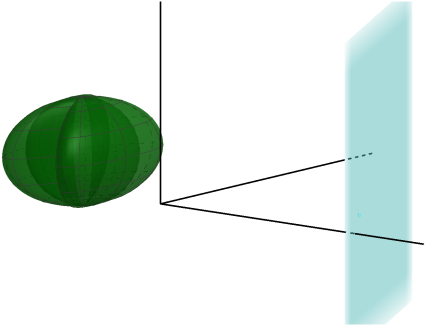

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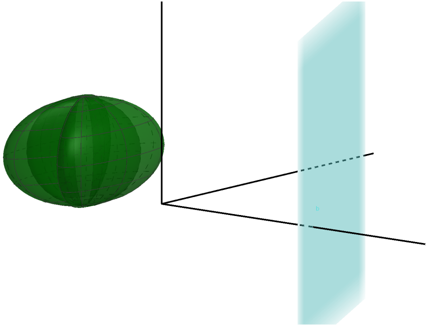

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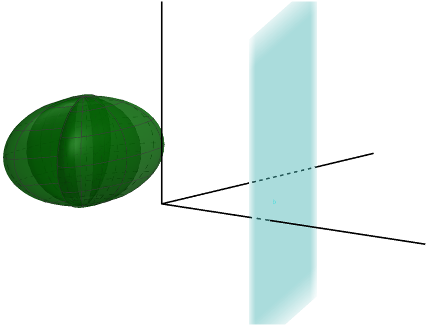

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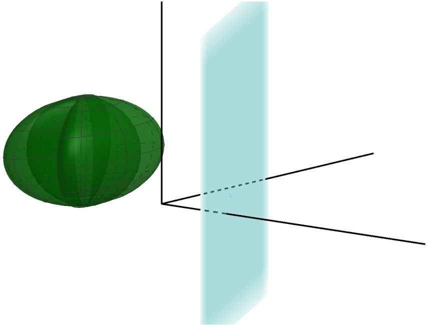

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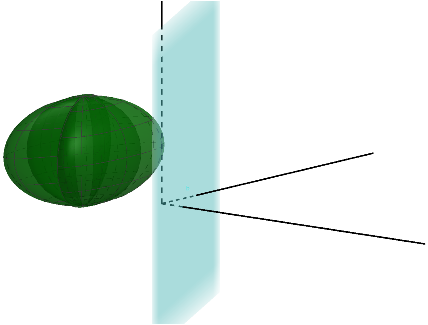

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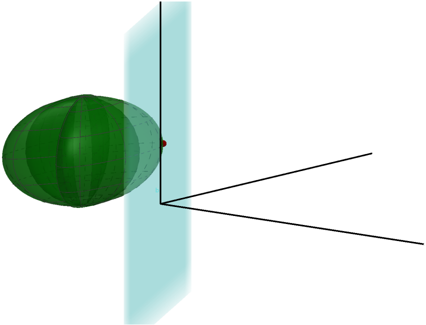

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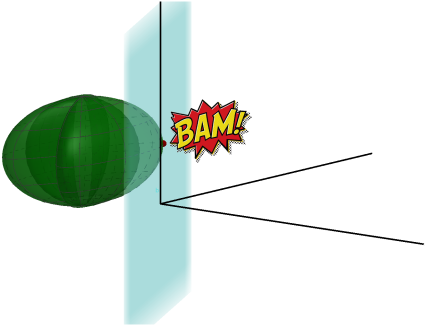

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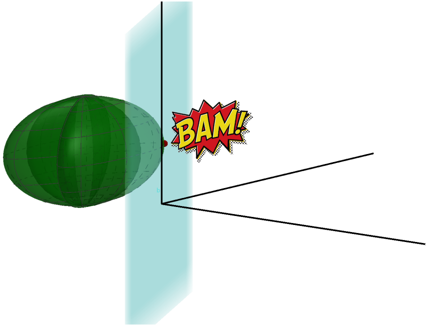

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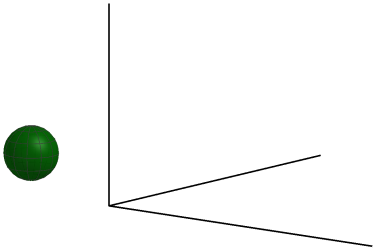

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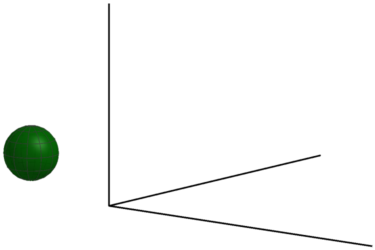

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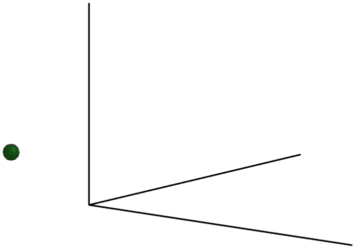

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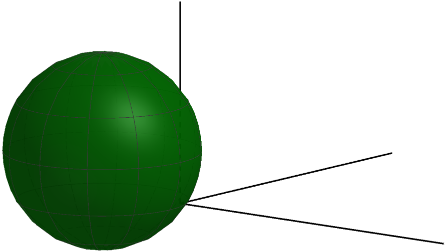

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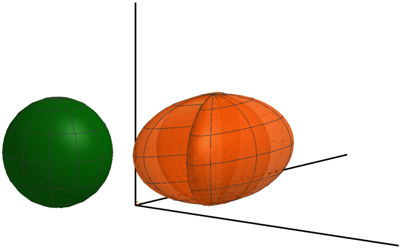

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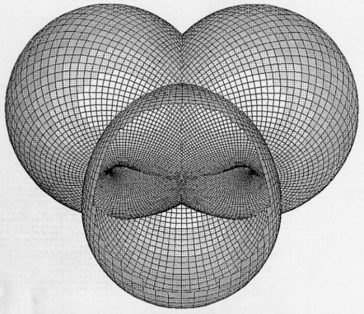

The Hopf problem.

In a given ambient space M, what are the **immersed H-spheres** in M?

Results

- (Hopf, 1971) If $M = \mathbb{R}^3$, $\mathbb{H}^3$ or $\mathbb{S}^3$, H-spheres are geodesic spheres.
- (Abresch-Rosenberg, 2004) If M is simply connected and $\dim(\mathrm{I}(\mathrm{M})) = 4$ (e.g. $\mathbb{H}^2 \times \mathbb{R}, \mathbb{S}^2 \times \mathbb{R}, \widetilde{\mathrm{SL}}(2, \mathbb{R}), {\mathbb{S}^3}_{\text{Berger}}$), then H-spheres are rotationally invariant and unique, up to ambient isometries.
- (Daniel-Mira, Meeks, 2013) If M is Sol_3 with its most symmetric left invariant metric, then H-spheres are invariant under (some) symmetries and are unique, up to ambient isometries.

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Theorem, Meeks-Mira-Pérez-Ros, 2017

Let M be a homogeneous 3-manifold. Then, any two **spheres** in M of the same absolute constant mean curvature differ, as sets, by an isometry of M. Moreover, if X is the universal covering space of M, it holds:

- If X is not diffeomorphic to $\mathbb{R}^3$, then, for every $H \in \mathbb{R}$, there exists a sphere of constant mean curvature H in M.
- If X is diffeomorphic to $\mathbb{R}^3$, then the values $H \in \mathbb{R}$ for which there exists a sphere of constant mean curvature H in M are exactly those with $|H| > \mathrm{Ch}(X)/2 = H(X)$.