Mini-course at IMPA on area, curvature and radius estimates for constant mean curvature surfaces

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Based on joint work with Mira, Pérez, Ros and Tinaglia,.

Lectures based on references appearing in last slide. Slides of talks appear on web page `http://professor.ufrgs.br/alvaro/pages/30-cbm`.

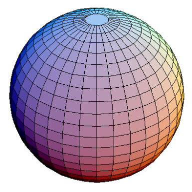
1. Lecture 1: Background material, statements of the main results.
2. **Lecture 2: Area estimates for embedded 3-periodic H-surfaces.**
3. Lecture 3: H-surfaces in homogeneous 3-manifolds
4. Lecture 4: Reduction of Hopf uniqueness theorem to area estimates
5. Lecture 5: Area estimates for H-spheres in Hopf uniqueness theorem

Definition 1

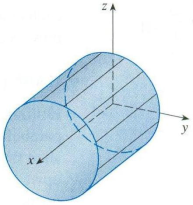
M is an **H-surface** means that it has constant mean curvature **H**.

Definition 2

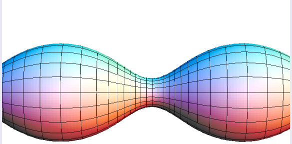
M is an **H-surface** $\iff$ **M** is a critical point for the area functional under compactly supported variations **preserving the volume**.



- Sphere



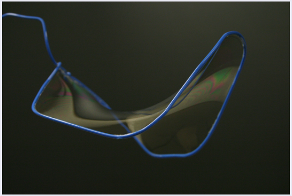
- Cylinder



- Delaunay surfaces

CMC surfaces in nature.

Soap films are minimal surfaces.

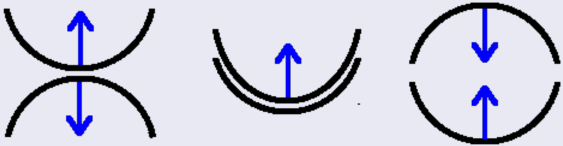


Soap bubbles are nonzero H-surfaces.

Mean Curvature Vector

$$\vec{\mathbf{H}} = \mathbf{H}\xi$$

What happens when 2 CMC surfaces get close to each other?



Local properties of embedded CMC surfaces $\mathbf{\Sigma}$ with $\mathbf{H} \neq 0$:

- 1-sided maximum principle;
- when $\mathbf{\Sigma}$ bounds a compact mean convex domain Ω then $\exists \varepsilon > 0$ and a 1-sided regular neighborhood $T_\varepsilon(\mathbf{\Sigma})$ of $\mathbf{\Sigma}$ in Ω;
- the minimum radius ε of $T_\varepsilon(\mathbf{\Sigma})$ only depends on $\mathbf{H}$, and upper bounds on $|\mathbf{A}_{\mathbf{\Sigma}}|$ and the geometry of the ambient space.

Definition (Injectivity Radius)

- Given a Riemannian manifold $\mathbf{M}$, the injectivity radius function $\mathbf{I_M} : \mathbf{M} \to (0, \infty]$ is defined by:

  $$\mathbf{I_M}(\mathbf{p}) = \sup\{R > 0 \mid \exp_{\mathbf{p}} : B(R) \subset T_{\mathbf{p}}\mathbf{M} \to \mathbf{M} \text{ is a diffeomorphism}\}$$

- The injectivity radius $\mathbf{Inj}(\mathbf{M})$ of $\mathbf{M}$ is the infimum of $\mathbf{I_M}$.

Let $\mathbf{X}$ be a complete homogeneously regular Riem. $\mathbf{3}$-manifold.

Theorem (Curvature Estimates for H-Disks, Meeks-Tinaglia)

Fix $\varepsilon, \mathbf{H}_0 > 0$. $\exists\ \mathbf{C} > 0$ s.t. for all embedded $(\mathbf{H} \geq \mathbf{H_0})$-disks $\mathbf{D}$:

$$|\mathbf{A_D}|(p) \leq \mathbf{C} \quad \text{for all} \quad p \in \mathbf{D} \text{ s.t. } \mathbf{dist_D}(p, \partial\mathbf{D}) \geq \varepsilon.$$

Theorem (Meeks-Tinaglia)

Let $\mathbf{M} \subset \mathbf{X}$ be a complete $\mathbf{H}$-surface with $\mathbf{H} > 0$.

$$\sup_{\mathbf{M}} |\mathbf{A}| < \infty \iff \mathbf{Inj}(\mathbf{M}) > 0.$$

(Note: open geodesic balls on $\mathbf{M}$ of radius $\mathbf{Inj}(\mathbf{M})$ are disks.)

What does a uniform bound on $|\mathbf{A}|$ imply?

Let $\mathbf{U}$ be an open set of $\mathbf{R^3}$ and let $\mathbf{M}_n$ be a sequence of minimal surfaces properly embedded in $\mathbf{U}$.

Well-known compactness result:

If there exist constants $C_1, C_2 < \infty$ so that

$$\sup_{\mathbf{M}_n} |\mathbf{A}| \leq C_1, \quad \mathrm{Area}(\mathbf{M}_n) < C_2$$

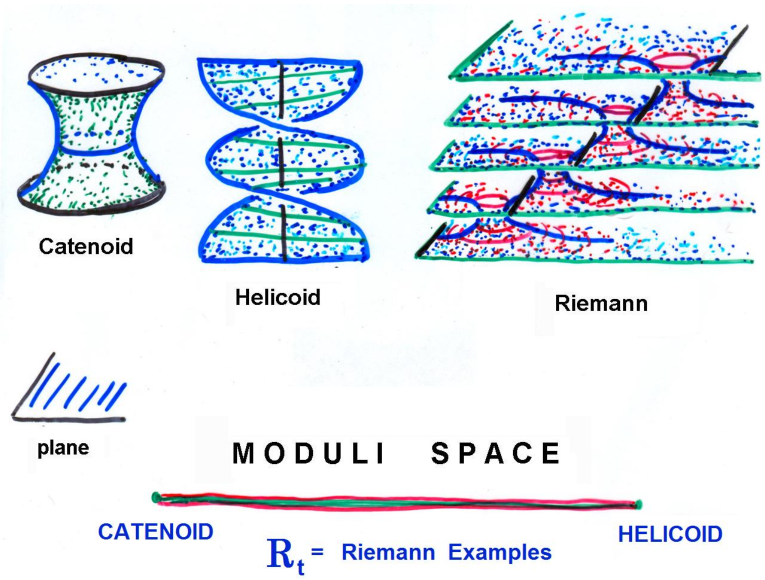
then, up to passing to a subsequence, $\mathbf{M}_n$ converges to a minimal surface $\mathbf{M}$ properly embedded in $\mathbf{U}$.

Analogous result true for $\mathbf{H}$-surfaces.

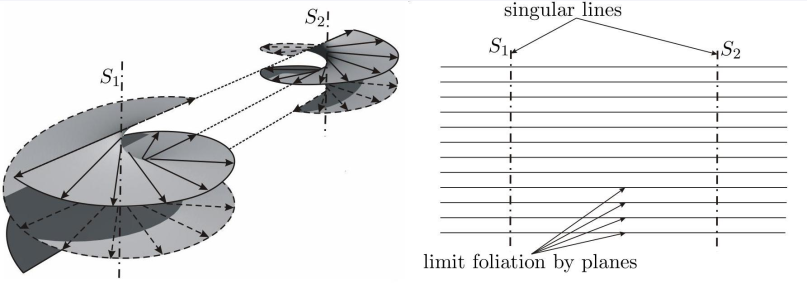
Proof of the well-known compactness result:

- $\sup_{\mathbf{M}_n} |\mathbf{A}| \leq C_1$ uniformly $\implies$ nearby a point $P \in \mathbf{U}$ we have a sequence of graphs $\mathbf{u}_n$ with $\|\mathbf{u}_n\|_{C^{2,\alpha}}$ uniformly bounded.
- Arzela-Ascoli $\implies$ subsequence converging C^2 to a minimal graph.
- $\ldots$

Proper genus-0 examples - Collin-Lopez-Meeks-Perez-Ros-Rosenberg



Riemann minimal examples near helicoid limits



- By appropriately scaling, the Riemann examples $\mathcal{R}_t$ converge as $t \to \infty$ to a foliation $\mathcal{F}$ of $\mathbf{R}^3$ by horizontal planes.
- The set of non-smooth convergence $\mathbf{S}(\mathcal{F})$ to $\mathcal{F}$ consists of **2** vertical lines $\mathbf{S_1}, \mathbf{S_2}$ perpendicular to the planes in $\mathcal{F}$.

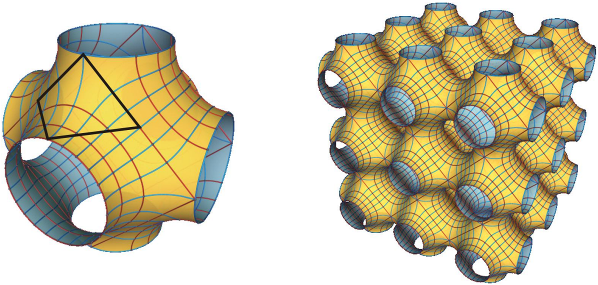


Figure: A body-centered cubic interface or Fermi surface in salt crystal.

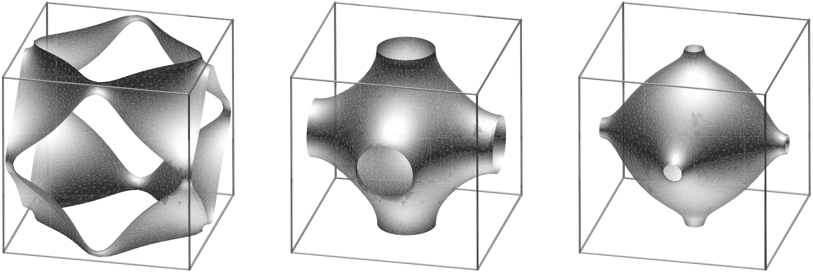
Next theorem is motivated by the study of **3**-periodic **H**-surfaces that appear as interfaces in material science or as equipotential surfaces in crystals. This result contrasts with the failure of area estimates for compact minimal surfaces of genus $\mathbf{g} > 2$ in any flat **3**-torus (**Traizet**).

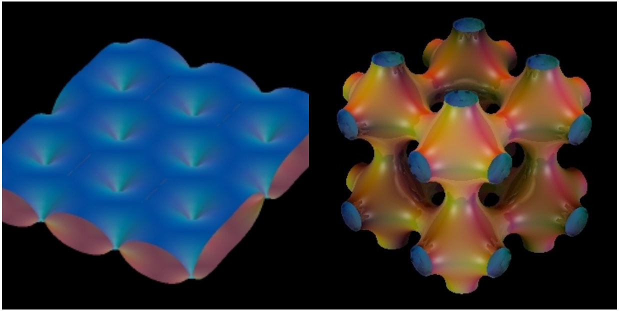
Theorem (Meeks-Tinaglia(2016))

Given a flat **3**-torus $\mathbb{T}^3$ and $\mathbf{H} > 0$, $\forall \mathbf{g} \in \mathbb{N}$, $\exists C(\mathbf{g}, \mathbf{H})$ s.t. a closed **H**-surface $\boldsymbol{\Sigma}$ embedded in $\mathbb{T}^3$ with genus at most $\mathbf{g}$ satisfies

$$\mathbf{Area}(\boldsymbol{\Sigma}) \leq C(\mathbf{g}, \mathbf{H}).$$

Closed H-surfaces in a flat 3-torus. By K. Grosse-Brauckmann (top) and N. Schmitt (bottom)





Theorem (Choi-Wang(1983), Choi-Schoen(1985))

Let **N** $=$ a closed Riemannian **3**-manifold with Ricci curvature > 0. Then:

1. The areas of closed, connected embedded minimal surfaces of fixed genus in **N** are bounded
2. The space of embedded closed minimal surfaces of fixed genus in **N** is **compact**.

Theorem (Meeks-Tinaglia(2017))

Let $0 < a \leq b$ and **N** $=$ closed Riem. **3**-manifold with $\mathbb{H}_2(\mathbf{N}) = 0$. Then:

1. The areas of closed, **connected** embedded **H**-surfaces of fixed genus **g** in **N** with $\mathbf{H} \in [a, b]$ are bounded and their indexes of stability are bounded.
2. For **every** closed Riemannian **3**-manifold **X** and any non-negative integer **g**, the space of strongly Alexandrov embedded closed surfaces in **X** of genus at most **g** and constant mean curvature $\mathbf{H} \in [a, b]$ is **compact**. (Similar compactness result holds for any fixed smooth compact family of metrics on **X**.)

Theorem (Meeks-Tinaglia)

Let $\mathbf{T}^3$ be a flat 3-torus. There exists a constant $C := C(\mathbf{T}^3, \mathbf{H}, \mathbf{g})$ such that the following holds. If $\mathbf{M}$ is a closed (compact with no boundary) $\mathbf{H}$-surface embedded in $\mathbf{T}^3$, $\mathbf{H} > 0$, of genus $\mathbf{g}$ then

$$\mathrm{Area}(\mathbf{M}) \leq C.$$

Proof

Arguing by contradiction, let $\mathbf{M}_n$ be a sequence of compact $\mathbf{H}$-surfaces embedded in $\mathbf{T}^3$, $\mathbf{H} > 0$, of genus $\mathbf{g}$ with

$$\mathrm{Area}(\mathbf{M}_n) > n.$$

Topological Fact (Ros-Rosenberg)

$\mathbf{M}_n$ is the boundary of a possibly disconnected domain in $\mathbf{T}^3$ which is mean convex.

Claim

$\lim_{n\to\infty} \inf_{\mathbf{M}_n} \mathbf{I}_{\mathbf{M}_n} = 0$

Proof

Arguing by contradiction, suppose that

$$\lim_{n\to\infty} \inf_{\mathbf{M}_n} \mathbf{I}_{\mathbf{M}_n} > \delta > 0, \text{ then}$$

- by curvature estimates for **H**-disks there exists a constant **C** such that
$$\sup_n \max_{\mathbf{M}_n} |\mathbf{A}| < \mathbf{C}.$$
- By the 1-sided regular neighborhood property $\mathbf{M}_n$ has a regular neighborhood $\mathbf{N}(n, \mathbf{C})$ on its mean convex side and there exists $\varepsilon = \varepsilon(\mathbf{C})$ such that
$$\varepsilon \mathrm{Area}(\mathbf{M}_n) < \mathrm{Volume}(\mathbf{N}(n, \mathbf{C})) < \mathrm{Volume}(\mathbf{T^3}).$$
- Contradiction:
$$n < \mathrm{Area}(\mathbf{M}_n) < \frac{\mathrm{Volume}(\mathbf{N}(n, \mathbf{C}))}{\varepsilon} < \frac{\mathrm{Volume}(\mathbf{T^3})}{\varepsilon}.$$

Definition

We say that $|\mathbf{A}_{\mathbf{M}_n}|$ blows-up at $p \in \mathbf{T^3}$ if (after replacing by a subsequence) there is a sequence of points $p_n \in \mathbf{M}_n$ such that

$$p_n \to p \quad \text{and} \quad |\mathbf{A}_{\mathbf{M}_n}|(p_n) \to \infty.$$

Such points are called **blow-up points** or **singular points** of convergence.

Note that $\mathbf{I}_{\mathbf{M}_n}(p_n) \to 0$. (Injectivity radius goes to zero)

By passing to a subsequence, one can prove that there is a (smallest) closed subset $\mathbf{\Delta} \subset \mathbf{T^3}$ such that (after replacing by any subsequence) the $|\mathbf{A}_{\mathbf{M}_n}|$ blows-up at each $p \in \mathbf{\Delta}$. By previous claim, $\mathbf{\Delta} \neq \emptyset$.

Let $\mathbf{B}$ be a compact set of $\mathbf{T^3} - \mathbf{\Delta}$. Then, by the 1-sided regular neighborhood property, there exists a constant $\varepsilon(\mathbf{B}) > 0$ such that

$$\text{Area}(\mathbf{M}_n \cap \mathbf{B}) < \frac{\text{Volume}(\mathbf{T^3})}{\varepsilon(\mathbf{B})}.$$

Claim

$\mathbf{\Delta}$ does not consist of a finite number of points $\mathbf{\Delta} = \{Q_1, \ldots, Q_m\}$.

Proof.

- Let $\mathbf{X}$ be a closed Riemannian manifold and $\mathbf{H} > 0$ be the given mean curvature that we have fixed.
- The monotonicity formula of Allard for proper surfaces $\mathbf{M}$ in $\mathbf{X}$ of absolute mean curvature at most $\mathbf{H}$ implies that for any $\delta \in (0, \mathrm{I}_{\mathbf{X}})$ sufficiently small (depending only on $\mathbf{X}$) for any point $p \in \mathbf{X}$: $\mathrm{Area}[\mathbf{M} \cap (B_{\mathbf{X}}(p, 2\delta) - B_{\mathbf{X}}(p, \delta))] \geq \mathrm{Area}(\mathbf{M} \cap B_{\mathbf{X}}(p, \delta))$.
- For δ is sufficiently small (and less than $\frac{1}{2}$ the minimum distance between points in $\mathbf{\Delta}$), define $\mathbf{B} = \mathbf{T}^3 - \bigcup_{i=1}^{m} B_{\mathbf{T}^3}(Q_i, \delta)$.
- Then $\mathrm{Area}(\mathbf{M}_n) = \mathrm{Area}(\mathbf{M}_n - \mathbf{B}) + \mathrm{Area}(\sum_{i=1}^{m} \mathbf{M}_n \cap B_{\mathbf{T}^3}(Q_i, \delta))$

$$\leq \frac{\mathrm{Volume}(\mathbf{T}^3)}{\varepsilon(\mathbf{B})} + \sum_{i=1}^{m} \mathrm{Area}[\mathbf{M} \cap (B_{\mathbf{X}}(Q_i, 2\delta) - B_{\mathbf{X}}(Q_i, \delta))]$$

$$\leq (m+1)\frac{\mathrm{Volume}(\mathbf{T}^3)}{\varepsilon(\mathbf{B})}. \quad \text{This is a contradiction!}$$

Claim

The number of points in $\mathbf{\Delta}$ is bounded.

What does $\mathbf{M}_n$ look like nearby a point P in $\mathbf{\Delta}$?

Let $P \in \mathbf{\Delta}$. Then, after replacing by a subsequence, there exists a sequence of points P_n such that $\mathbf{I}_{\mathbf{M}_n}(P_n) < \frac{1}{n}$ (which are points of almost-minimal injectivity radius) and converging to P. Let

$$\mathbf{\Sigma}_n := \frac{1}{\mathbf{I}_{\mathbf{M}_n}(P_n)}[\mathbf{M}_n - P_n].$$

Let $\Sigma_n := \frac{1}{I_{M_n}(P_n)}[M_n - P_n]$. Note that $I_{\Sigma_n}(\vec{0}) = 1$ and $H_n \to 0$.

Claim

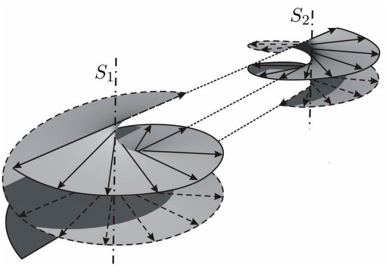
Σ_n converges to a catenoid or to a complete properly embedded minimal surface of positive genus.

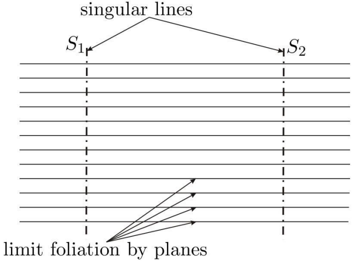
Case 1: Σ_n has uniformly bounded $|A|$ in $\mathbf{R}^3$. (No blow-up points.)

Case 2: Σ_n does NOT have uniformly bounded $|A|$ in $\mathbf{R}^3$.

Case 2: $\mathbf{\Sigma}_n$ does NOT have uniformly bounded $|\mathbf{A}|$ in $\mathbf{R}^3$.

$\mathbf{\Sigma}_n$ "converges" to parking garage structure with 2 columns.





Case 1: Σ_n has uniformly bounded $|A|$ in $\mathbb{R}^3$.

- "Well-known Compactness Theorem";
- Classification of complete properly embedded minimal surfaces of finite genus;

$$\Downarrow$$

Σ_n converges to a complete properly embedded minimal surface Σ of finite genus at most **g**:

- a flat plane or a helicoid (**Colding-Minicozzi**, **Meeks-Rosenberg**);
- a catenoid (**Lopez-Ros**);
- a Riemann minimal example (**Meeks-Perez-Ros**);
- a surface with **positive** genus at most **g**.

- a flat plane or a helicoid (**Colding-Minicozzi**, **Meeks-Rosenberg**);
- a catenoid (**Lopez-Ros**);
- a Riemann minimal example (**Meeks-Perez-Ros**);
- a surface with **positive** genus at most **g**.

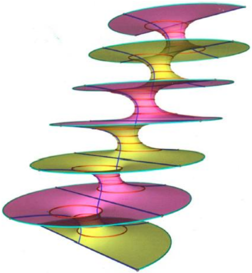
Remark

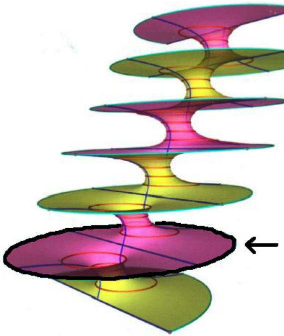
The occurrence of a plane or a helicoid can be immediately ruled out because $\mathbf{I}_{\mathbf{\Sigma}}(\vec{0}) = 1$.

Recall we want to prove that $\mathbf{\Sigma}_n$ converges to a catenoid or to a complete properly embedded minimal surface of positive genus.

- a catenoid or a surface with positive genus;
- a Riemann minimal example;
- a parking garage structure.

We need to rule out the occurrence of a Riemann minimal example and of a parking garage structure.



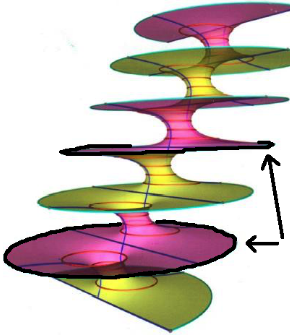


Claim

This curve is not homotopically trivial in the local picture and it is not homotopically trivial in $\mathbf{M}_n$.

Proof

Recall that $\mathbf{M}_n$ bounds a mean convex domain in $\mathbf{T^3}$. Contradiction: solving Plateau Problem contradicts Convex Hull Property for minimal surfaces.



Claim

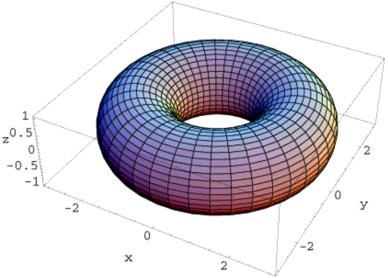
This pair of curves do not bound an annulus in the local picture and they do not bound an annulus in $\mathbf{M}_n$.

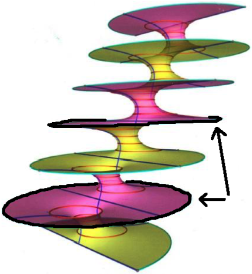
Proof

Recall that $\mathbf{M}_n$ bounds a mean convex domain in $\mathbf{T}^3$. Contradiction: solving Plateau Problem for annuli and lifting to $\mathbf{R}^3$ contradicts the isoperimetric inequality for minimal surfaces.

Claim

Let $\boldsymbol{\Sigma}$ be a compact Riemann surface of genus $\mathbf{g}$ and let $\boldsymbol{\Gamma}$ be a collection of simple closed curves in $\boldsymbol{\Sigma}$ that are not homotopically trivial and are pair-wise disjoint. If the number of curves in $\boldsymbol{\Gamma}$ is greater than $3\mathbf{g} - 2$ then there exists at least a pair of distinct curves that bounds an annulus in $\boldsymbol{\Sigma}$.





- The genus of $\mathbf{M}_n$ is bounded by $\mathbf{g}$.
- The numbers of curves in $\mathbf{M}_n$ that are not homotopically trivial is becoming arbitrarily large.
- Contradiction: by the claim there must exist a pair that bounds an annulus in $\mathbf{M}_n$.

This rules out the appearance of a Riemann minimal example and the exact same argument works to rule out a minimal parking garage structure.

We now know that the following claim holds:

Claim

Σ_n converges to a catenoid or to a properly embedded minimal surface of positive genus.

Recall that we are trying to prove that the number of singular points, that is points in Δ, is bounded.

Since the genus is additive, the number of singular points where a catenoid does NOT appear is bounded by the genus and finite total curvature.

Given $p \in \mathbf{\Delta}$ where catenoids form, let $\mathbf{\Gamma}_n(p)$ be a sequence of closed loops converging to the closed geodesic of the catenoid forming at p.

$\mathbf{\Gamma}_n(p)$ is NOT homotopically trivial in $\mathbf{M}_n$ because a catenoid has non-zero flux and flux is a homological invariant.

Suppose $p, q \in \mathbf{\Delta}$ and $\mathbf{\Gamma}_n(p) \cup \mathbf{\Gamma}_n(q)$ bounds an annulus $\mathbf{A}_n(p, q)$.

Claim

The number m of such annuli is bounded.

- $\mathbf{A}_n(p, q)$ lifts to $\mathbf{R^3}$ and it is large;
- By using the Alexandrov reflection principle, $\mathbf{A}_n(p, q)$ corresponds to a region of space $\mathbf{V}_n(p, q)$ in $\mathbf{T^3}$ such that $\mathrm{Volume}(\mathbf{V}_n(p, q)) > \varepsilon > 0$ where ε does NOT depend on p, q or n.
- Therefore the number of such annuli is bounded.

$$\varepsilon m \leq \sum_{i=1}^{m} \mathrm{Volume}(\mathbf{V}_n(p_i, q_i)) \leq \mathrm{Volume}(\mathbf{T^3}).$$

Claim

The number of singular points, that is points in $\mathbf{\Delta}$, is bounded.

- If $\mathbf{\Delta}$ is infinite then there is an arbitrarily large number of curves $\mathbf{\Gamma}_n(p)$ that are not homotopically trivial.
- Since the genus of $\mathbf{M}_n$ is uniformly bounded, this creates an arbitrarily large number of curves bounding annuli.
- This contradicts the fact that the number of such annuli is bounded.

Theorem (Meeks-Tinaglia)

Let $\mathbf{T^3}$ be a flat 3-torus. There exists a constant $C := C(\mathbf{T^3}, \mathbf{H}, \mathbf{g})$ such that the following holds. If $\mathbf{M}$ is a closed (compact with no boundary) $\mathbf{H}$-surface in $\mathbf{T^3}$, $\mathbf{H} > 0$, of genus $\mathbf{g}$ then

$$\mathrm{Area}(\mathbf{M}) \leq C.$$

Corollary (Meeks-Tinaglia)

- Let $\mathbf{T^3}$ be a flat 3-torus.
- If $\mathbf{M}_n$ is a sequence of compact $\mathbf{H}$-surfaces in $\mathbf{T^3}$, $\mathbf{H} > 0$, of genus at most $\mathbf{g}$ then, up to a subsequence, it converges to a non-empty, possibly disconnected, strongly Alexandrov embedded surface $\mathbf{M}_\infty$ of constant mean curvature $\mathbf{H}$ and genus at most $\mathbf{g}$.
- The convergence is smooth away from a finite set of points, $\mathbf{\Delta}$.
- The set $\mathbf{\Delta}$ is a subset of the set of points where $\mathbf{M}_\infty$ is not embedded.

Definition

- Suppose $f\colon \mathbf{\Sigma} \to \mathbf{N}$ is a closed immersed surface positive mean curvature in a Riemannian 3-manifold $\mathbf{N}$.
- $\mathbf{\Sigma}$ is called **strongly Alexandrov embedded** if f extends to an immersion $F\colon \mathbf{W} \to \mathbf{N}$ of a compact 3-manifold $\mathbf{W}$ with $\mathbf{\Sigma} = \partial \mathbf{W}$, where the extended immersion is injective on the interior of $\mathbf{W}$.

Corollary (Meeks-Tinaglia)

- Let $\mathbf{T^3}$ be a flat 3-torus.
- If $\mathbf{M}_n$ is a sequence of compact $\mathbf{H}$-surfaces in $\mathbf{T^3}$, $\mathbf{H} > 0$, of genus at most $\mathbf{g}$ then, up to a subsequence, it converges to a non-empty, possibly disconnected, strongly Alexandrov embedded surface $\mathbf{M}_\infty$ of constant mean curvature $\mathbf{H}$ and genus at most $\mathbf{g}$.
- The convergence is smooth away from a finite set of points, $\mathbf{\Delta}$.
- The set $\mathbf{\Delta}$ is a subset of the set of points where $\mathbf{M}_\infty$ is not embedded.

- In the previous arguments, to bound the number of singular points we did not use that the area was going to infinity.
- Area estimate and bounded genus imply that the singularities are removable.
- Density arguments show that $\mathbf{\Delta}$ is a subset of the set of points where $\mathbf{M}_\infty$ is not embedded.
- Density arguments show that the singular points can only be catenoid singular points.

Theorem (LRST for H-laminations, Meeks-Perez-Ros)

Let $\mathcal{S} \subset \mathbf{X}$ be a closed countable set in a Riemannian **3**-manifold **X** and let $\mathcal{L}$ be a weak **H**-lamination of $\mathbf{X} - \mathcal{S}$.
Suppose for each p of $\mathcal{S}$, there exist a $\mathbf{C_p} > 0$ and some ball $\mathbf{B}_p$ centered at p such that for $x \in \mathbf{B}_p - \mathcal{S}$:

$$\boxed{|\mathbf{A}_{\mathcal{L}}|(x) \leq \frac{\mathbf{C_p}}{\mathbf{d_X(x,}\mathcal{S}\mathbf{)}}},$$

where $|\mathbf{A}_{\mathcal{L}}|$ is the norm of the second fundamental form of the leaves of $\mathcal{L}$. Then:

$\mathcal{L}$ extends across $\mathcal{S}$ to weak H-lamination $\overline{\mathcal{L}}$ of X.

A few more details.

What does $\mathbf{M}_n$ look like nearby a point P in $\mathbf{\Delta}$?

Let $P \in \mathbf{\Delta}$. Then there exists a sequence of points $P_n \to P$ such that $\lim_{n\to\infty} \mathsf{I}_{\mathbf{M}_n}(P_n) = 0$.
In fact, the sequence of points P_n converging to P can be taken to be a sequence of points of **almost minimal injectivity radius**.
Namely, there exists a sequence of numbers $\delta_n > 0$ such that

1. $\lim_{n\to\infty} \mathsf{I}_{\mathbf{M}_n}(P_n) = 0$ and $\lim_{n\to\infty} \delta_n = 0$;
2. $\lim_{n\to\infty} \frac{\delta_n}{\mathsf{I}_{\mathbf{M}_n}(P_n)} = \infty$;
3. Genus of $\mathcal{B}_{\delta_n}(P_n)$ is at most $\mathbf{g}$;
4. $\sup_{\mathcal{B}_{\delta_n}(P_n)} \mathsf{I}_{\mathbf{M}_n}(Q) \geq 4\mathsf{I}_{\mathbf{M}_n}(P_n)$.

Let $\boldsymbol{\Sigma}_n := \frac{1}{\mathbf{I}_{\mathbf{M}_n}(P_n)}[\mathcal{B}_{\delta_n}(P_n) - P_n]$.

Claim

$\boldsymbol{\Sigma}_n$ converges to a catenoid.

Proof.

- $\boldsymbol{\Sigma}_n$ converges to a catenoid or to a properly embedded minimal surface of finite total curvature, positive genus and with more than two ends.
- But a properly embedded minimal surface of finite total curvature, positive genus and with more than two ends has the area density of at least $2\frac{1}{2}$ planes in some ball around the origin, which contradicts that $\mathbf{M}_\infty$ has area density 2 at its nonembedded points.

□

- If $\mathbf{M}_n \subset \mathbf{T^3}$ is a sequence of $(\mathbf{H} > 0)$-surfaces that converges to the limit surface $\mathbf{M}_\infty$ given in its conclusion, then the following holds:
- Let $q \in \mathbf{T^3}$ be a singular point of convergence of the $\mathbf{M}_n$ to $\mathbf{M}_\infty$.
- Then for any $\varepsilon > 0$ sufficiently small, there exists an $N_0 = N_0(\varepsilon)$ such that for $n \geq N_0$, $\mathbf{\Sigma}_n = \overline{B}_{\mathbf{T^3}}(q, \varepsilon) \cap \mathbf{M}_n$ is a connected compact surface with two boundary components.

Conjecture (Genus-zero Singular Points of Convergence Conjecture)

For $\varepsilon > 0$ sufficiently small and n sufficiently large, the compact surface $\mathbf{\Sigma}_n = \overline{B}_{\mathbf{T^3}}(q, \varepsilon) \cap \mathbf{M}_n$ is annulus of total absolute Gaussian curvature $C(\Sigma_n) \in (4\pi - \varepsilon, 4\pi + \varepsilon)$.

Conjecture (Finiteness Conjecture)

For any $\mathbf{H} > 0$ and $g \in \mathbb{N} \cup \{0\}$, the moduli space of non-congruent, connected closed $\mathbf{H}$-surfaces of at most genus $\mathbf{g}$ in a fixed flat 3-torus is finite.

Remark

This conjecture is false for closed minimal surfaces for every finite genus $\mathbf{g} \neq 0, 2$ by Traizet.