Mini-course at IMPA on area, curvature and radius estimates for constant mean curvature surfaces

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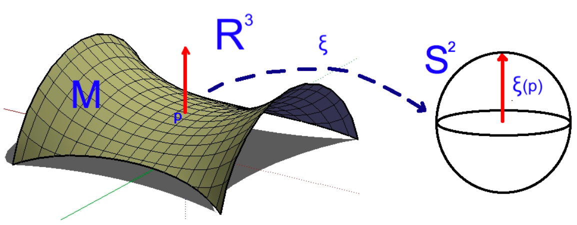
Based on joint work with Mira, Pérez, Ros and Tinaglia,.

Lectures based on references appearing in last slide. Slides of talks appear on web page `http://professor.ufrgs.br/alvaro/pages/30-cbm`.

1. Lecture 1: Background material, statements of the main results.
2. Lecture 2: Area estimates for embedded **3**-periodic **H**-surfaces.
3. Lecture 3: **H**-surfaces in homogeneous **3**-manifolds
4. Lecture 4: Reduction of Hopf uniqueness theorem to area estimates
5. Lecture 5: Area estimates for **H**-spheres in Hopf uniqueness theorem

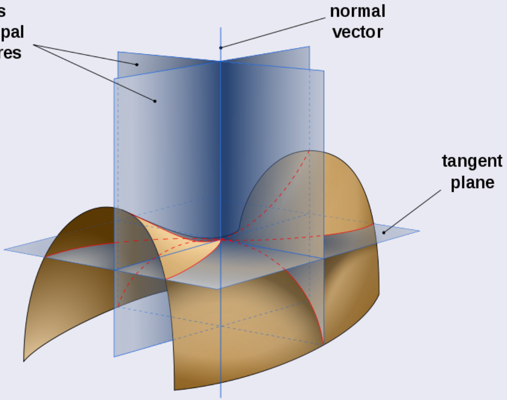


- Let **M** be an oriented surface in $\mathbf{R}^3$, let ξ be the unit vector field normal to **M**:

$$\mathbf{A_p} = -\mathbf{d}\xi_{\mathbf{p}} \colon T_{\mathbf{p}}\mathbf{M} \to T_{\xi(\mathbf{p})}\mathbf{S}^2 \simeq T_p\mathbf{M}$$

  is the **shape operator** of **M**.

- The map $\xi \colon \mathbf{M} \to \mathbf{S}^2$ is called the **Gauss map** of **M** at **p**.



- Let $\mathbf{M}$ be an oriented surface in $\mathbf{R}^3$, let ξ be the unit vector field normal to $\mathbf{M}$:

$$\mathbf{A_p} = -\mathbf{d}\xi_{\mathbf{p}} \colon T_{\mathbf{p}}\mathbf{M} \to T_{\xi(\mathbf{p})}\mathbf{S}^2 \simeq T_p\mathbf{M}$$

  is the **shape operator** of $\mathbf{M}$. Here $\mathbf{d}\xi_{\mathbf{p}}$ is the derivative map.

- The trace of $\mathbf{A_p}$ is twice the mean curvature $\mathbf{H}(p)$ at $p \in \mathbf{M}$.

Definition

- **Principal curvatures** $k_1(\mathbf{p}), k_2(\mathbf{p})$ at $\mathbf{p}$ are the eigenvalues of $\mathbf{A_p}$.
- $\mathbf{K} = \det(\mathbf{A}) = k_1 k_2 =$ **Gauss curvature** function.
- $\mathbf{H} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) = \frac{k_1+k_2}{2} =$ **mean curvature** function.
- $|\mathbf{A}| = \sqrt{k_1^2 + k_2^2} =$ **norm of 2nd fundamental form (shape op).**

Gauss equation

$$4\mathbf{H}^2 = |\mathbf{A}|^2 + 2\mathbf{K} \quad (\mathbf{K} = \text{Gaussian curvature})$$

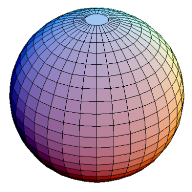
So, when $\mathbf{H}(\mathbf{p}) = 0$, then $\mathbf{K}(\mathbf{p}) \leq 0$.

Definition 1

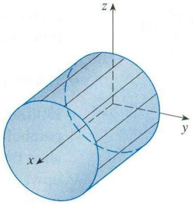
M is an **H-surface** means that it has constant mean curvature **H**.

Definition 2

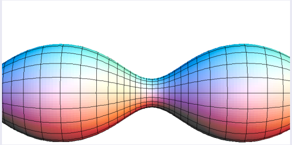
M is an **H-surface** $\iff$ **M** is a critical point for the area functional under compactly supported variations **preserving the volume**.



- Sphere



- Cylinder



- Delaunay surfaces

Definition

Let $\mathbf{M} \subset \mathbf{R}^3$ be a compact surface with boundary and unit normal field N.

- A function $f \colon \mathbf{M} \to \mathbb{R}$ is said to have mean value zero if $\int_{\mathbf{M}} f = 0$.
- A (normal) variation of $\mathbf{M}$ which **preserves volume** is a family $F_t(x) = x + tu(x)N(x)$ such that u has mean value zero and $u|_{\partial \mathbf{M}} = 0$.
- The first variation of area with respect to such an F_t is the derivative of the area of the surfaces $F_t \colon \mathbf{M} \to \mathbf{R}^3$ at $t = 0$.
- The stability operator of $\mathbf{M}$ is

$$S = \mathbf{\Delta} + |\mathbf{A_M}|^2 + \mathrm{Ric}(N),$$

  where $\mathbf{\Delta}$ is the Laplacian, $\mathbf{A_M}$ is the second fundamental form of $\mathbf{M}$ and $\mathrm{Ric}(\cdot)$ is the Ricci curvature.
- If $\mathbf{M}$ is an $\mathbf{H}$-surface, then the **nullity** of S is the dimension of the kernel of S and the **index** is the dimension of the kernel of S (with respect to functions u with $u|_{\partial \mathbf{M}} = 0$).

Definition (Isoperimetric Problem)

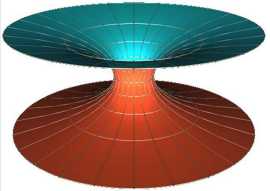
- A smooth compact domain Ω in a 3-dimensional Riemannian manifold X is called a **solution of the isoperimetric problem** in X if any other smooth compact subdomain $\Omega' \subset X$ with the same volume as Ω satisfies

$$\text{Area}(\partial\Omega') \geq \text{Area}(\partial\Omega).$$

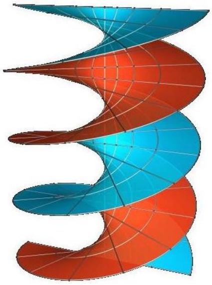
- Note that the boundary of a solution to the isoperimetric problem in X is a critical point to volume preserving variations and hence must have constant mean curvature.
- Fact: if X is homogenous or closed, then for every $V > 0$, there exists a smooth domain Ω_V with volume V that is a solution to the isoperimetric problem.

Definition

An **H**-surface **M** is a **minimal surface** $\iff$ **H** $\equiv 0$ $\iff$ **M** is a critical point for the area functional under compactly supported variations.



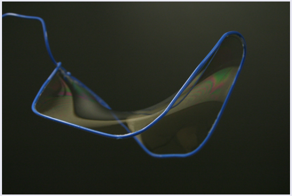
- Catenoid



- Helicoid

CMC surfaces in nature.

Soap films are minimal surfaces.



Soap bubbles are nonzero H-surfaces.

Notation and Language

- $\mathrm{Ch}(\mathbf{Y}) = \mathbf{Inf}_{\mathbf{K}\subset\mathbf{Y}\ \text{compact}} \dfrac{\mathbf{Area}(\partial\mathbf{K})}{\mathbf{Volume}(\mathbf{K})}$ = Cheeger constant of $\mathbf{Y}$.
- $\mathbf{H}(\mathbf{Y}) = \mathbf{Inf}\{\max|\mathbf{H}_{\mathbf{M}}| : \mathbf{M}$ = immersed closed surface in $\mathbf{Y}\}$, where $\max|\mathbf{H}_{\mathbf{M}}|$ denotes max of absolute mean curvature function $\mathbf{H}_{\mathbf{M}}$.
- The number $\mathbf{H}(\mathbf{Y})$ is called the **critical mean curvature** of $\mathbf{Y}$.

Theorem (Meeks-Mira-Pérez-Ros)

- If $\mathbf{Y}$ is a simply connected homogeneous $\mathbf{3}$-manifold, then:

$$2\mathbf{H}(\mathbf{Y}) = \mathrm{Ch}(\mathbf{Y})$$

Remark

Proof uses $\mathbf{H}(\mathbf{Y})$-foliations of $\mathbf{Y}$ to show that if $\mathbf{\Omega}(n) \subset \mathbf{Y}$ is a sequence of isoperimetric domains in $\mathbf{Y}$ with $\mathbf{Volume}(\mathbf{\Omega}(n)) \to \infty$, then

$$\mathbf{H}_{\partial\mathbf{\Omega}(n)} \geq \mathbf{H}(\mathbf{Y}) \quad \textbf{and} \quad \lim_{n\to\infty} \mathbf{H}_{\partial\mathbf{\Omega}(n)} = \mathbf{H}(\mathbf{Y}).$$

Classification Fact:

Simply connected homogeneous **3**-manifolds **X** are either isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$ or to a metric Lie group.

Let **M** be a Riemannian homogeneous **3**-manifold, **X** denote its Riemannian universal cover, $\mathrm{Ch}(\mathbf{X})$ denote the Cheeger constant of **X**.

The next theorem solves the so called **Hopf Uniqueness Problem.**

Theorem (Hopf Uniqueness Problem, Meeks-Mira-Pérez-Ros)

Any two spheres in **M** of the same absolute constant mean curvature differ by an isometry of **M**. Moreover:

(1) If **X** is not diffeomorphic to $\mathbb{R}^3$, then, for every $\mathbf{H} \in \mathbb{R}$, there exists a sphere of constant mean curvature **H** in **M**.

(2) If **X** is diffeomorphic to $\mathbb{R}^3$, then the values $\mathbf{H} \in \mathbb{R}$ for which there exists a sphere of constant mean curvature **H** in **M** are exactly those with $|\mathbf{H}| > \mathrm{Ch}(\mathbf{X})/2$.

Theorem (Geometry of **H**-spheres, Meeks-Mira-Pérez-Ros)

Let S be an **H**-sphere in **M**.

1. If **H** $= 0$ and **X** is a product $\mathbb{S}^2(\kappa) \times \mathbb{R}$, where $\mathbb{S}^2(\kappa)$ is a sphere of constant curvature κ, then S is totally geodesic, stable and has nullity 1 for its Jacobi operator.
2. **Otherwise**, S has index 1 and nullity 3 for its Jacobi operator and the immersion of S into **M** extends as the boundary of an isometric immersion $F\colon$ **B** $\to$ **M** of a Riemannian **3**-ball **B** which is mean convex.
3. There is a point $\mathbf{p}_S \in$ **M**, called the **center of symmetry** of S, such that every isometry of **M** that fixes $\mathbf{p}_S$ also leaves invariant S.

Previous influential results on the **Hopf Uniqueness Problem**:

Theorem (Hopf, 1951)

H-spheres in $\mathbf{R}^3$ are round. In fact if **M** has constant sectional curvature, then **H**-spheres in **M** are geodesic spheres of fixed radii.

Theorem (Abresch-Rosenberg, 2004)

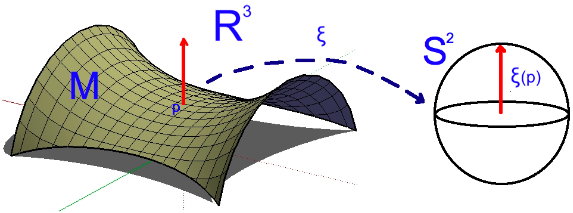
If **M** has a **4**-dimensional isometry group, then **H**-spheres in **M** are surfaces of revolution and they are unique.

Theorem (Daniel-Mira (2013), Meeks (2013))

- If **X** is the Lie group $\mathbf{Sol_3}$ with the left invariant metric

$$e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

  then **H**-spheres in **X** are unique, embedded and have index 1.
- After left translation, these spheres have ambient symmetry group generated by reflections in the (x, z) and (y, z)-planes and rotations by π around the two lines $y = \pm x$ in the (x, y)-plane.



Definition (Left invariant Gauss map)

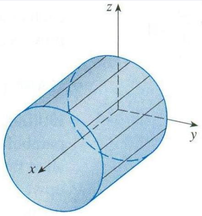
- Let $\mathbf{X}$ be a $\mathbf{3}$-dimensional metric Lie group.
- Given an oriented immersed surface $f\colon \mathbf{\Sigma} \to \mathbf{X}$ with unit normal vector field $N\colon \mathbf{\Sigma} \to T\mathbf{X}$, the **left invariant Gauss map** of $\mathbf{\Sigma}$ is the map $G\colon \mathbf{\Sigma} \to \mathbb{S}^2 \subset T_e\mathbf{X}$ that assigns to each $p \in \mathbf{\Sigma}$, the unit tangent vector to $\mathbf{X}$ at the identity element e given by left translation:

$$(dl_{f(p)})_e(G(p)) = N_p.$$

Question

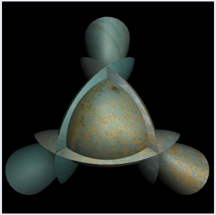
Is the round sphere the only complete simply connected surface **embedded** in $\mathbf{R}^3$ with **non-zero** constant mean curvature?

NOT simply connected



- Cylinder

NOT embedded



- Smyth surface conformally $\mathbb{C}$

New uniqueness results for CMC surfaces.

Theorem (Meeks-Tinaglia)

Round spheres are the only complete simply connected surfaces **embedded** in $\mathbf{R}^3$ with **non-zero** constant mean curvature.

1986 - Above result proved by **Meeks** for **properly embedded**.

2007 - Work of **Colding-Minicozzi** and **Meeks-Rosenberg** for $\mathbf{H} = 0$ shows that if $\mathbf{M}$ is a complete, simply connected $\mathbf{0}$-surface (minimal surface) **embedded** in $\mathbf{R}^3$, then $\mathbf{M}$ is either

a plane or **a helicoid**.

Theorem (Meeks-Tinaglia)

Let $M \subset \mathbb{R}^3$ be a complete, connected embedded H-surface.

1. M has positive injectivity radius $\Longrightarrow$ M is properly embedded in $\mathbb{R}^3$.
2. M has finite topology $\Longrightarrow$ M has positive injectivity radius.
3. Suppose $H > 0$. Then:

$$|A_M| \text{ is bounded} \iff M \text{ has positive injectivity radius.}$$

When $H = 0$, items 1 and 2 were proved by Meeks-Rosenberg, based on:

Colding-Minicozzi: M has finite topology and $H = 0 \Longrightarrow M$ is proper.

Item 3 in the above theorem holds for 3-manifolds which are homogeneously regular; in particular it holds in closed Riemannian 3-manifolds.

Theorem (Radius Estimates for H-Disks, Meeks-Tinaglia)

$\exists\, \mathbf{R_0} \geq \pi$ such that every embedded $\mathbf{H}$-disk in $\mathbf{R}^3$ has radius $< \mathbf{R_0}/\mathbf{H}$.

Corollary (Meeks-Tinaglia)

A complete simply connected $\mathbf{H}$-surface embedded in $\mathbf{R}^3$ with $\mathbf{H} > 0$ is a round sphere.

Theorem (Curvature Estimates for H-Disks, Meeks-Tinaglia)

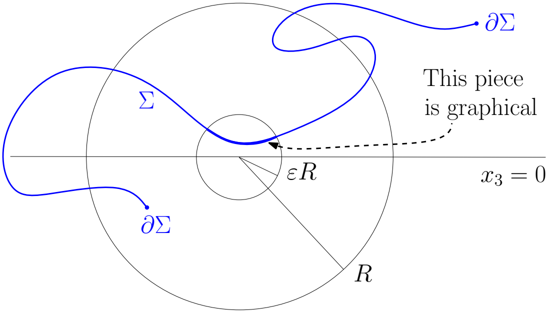
Fix $\varepsilon, \mathbf{H}_0 > 0$ and a complete locally homogenous $\mathbf{3}$-manifold $\mathbf{X}$. $\exists\, \mathbf{C} > 0$ s.t. for all embedded $(\mathbf{H} \geq \mathbf{H_0})$-disks $\mathbf{D}$:

$$|\mathbf{A_D}|(p) \leq \mathbf{C} \quad \text{for all} \quad p \in \mathbf{D} \ \text{s.t.}\ \mathbf{dist_D}(p, \partial \mathbf{D}) \geq \varepsilon.$$

Theorem (One-sided curvature estimate for H-disks, Meeks-Tinaglia)

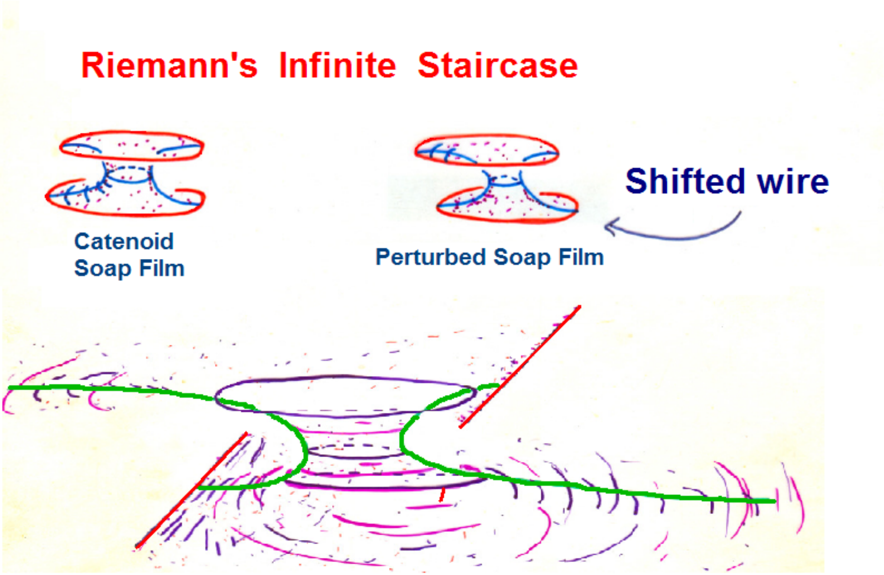
$\exists \mathbf{C}, \varepsilon > 0$ s.t. for any **H**-disk $\mathbf{\Sigma} \subset \mathbf{R^3}$ as in the figure below:

$$|\mathbf{A_\Sigma}| \leq \frac{\mathbf{C}}{R} \text{ in } \mathbf{\Sigma} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}.$$

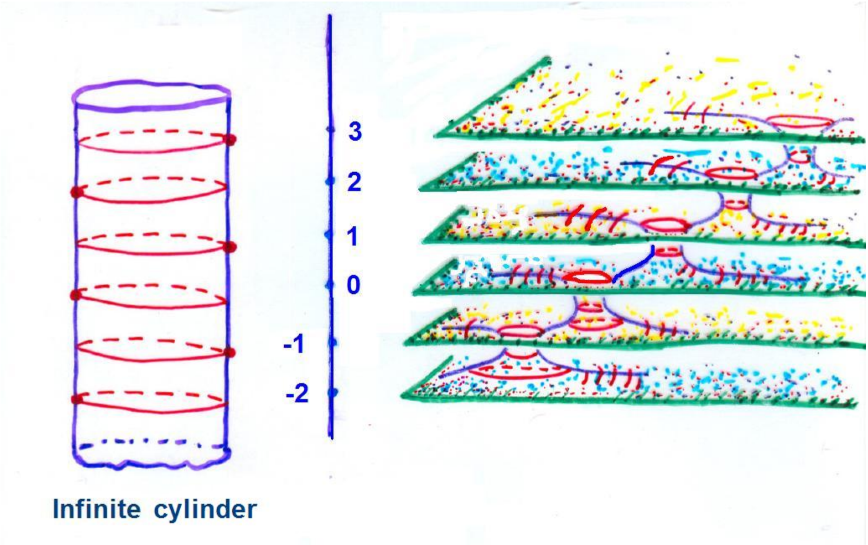


This result generalizes the one-sided curvature estimates for minimal disks by Colding-Minicozzi, and uses their work in its proof.

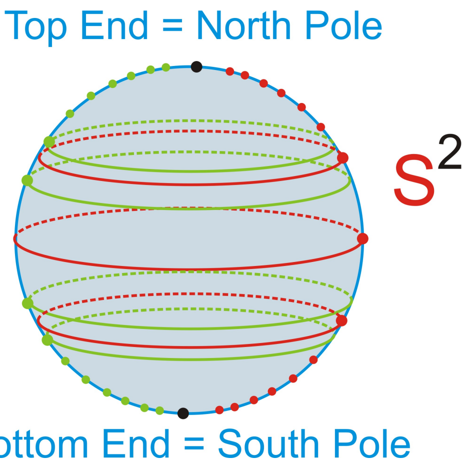
The family $\mathcal{R}_t$ of Riemann minimal examples



Cylindrical parametrization of a Riemann minimal example

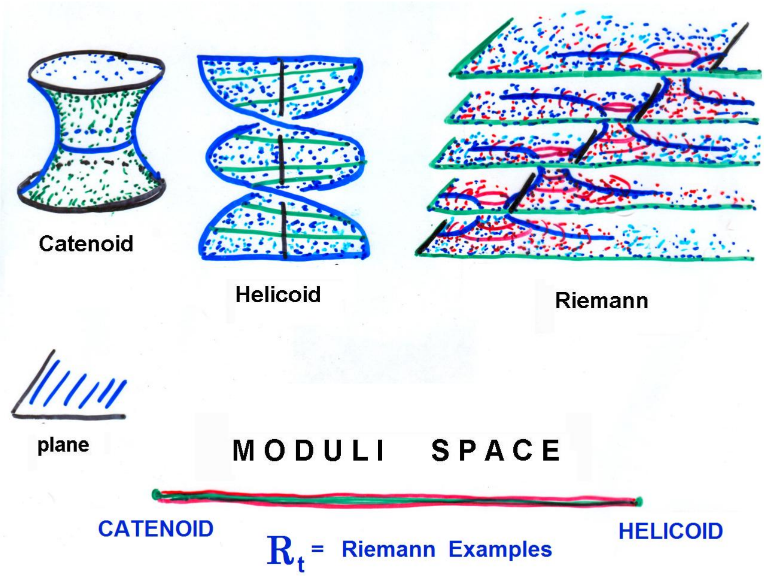


Conformal compactification of a Riemann minimal example

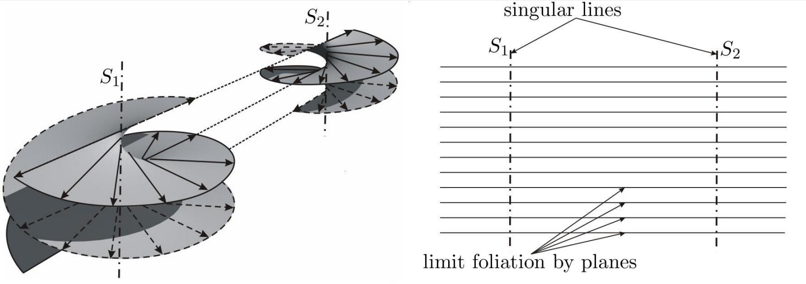


Example

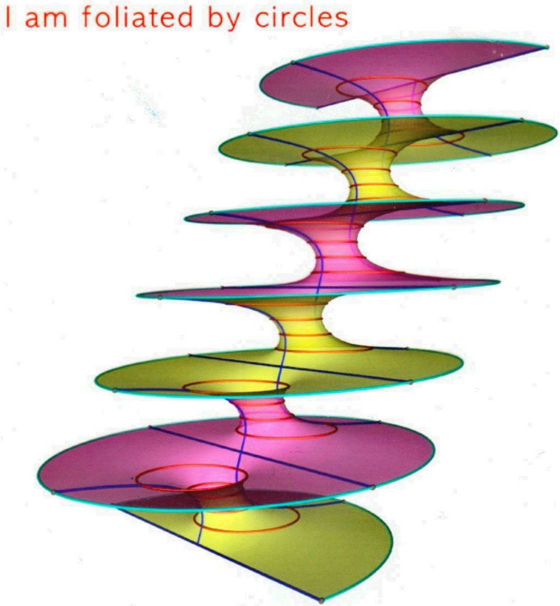
Topologically there is **only one** connected **genus-zero** surface with **two limit ends**. Riemann minimal examples have this property.


Catenoid
Helicoid
Riemann
plane
MODULI SPACE
CATENOID
HELICOID
R_t = Riemann Examples

Riemann minimal examples near helicoid limits



- By appropriately scaling, the Riemann examples $\mathcal{R}_t$ converge as $t \to \infty$ to a foliation $\mathcal{F}$ of $\mathbf{R}^3$ by horizontal planes.
- The set of non-smooth convergence $\mathbf{S}(\mathcal{F})$ to $\mathcal{F}$ consists of **2** vertical lines $\mathbf{S_1}, \mathbf{S_2}$ perpendicular to the planes in $\mathcal{F}$.


I am foliated by circles

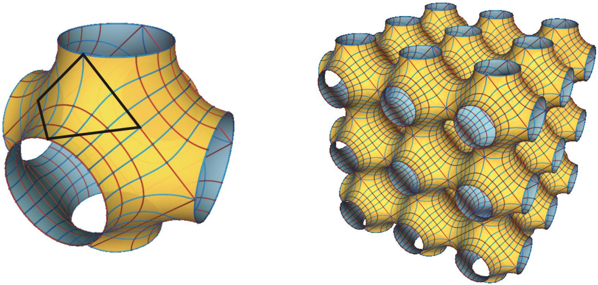


Figure: A body-centered cubic interface or Fermi surface in salt crystal.

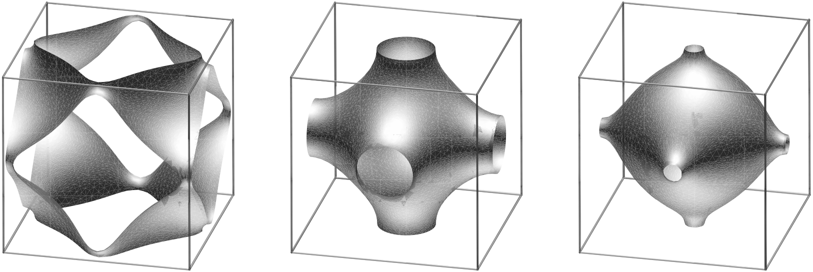
Next theorem is motivated by the study of **3**-periodic **H**-surfaces that appear as interfaces in material science or as equipotential surfaces in crystals. This result contrasts with the failure of area estimates for compact minimal surfaces of genus $\mathbf{g} > 2$ in any flat **3**-torus (**Traizet**).

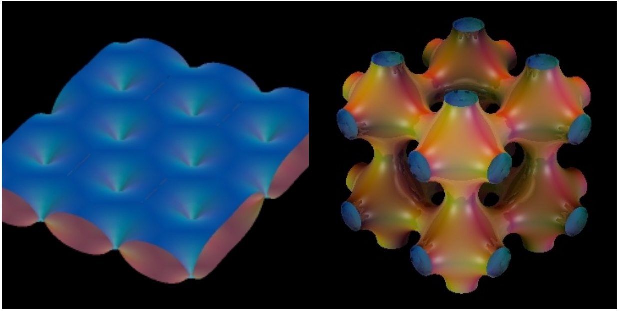
Theorem (Meeks-Tinaglia(2016))

Given a flat **3**-torus $\mathbb{T}^3$ and $\mathbf{H} > 0$, $\forall \mathbf{g} \in \mathbb{N}$, $\exists C(\mathbf{g}, \mathbf{H})$ s.t. a closed **H**-surface $\mathbf{\Sigma}$ embedded in $\mathbb{T}^3$ with genus at most $\mathbf{g}$ satisfies

$$\mathbf{Area}(\mathbf{\Sigma}) \leq C(\mathbf{g}, \mathbf{H}).$$

Closed H-surfaces in a flat 3-torus. By K. Grosse-Brauckmann (top) and N. Schmitt (bottom)





Theorem (Choi-Wang(1983), Choi-Schoen(1985))

Let **N** = a closed Riemannian **3**-manifold with Ricci curvature > 0. Then:

1. The areas of closed, connected embedded minimal surfaces of fixed genus in **N** are bounded
2. The space of embedded closed minimal surfaces of fixed genus in **N** is **compact**.

Theorem (Meeks-Tinaglia(2017))

Let $0 < a \leq b$ and **N** = closed Riem. **3**-manifold with $\mathbb{H}_2(\mathbf{N}) = 0$. Then:

1. The areas of closed, **connected** embedded **H**-surfaces of fixed genus **g** in **N** with $\mathbf{H} \in [a, b]$ are bounded and their indexes of stability are bounded.
2. For **every** closed Riemannian **3**-manifold **X** and any non-negative integer **g**, the space of strongly Alexandrov embedded closed surfaces in **X** of genus at most **g** and constant mean curvature $\mathbf{H} \in [a, b]$ is **compact**. (Similar compactness result holds for any fixed smooth compact family of metrics on **X**.)

References

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