On groupoids and inverse semigroupoids actions

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Abstract

Definition 0.1. *Let G be a semigroupoid. G is said to be an inverse semigroupoid if for each $g \in G$ there exists an unique $g^* \in G$ such that $gg^*g = g$ and $g^*gg^* = g^*$.*

Definition 0.2. *Let G be a groupoid. We define the universal semigroupoid $S(G)$ generated by $[g]$, where $s([g]) = [s(g)]$ and $t([g]) = [t(g)]$ with the following relations:*

a) $[g^{-1}][g][h] = [g^{-1}][gh]$ when $t(g) = s(h)$.
b) $[h][g][g^{-1}] = [hg][g^{-1}]$, when $t(h) = s(g)$
c) For each $g \in G$, $[s(g)][g] = [g][t(g)] = [g]$.

Remark 0.3. *Note that $[g][g^{-1}][g] = [g]$, for each $g \in G$. Thus we have a suspicious that $S(G)$ is an inverse semigroupoid.*

Proposition 0.4. *For every $g \in G$, we define $\epsilon_g = [g][g^{-1}]$. The following statements hold.*

a) $\epsilon_g = \epsilon_g^2$, for any $g \in G$.
b) When the product $h.g$ exists we have that $[h]\epsilon_g = \epsilon_{gh}[h]$.
c) ϵ_g and ϵ_h commutes when $s(g) = s(h)$.

Proof. a) $\epsilon_h^2 = [h][h^{-1}[h][h^{-1}] = [hh^{-1}][h][h^{-1}] = [s(h)][h][h^{-1}] = [h][h^{-1}] = \epsilon_h$.

b) $[h]\epsilon_g = [h][g][g^{-1}] = [hg][g^{-1}] = [hg][g^{-1}h^{-1}][hg][g^{-1}] = [hg][g^{-1}h^{-1}][h] = \epsilon_{hg}[h]$.

c) $[g][g^{-1}]\epsilon_h = [g]\epsilon_{g^{-1}h}[g^{-1}] = \epsilon_h[g][g^{-1}] = \epsilon_h\epsilon_g$.

□

Proposition 0.5. *Every element $\alpha \in S(G)$ admits a decomposition $\alpha = \epsilon_{r_1}\epsilon_{r_2}...\epsilon_{r_n}[g]$, where $[s(r_i)] = [s(r_{i+1})]$, for each $i \in \{1,...,n-1\}$ and $[s(r_n)] = [s(g)]$. In addition we can assume that*

(i) $r_i \neq r_j$, for $i \neq j$.

(ii) $r_i \neq g$ and either $r_i \neq s(r_i)$ or $r_i \neq t(r_i)$.

Proof. Let S be the subset of $S(G)$ consisting of those α that admits a decomposition as above. Note that $n = 0$ is allowed since $[g] \in S$. We claim that S is a right ideal. In fact, let $\alpha = \epsilon_{r_1}...\epsilon_{r_n}[g]$ and $[z] \in S$ such that gz exists. Then $[g][z] = \epsilon_g[gz]$. Thus, $\alpha[z] \in S$. Let $g \in G$. Then we have that $[s(g)][g] = [g] \in S$ and we have that $S = S(G)$.

Now, if we have $\alpha = \epsilon_{r_1}...\epsilon_{r_i}....\epsilon_{r_{j-1}}\epsilon_{r_i}\epsilon_{r_{j+1}}...\epsilon_{r_n}[s]$, then by Proposition 0.4 $\epsilon_{r_1}...\epsilon_{r_i}....\epsilon_{r_{j-1}}\epsilon_{r_{j+1}}...\epsilon_{r_n}[g]$. Hence, the repeated ϵ_{r_j} can be dropped.

Next, if $\epsilon_{r_i} = \epsilon_f$, $f \in G_0$, then $\epsilon_{r_{i-1}}\epsilon_f = [r_{i-1}][r_{i-1}^{-1}][f][f] = [r_{i-1}][r_{i-1}^{-1}]$. Also, if some $r_i = s$, then $\epsilon_{r_i} = [s][s^{-1}]$ and again by Proposition 0.4 we have that $\alpha = \epsilon_{r_1}..\widehat{\epsilon_{r_i}}......\epsilon_{r_n}[s][s^{-1}][s] = \epsilon_{r_1}..\widehat{\epsilon_{r_i}}......\epsilon_{r_n}[s]$. □

Definition 0.6. *If α is written as $\alpha = \epsilon_{r_1}..\epsilon_{r_n}[g]$, where $[s(r_i)] = [s(r_{i+1}]$ and $[s(r_n)][s(g)]$, we say that α is in a standard form.*

Proposition 0.7. *For each $\alpha \in S(G)$, there exists $\gamma \in S(G)$ such that $\alpha\gamma\alpha = \alpha$ and $\gamma\alpha\gamma = \gamma$.*

Proof. Let $\alpha = \epsilon_{r_1}...\epsilon_{r_n}[g]$, where $[s(r_i)] = [s(r_{i+1})]$, $i \in \{1,...,n-1\}$ and $[s(r_n)] = [s(g)]$. Then take $\alpha^* = [g^{-1}]\epsilon_{r_n}...\epsilon_{r_1}$ and using the Proposition $\alpha\alpha^*\alpha = \epsilon_{r_1}...\epsilon_{r_n}[g][g^{-1}]\epsilon_{r_n}...\epsilon_{r_1}\epsilon_{r_1}...\epsilon_{r_n}[g] = \epsilon_{r_1}...\epsilon_{r_n}\epsilon_{r_n}...\epsilon_{r_1}\epsilon_{r_n}...\epsilon_{r_1}[g][g^{-1]}[g] = \epsilon_{r_1}...\epsilon_{r_n}[g]$. By similar methods we show that $\alpha^*\alpha\alpha^* = \alpha^*$. □

Definition 0.8. *Let G be a groupoid, H a semigroupoid and $\pi : G \to H$ a map. We say that π is a partial morphism if the following conditions are satisfied:*

a) $\pi(G_0) \subseteq H_0$

b) For each $g, h \in G$ such that the product $g.h$ and $\pi(g).\pi(h)$ exists we have that $\pi(g^{-1})\pi(g)\pi(h) = \pi(g^{-1})\pi(gh)$.

c) For each $g, h \in G$ such that $h.g$ and $\pi(h).\pi(g)$ exists we have that $\pi(h)\pi(g)\pi(g^{-1}) = \pi(hg)\pi(g^{-1})$.

d) $\pi(s(g))\pi(g) = \pi(g)\pi(s(g)) = \pi(g)$, for every $g \in G$.

Lemma 0.9. *Let G be a groupoid, H a semigroupoid and $\pi : G \to H$ a partial morphism. The following statements hold.*

(i) $\pi(g)\pi(g^{-1})$ and $\pi(g^{-1})\pi(g)$ are idempotents.

b) For each $g, h \in G$ such that $s(h) = s(g)$ we have that $\pi(g)\pi(g^{-1})\pi(h)\pi(h^{-1}) = \pi(h)\pi(h^{-1})\pi(g)\pi(g^{-1})$.

c) Let $g, h \in G$ be such that $t(h) = s(g)$. Then $\pi(h)\pi(g)\pi(g^{-1}) = \pi(hg)\pi((hg)^{-1})\pi(h)$.

Proof. The proof is similar to the proof of Proposition 0.4. □

Proposition 0.10. *Let G be a groupoid, H a semigroupoid and $\pi : G \to H$ a partial morphism, Then there exists a unique homomorphism $\bar{\pi} : S(G) \to H$ such that $\bar{\pi} \circ i = \pi$..*

Proof. We define $\pi(\epsilon_{r_1}...\epsilon_{r_n}[g] = \epsilon_{\pi(r_1)}...\epsilon_{\pi(r_n)}\pi(g)$, where $\epsilon_{\pi(r_i)} = \pi(r_i)\pi(r_i^{-1})$, for all $i \in \{1, ..., n\}$. We claim that $\pi(\epsilon_{r_1}...\epsilon_{r_n}[g]\epsilon_{m_1}...\epsilon_{m_n}[h]) = \pi(\epsilon_{r_1}...\epsilon_{r_n}[g])\pi(\epsilon_{m_1}...\epsilon_{m_n}[h])$ when $[t(g)] = [s(m_1)]$. In fact,

$$\pi(\epsilon_{r_1}...\epsilon_{r_n}[g])\pi(\epsilon_{m_1}...\epsilon_{m_n}[h]) = \epsilon_{\pi(r_1)}...\epsilon_{\pi(r_n)}\pi(g)\epsilon_{\pi(s_1)}...\epsilon_{\pi(s_n)}\pi(h) =$$
$$\epsilon_{\pi(r_1)}...\epsilon_{\pi(r_n)}\epsilon_{\pi(gm_1)}...\epsilon_{\pi(gm_n)}\pi(g)\pi(h) =$$
$$\epsilon_{\pi(r_1)}...\epsilon_{\pi(r_n)}\epsilon_{\pi(gm_1)}...\epsilon_{\pi(gm_n)}\epsilon_{\pi(g)}\pi(gh).(1)$$

Note that

$$\pi(\epsilon_{r_1}...\epsilon_{r_n}[g]\epsilon_{m_1}...\epsilon_{m_n}[h]) = \pi(\epsilon_{r_1}...\epsilon_{r_n}\epsilon_{gm_1}...\epsilon_{gm_n}\epsilon_g[gh]) =$$
$$\epsilon_{\pi(r_1)}...\epsilon_{\pi(r_n)}\epsilon_{gm_1}...\epsilon_{gm_n}\epsilon_{\pi(g)}\pi(gh).\ (2)$$

We easily see that (1) = (2). Thus, π is an homomorphism of semigroupoids. Moreover, $\bar{\pi} \circ i(g) = \pi([g]) = \pi(g)$. □

Remark 0.11. *1) For each groupoid G, we have the canonical partial morphism $i : G \to S(G)$ is defined by $i(g) = [g]$.*

2) We clearly have that the identity map $j : G \to G$ is a partial morphism. Then by Proposition 0.10 there exists a morphism $\omega : S(G) \to G$ defined by $\omega([g]) = g$.

Theorem 0.12. *$S(G)$ is an inverse semigroupoid.*

Proof. Let $\alpha = \epsilon_{r_1}...\epsilon_{r_n}[g]$ be an idempotent. Then $\omega(\alpha) = g$ and we have that g is an idempotent. Thus, $g = s(g)$ since G is a groupoid and it follows that $\alpha = \epsilon_{r_1}...\epsilon_{r_n}$. So, given α, γ such that $\alpha\gamma$ and $\gamma\alpha$ exists, we have, by Proposition 0.4 that $\alpha\gamma = \gamma\alpha$. Therefore, $S(G)$ is an inverse semigroupoid. □

Example 0.13. *Let $G = \{g, g^{-1}, s(g), t(g)\}$ with $s(g) \neq t(g)$. Then we have that G is a groupoid and the inverse semigroupoid $S(G)$ is*

$$S(G) = \{[g], [g^{-1}], [g][g^{-1}], [g^{-1}][g], [s(g)], [t(g)]\}.$$

Let A be a K-algebra, where K is a commutative ring. We set $I(A)$ as the inverse semigroup of the isomorphisms between ideals of A. We consider $(I(A), .)$ the restricted groupoid, see [Lawson] for more details.

Proposition 0.14. *Let G be a groupoid and A a K-algebra. Then a map $\theta : G \to (I(A), .)$ is a partial morphism if and only if we have a partial action of the groupoid G on A.*

Proof. Note that $\theta_{g^{-1}} \circ \theta_g \circ \theta_{g^{-1}} = \theta_{g^{-1}}$ and $\theta_g \circ \theta_{g^{-1}} \circ \theta_g = \theta_g$ which implies that $\theta_g^* = \theta_{g^{-1}}$. Denote $D_g = Im(\theta_g)$ and $D_{g^{-1}} = Im(\theta_{g^{-1}}) = dom(\theta_g)$. In this case, $\theta_g : D_{g^{-1}} \to D_g$. For each $g, h \in G$ such that $\exists g^{-1}h^{-1}$ we have $\theta_{g^{-1}}\theta_{h^{-1}}\theta_h\theta_h\theta_{h^{-1}} = \theta_{g^{-1}h^{-1}}\theta_h\theta_{h^{-1}}$. Since $dom(\theta_{g^{-1}})\theta(\theta_{h^{-1}}) = \theta_h(D_{h^{-1}} \cap D_g)$ and $Dom(\theta_{g^{-1}h^{-1}}\theta_h\theta_{h^{-1}}) = D_h \cap D_{hg}$. Thus, $\theta_h(D_{h^{-1}} \cap D_g) = D_h \cap D_{hg}$. Now, to prove that the third condition of partial actions of groupoids is standard. We claim that for each $e \in G_0$, $dom(\theta_e) = Im(\theta_e) = D_e, \theta_e = id_{D_e}$. In fact, for each $e \in G_0$, we have that $\theta_e \circ \theta_e = \theta_e$. Thus, $Dom(\theta_e) = Im(\theta_e) = D_e$ and $\theta_e = id_{D_e}$. Moreover, we have $\theta_{t(g)} \circ \theta_{g^{-1}} = \theta_{g^{-1}}$, which implies that $Dom(\theta_{t(g)} \circ \theta_{g^{-1}}) = \theta_g(D_{g^{-1}} \cap D_{t(g)}) = D_g \cap D_{t(g)} = Dom(\theta_{g^{-1}}) = D_g$ and it follows that $D_g \subseteq D_{t(g)}$.

The converse is straightforward. □

A global action of $S(G)$ os a K-algebra A is a morphism of inverse semigroupoids $\theta : S(G) \to (I(A), .)$

Theorem 0.15. *There are a bijection between the partial actions of a groupoid G on a K-algebra A and the global actions of $S(G)$ on A.*

Proof. Let γ be a global action of $S(G)$ on a K-algebra A. We claim that $(\{D_{[g]}\}_{g \in G}, \{\gamma_{[g]}\}_{g \in G})$ is a partial action of G on A. In fact, note that

$$D_{[g][h]} = D_{[g][g^{-1}][g][h]} = \gamma_{[g]}(D_{[g^{-1}][gh]} \cap D_{[g^{-1}]}) = \gamma_{[g]}(\gamma_{[g^{-1}]}(D_{[gh]} \cap D_{[g]}) = D_{[gh]} \cap D_{[g]} \quad (1)$$

when $\exists gh$. Thus, using the fact that $\gamma_{[g]} \circ \gamma_{[h]} = \gamma_{[g][h]}$ and (1) we get that $\gamma_{[g]}(D_{[g^{-1}]} \cap D_{[h]}) = D_{[g]} \cap D_{[g][h]}$. Hence, the item (ii) is satisfied. The item (i) and (iii) are easily satisfied. Moreover, we clearly have that $D_{[g]} \subseteq D_{[t(g)]}$ and for each $e \in G_0$, $\gamma_{[e]} = id_{D_{[e]}}$.

On the other hand, if we have a partial action of α of G on A, then by Proposition 0.14 we a partial morphism and by Proposition 0.10 we have a morphim between $S(G)$ on $I(A)$, that is, we have an action of $S(G)$ on A. □

Let β an action of $S(G)$ on A. We consider L the set of all finite formal sums $\sum_{s \in S(G)} a_s\delta_s$ with usual sum and multiplication rule is $(a_s\delta_s)(a_z\delta_z) =$

$a_s\beta_s(a_z)\delta_{sz}$ when $\exists sz$ and $(a_s\delta_s)(a_z\delta_z) = 0$ when sz does not exist. Let $M = \langle a\delta_s - a\delta_z : s \leq z \text{ and } a \in D_s\rangle$, that is, the ideal generated by $a\delta_s - a\delta_z$. We define the algebraic crossed product of β as $A *_\beta S(G) = L/M$.

The following lemma can be similarly proved as in Lemma 3.6 of R. Exel and F. Vieira

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Lemma 0.16. *Let β be an action of $S(G)$ on A. For $r_1, ..., r_n, g, h \in G$ we have*

(i) $\overline{a\delta_{[g][h]}} = \overline{a\delta_{gh]}}$, for $a \in D_{[g][h]}$ when $\exists gh$.

(ii) $\overline{a\delta_{\epsilon_{r_1},...,\epsilon_{r_n}[g]}} = \overline{a\delta_{[g]}}$, for $a \in D_{\epsilon_{r_1}...\epsilon_{r_n}[g]}$.

Theorem 0.17. *Let α be a partial action of the groupoid G on a K-algebra A and consider the action γ related to α as in the last theorem. Then $A *_\alpha G \simeq A *_\gamma S(G)$.*

Proof. First, we easily have that $\overline{a\delta_{[g][h]}} = \overline{a\delta_{[gh]}}$, for $a \in D_{[g][h]}$ when $\exists gh$, because of $[g][h] = [g][h][h^{-1}][h] = [gh][h^{-1}][h]$ and we have that $[g][h] \leq [gh]$. Define $\varphi : A *_\alpha G \to A *_\gamma S(G)$ by $\varphi(a\delta_g) = \overline{a\delta_{[g]}}$. Now, we easily obtain as in Theorem 3.7 of R. Exel and F. Vieira that φ is an homomorphism.

Define $\Psi : L \to A *_\alpha G$ by $\Psi(a\delta_s) = a\delta_{w(s)}$. Note that Ψ is an homomorphism and $\Psi(M) = 0$. Hence, we can extend Ψ to $\eta : A *_\alpha G \to A *_\alpha S(G)$ by $\eta(\overline{a\delta_s} = a\delta_{w(s)}$. We easily have that η and φ are inverses of each other. □