New insight in the dispersion characteristics of electrostatic waves in ultra-dense plasmas: electron degeneracy and relativistic effects

I. Kourakis1, M. McKerr1, I.S. Elkamash1,2 and F. Haas3
1 Centre for Plasma Physics, Queen’s University Belfast, BT7 1NN Northern Ireland, UK
2 Physics Department, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
3 Instituto de Física, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9500, Porto Alegre, RS, Brazil

The dispersion properties of electrostatic waves propagating in ultrahigh density plasma are investigated, from first principles, in a one-dimensional geometry. A self-consistent multispecies plasma fluid model is employed as starting point, incorporating electron degeneracy and relativistic effects. The inertia of all plasma components is retained, for rigor. Exact expressions are obtained for the oscillation frequency, and the phase and group velocity of electrostatic waves is computed. Two branches are obtained, namely an acoustic low-frequency dispersion branch and an upper (optic-like) branch: these may be interpreted as ion-acoustic and electron-plasma (Langmuir) waves, respectively, as in classical plasmas, yet bearing an explicit correction in account of relativistic and electron degeneracy effects. The electron-plasma frequency is shown to reduce significantly at high values of the density, due to the relativistic effect. The result is compared with approximate models, wherein either electrons are considered inertialess (low-frequency ionic scale) or ions are considered to be stationary (Langmuir-wave limit).

I. INTRODUCTION

The dynamics of dense plasmas in one-dimensional (1D) geometry is important, from a fundamental point of view, but also for practical applications [1]. Such systems are of relevance to the target normal sheath acceleration (TNSA) mechanism [2], during ultrahigh-intensity laser-matter interaction [3]. Applications range from dense quantum diodes [4] and electron-holes injected into quantum wires [5] to 1D fermionic Luttinger liquids [6]. Earlier studies have also focused on breather-modes in 1D semiconductor quantum wells [7] and Lagrangian structures in dense 1D plasmas [8].

The linear response of relativistically degenerate plasmas has been discussed, from first principles, in a number of earlier works. Tsytovich [8] discussed longitudinal and transverse wave propagation in a relativistic electron gas at high densities and temperatures, with particular attention to absorption due to pair production. Jancovici [10] discussed the dielectric properties of a high density relativistic electron gas at zero temperature in a positive background, using a quasi-boson formalism, which is equivalent to the random phase approximation. Hakim and Heyvaerts [11] extend the above descriptions by considering a covariant Wigner function formalism for a relativistic electron gas. (In the particular case of a fully degenerate equilibrium, the results of Ref. [10] are recovered.) We point out, to introduce our scope, here, that the above works have only focused on the electron gas dynamics (against a positive ion background), thus neglecting the ionic inertia and inevitably overlooking low-frequency ion-acoustic electrostatic waves. These have been included in later works [12, 13], where the ion component dynamics was properly taken into account.

From a fundamental point of view, the large value of the Fermi momentum in ultradense plasma configurations [14] may result in significant increase in the relativistic parameter $p_F/mc$ [15], where $p_F$ and $m$ are respectively the Fermi momentum and the mass of the charge carriers, and $c$ is the speed of light. Relativistic features and electron degeneracy effects may therefore operate hand in hand in such high density systems.

In this article, we investigate the dispersion characteristics of electrostatic waves in dense plasmas. To this end, we generalize a relativistic model for electrostatic excitations introduced recently [16, 17] by taking into account the inertia of both ion and electron component (fluids). Electron degeneracy is taken into account by an equation of state similar to Chandrasekhar’s [18], albeit adapted to a 1D geometry [19, 20]. A cold (classical) ion fluid is considered, for simplicity.

Our work here focuses on plasma dynamics by adopting a one-dimensional (1D) geometry. In a broad context, 1D plasmas have been traditionally understood within the theory of 1D Coulomb systems, since the works by Lenard [21], Dawson [22] and Feix and co-workers [23, 24]. Originally, the adoption of an 1D geometry was justified to save effort and time in numerical computation. Recently, the exact analytical expression of the partition function in 1D colloidal systems has been obtained [24], following an earlier investigation of charge screening in 1D Coulomb systems [25]. The comparison
between quantitative prediction based on 1D theoretical considerations and their 3D counterpart inevitably involves certain ambiguities in interpretation, and requires a certain subtlety in manipulating dimensional analysis (and units). From a formal point of view, a correspondence between 1D and 3D plasmas is achieved by an assembly of parallel planes of charge. In this sense, the elementary charge becomes a surface density of charge, viz. \( e = 1.6 \times 10^{-19} \, C/m^2 \).

A relativistic multi-fluid plasma model is presented in the following Section II and its physical limitations are discussed. The dispersion properties of electrostatic waves are obtained and analyzed in Section III. A critical discussion of the relation to kinetic theory for electrostatic excitations is presented in Section XV. Simplified (approximate) versions for the dispersion relation are obtained in Section VI by considering various limits. The quantum-relativistic analogues of the ion-acoustic and electron plasma (Langmuir) mode(s) are obtained and analyzed in Sections XV and VII respectively. Finally, our results are discussed and summarized in the concluding Section VIII.

II. MULTIFLUID PLASMA MODEL

We consider an electron-ion plasma consisting of ions (mass \( m_i \), charge \( +e \)) and relativistically-degenerate electrons (mass \( m_e \), charge \( -e \)) A one-dimensional (1D) geometry is adopted for simplicity. The equations of motion from the outset that magnetic field generation may be neglected in the highly relativistic limit. We assume for the electrons, providing the appropriate degeneracy to kinetic theory for electrostatic excitations is presented in Section XV. Simplified (approximate) versions for the dispersion relation are obtained in Section VI by considering various limits. The quantum-relativistic analogues of the ion-acoustic and electron plasma (Langmuir) mode(s) are obtained and analyzed in Sections XV and VII respectively. Finally, our results are discussed and summarized in the concluding Section VIII.

The above (1D ) equation of state for a degenerate relativistic electron gas, which is reminiscent of the Chandrasekhar (3D) equation of state used to describe equilibria in dense stars [18], was also derived by Chavanis in the context of white dwarf plasma equilibria [18]. The non-relativistic equation of state \( P_e = \frac{2n_e^2 c^3}{h} \left[ (1 + \xi^2)^{1/2} - \sinh^{-1} \xi \right] \) is recovered for \( \xi \ll 1 \) (i.e., in the limit \( c \to \infty \)). Here, the subscript \( 0 \) denotes the value at equilibrium and \( p_e = \frac{2n_e^2}{h} (3m_e n_0^3) \) is the pressure of state \( P_e \). In fact, one could formulate the basic equations in terms of \( N_e \), but in this case, for the sake of coherence, one would be obliged to insert gamma factors inside the equations of state - a cumbersome option in our view.

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The system is closed by Poisson’s equation:

\[
\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\epsilon_0} (\gamma_e n_e - \gamma_i n_i), \tag{6}
\]

where \(\epsilon_0\) denotes the permittivity of vacuum. Note that charge neutrality at equilibrium imposes \(n_{i0} = n_{e0} = n_{0}\), where the subscript ‘0’ indicates the equilibrium value (of the density, here).

### III. LINEAR DISPERSION RELATION (EXACT)

We shall assume small harmonic variations around equilibrium, i.e. setting \(n_{i,e} = n_{0} + \tilde{n}_{i,e} e^{i(kx - \omega_0 t)} \) (cc), \(v_{i,e} = \tilde{v}_{i,e} e^{i(kx - \omega_0 t)} \) (cc) and \(\phi = \tilde{\phi} e^{i(kx - \omega_0 t)} \) (cc), with the understanding that the tilded quantities are very small (compared to appropriate characteristic scales, e.g. \(n_{e} \ll n_{0}\) and so forth) and that they are practically constant in space and time. (The usual acronym “cc” has been used to denote the complex conjugate.) One thus obtains a linear (Cramer) system for the amplitudes, in the form:

\[
-\omega \tilde{n}_i + n_0 k \tilde{v}_i = 0, \\
-m_{i} \omega \tilde{v}_i + ek \tilde{\phi} = 0, \\
-\omega \tilde{n}_e + n_0 k \tilde{v}_e = 0, \\
-m_{e} H_0 \omega \tilde{v}_e + m_{e} \lambda k \tilde{n}_e - ek \tilde{\phi} = 0, \\
-k^2 \tilde{\phi} + \frac{e}{\epsilon_0} \tilde{n}_i - \frac{e}{\epsilon_0} \tilde{n}_e = 0
\]

\[
(7)
\]

where \(H_0 = \sqrt{1 + \xi_0^2}, \lambda = \frac{c^2 \xi_0^2}{n_0 \sqrt{1 + \xi_0^2}}\) and \(\xi_0 = \frac{h n_0}{4 m_e e} = \frac{p_{e,0}}{m_e}\).

The condition for non-trivial solutions to exist is then expressed, after an algebraic manipulation, as

\[
\omega^4 - \left[ \frac{\omega_{p,e}^2}{H_0} + \mu \right] + \frac{\xi_0^2}{1 + \xi_0^2} \omega^2 + \omega_{p,i}^2 \frac{c^2 k^2 \xi_0^2}{1 + \xi_0^2} = 0, \tag{8}
\]

where we have defined the mass ratio \(\mu = m_e/m_i \ll 1\) and the (ion or electron, respectively, for \(j = i, e\)) plasma frequency \(\omega_{p,j} = \left( \frac{c^2 n_{0,j}}{\epsilon_0 m_j} \right)^{1/2} \). It is worth recalling that \(\omega_{p,e,0}^2 = \omega_{p,i,0}^2\). Solving this general relation, we obtain two branches: see Figs. 5 and 6. The lower curve (\(\omega_-\)) corresponds to an acoustic mode (viz., \(\omega_- = 0\) for \(k = 0\)). The upper curve (\(\omega_+\)) corresponds to an optical-like mode, characterized by a frequency gap at \(k = 0\):

\[
\omega_+^2 (k = 0) = \omega_{p,e}^2 \left( \frac{1}{H_0} + \frac{m_e}{m_i} \right) = \omega_{p,e}^2 \left( \frac{1}{1 + \xi_0^2} + \frac{m_e}{m_i} \right)
\]

\[
(9)
\]

This is essentially a modified Langmuir (electron plasma) mode. The cutoff frequency \(\omega_0 = \omega_+ (k = 0)\) is slightly higher than the electron plasma frequency \(\omega_{p,e}\) in the classical limit, viz. \(\omega_0 \rightarrow \omega_{p,e} [1 + m_e/(2m_i)] \approx 1.00025 \omega_{p,e}\) in the “classical” limit \(\xi_0 \rightarrow 0\), due to the finite electron inertia being retained (see that the latter expression would reduce to \(\omega_0 \approx \omega_{p,e}\) should one neglect the electron inertia, as expected, since \(m_e \ll m_i\). Furthermore, in the weakly relativistic case \(\xi_0 \ll 1\), \(\omega_0 \approx \omega_{p,e} [1 - \xi_0^2/4 + m_e/(2m_i)]\). However, at high densities, viz. \(\xi_0 \gg 1\), the cutoff frequency (representing essentially non-propagating electrons oscillations) reduces dramatically, due to the relativistic effect: see that \(\omega_0 \approx \omega_{p,e}/\xi_0^{1/2}\) for \(\xi_0 \gg 1\). The cutoff frequency expressed in (1) may take the alternative form

\[
\omega_+^2 (k = 0) = \omega_{p,i}^2 + \omega_{p,e}^2 \sqrt{1 + \xi_0^2}, \tag{10}
\]

reflecting the fact that the electron-plasma oscillation eigenfrequency is modified due to quantum-relativistic corrections, but also due to the finite ionic inertia having been retained.

The latter relations may be simplified further, recalling the definition \(\xi_0 = \hbar n_0/(4m_e e)\), viz. substituting with \(\xi_0^2 c^2 = h^2 n_0^2/(4m_e e^2)\) where appropriate. It is straightforward to see that the wavenumber \(k\) enters the dispersion relation only via the function, say \(\Omega_k^2\), defined by:

\[
\Omega_k^2 = \frac{c^2 k^2 \xi_0^2}{1 + \xi_0^2} = \frac{(\frac{\hbar n_0}{4m_e})^2 c^2 k^2}{1 + (\frac{\hbar n_0}{4m_e})^2} = \frac{c^2 k^2}{1 + (\frac{\xi_0}{2})^2}.
\]

It was simply a matter of physical intuition to define the quantity \(c_0 = \hbar n_0/(4m_e e)\) \((= \frac{p_{e,0}}{m_e})\) \([13]\) as a characteristic speed, essentially representing here the equivalent of an acoustic (sound) speed, in the quantum-relativistic model.

The above two branches are shown in Fig. 6. We see that both frequencies (acoustic and Langmuir-like branch) increase, for higher density values.

It is worth pointing out that, for ease of algebraic manipulation, one may cast the dispersion relation in the
FIG. 2: The acoustic “lower” mode (top panel) and the Langmuir- (optical-)like “upper” mode (bottom panel) are depicted, as derived numerically from relation (8). The angular frequency $\omega$ is shown versus the wavenumber $k$ for different values of the equilibrium density $n_0$. Top to bottom, the values of $\xi_0$ are: $\xi_0 \approx 0.6, 0.3, 0.06$, respectively, for $n_0 = 10^{12}, 5 \times 10^{11}, 10^{10} \text{m}^{-1}$.

form:

$$\omega^4 - (\tilde{\omega}_{p,e}^2 + \omega_{p,i}^2 + \Omega_k^2) \omega^2 + \omega_{p,i}^2 \Omega_k^2 = 0,$$  
(12)

where all quantities were defined above except $\tilde{\omega}_{p,e}^2 = \omega_{p,e}^2 / H_0 = \omega_{p,e}^2 / \sqrt{1 + \xi_0^2}$.

IV. COMPARISON WITH KINETIC THEORY

To validate our results presented above, based on the fluid model, it would be interesting to compare with the results from the 1D relativistic Vlasov-Poisson system, which reads

$$\frac{\partial f_{e,i}}{\partial t} + \frac{p}{\Gamma_{e,m_e}} \frac{\partial f_{e,i}}{\partial x} - eE \frac{\partial f_{e,i}}{\partial p} = 0,$$  
(13)

$$\frac{\partial f_{e,i}}{\partial t} + \frac{p}{\Gamma_{m_i}} \frac{\partial f_{e,i}}{\partial x} + eE \frac{\partial f_{e,i}}{\partial p} = 0,$$  
(14)

$$\frac{\partial E}{\partial t} = \frac{e}{\xi_0} \left( \int_{-\infty}^{\infty} f_{e,i} dp - \int_{-\infty}^{\infty} f_{e} dp \right),$$  
(15)

where $f_{e,i}(x, p, t)$ denote the phase space electron and ion probability distribution functions and $\Gamma_{e,i} = \sqrt{1 + p^2/(m_{e,i} c^2)}$. As above, we neglect ion temperature effects, so that the equilibrium ion distribution function will be $f_{i}^0 = n_0 \delta(p)$. Denoting $f_{e}^0 = f_{e}^0(p)$ as the equilibrium electron distribution function and linearizing by following the usual procedure [33, 34], one derives the relativistic dispersion relation

$$1 = \frac{\omega_{pe}^2}{\omega^2} + \omega_{pe}^2 \int_{-\infty}^{\infty} \frac{dp f_{e}^{0}(p)}{\Gamma^3_e (\omega - k p / m_e c)}.$$  
(16)

We are especially concerned with the real part of the kinetic dispersion relation, so that the principal value (denoted as $\tilde{f}$) is understood in the integral in Eq. (16).

Considering the 1D degenerate normalized electronic equilibrium: $f_{e}^0 = n_0/(2p_F)$ for $|p| < p_F$; $f_{e}^0 = 0$ for $|p| > p_F$, we find

$$1 = \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pe}^2 \sqrt{1 + \xi_0^2}}{(1 + \xi_0^2) \omega^2 - \xi_0^2 \omega^2 k^2},$$  
(17)

which is exactly the same as the fluid dispersion relation, namely

$$1 = \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pe}^2}{\sqrt{1 + \xi_0^2 \omega^2 - (k^2/m_e) (dp E/df_{e})_0}}$$  
(18)

also shown in Eq. (8), as can be verified using the 1D equation of state (6). It should be noted that the full agreement observed here between the fluid dispersion relation and the (real part of) the kinetic dispersion relation is a special feature of the 1D geometry.

V. APPROXIMATE EXPRESSIONS

A. Small frequency limit

Assuming $\omega \ll (\tilde{\omega}_{p,e}^2 + \omega_{p,i}^2 + \Omega_k^2)^{1/2}$ (see definitions above), the quartic term $\omega^4$ may be neglected. The dispersion relation then simplifies to:

$$\omega_{\text{approx}}^2 \approx \frac{\omega_{p,i}^2 \Omega_k^2}{\tilde{\omega}_{p,e}^2 + \omega_{p,i}^2 + \Omega_k^2} = \frac{\omega_{p,i}^2 \xi_0^2 k^2}{1 + \left( \frac{\xi_0^2}{4} \right)^2 \tilde{\omega}_{p,e}^2 + \omega_{p,i}^2 + \frac{\xi_0^2 k^2}{1 + \left( \frac{\xi_0^2}{4} \right)^2}}.$$  
(19)

This is essentially the analogue of the ion-acoustic mode in the quantum-relativistic model, where the thermal pressure of the electrons (reflected in the classical textbook [34] definition of the sound speed, $c_s = (k_B T_e / m_e)^{1/2}$, which is related to the electron temperature $T_e$; $k_B$ being the Boltzmann constant) is here substituted by the Fermi-energy related pressure (manifested in the above expression for the characteristic speed $c_0 = (E_{F,e} / m_i)^{1/2}$).

B. High frequency limit

It can be verified that, for all density values high enough to be of relevance in our model, the coefficient of
the quadratic term in the bi-quadratic polynomial equation (12) above (and in all of its preceding alternative forms) exceeds the constant term by far. It can thus be shown that the upper (Langmuir-type) frequency mode is approximately given by:

\[
\omega_{\text{IA}}^2 = \frac{\omega_{\text{pe}}^2}{1 + \xi_0^2} \approx \frac{\omega_{\text{pi}}^2 + \omega_{\text{pe}}^2 + c^2k^2\xi_0^2}{1 + \xi_0^2} + \frac{\omega_{\text{pi}}^2}{1 + \xi_0^2} \left( \frac{c^2k^2 + \omega_{\text{pe}}^2}{\xi_0^2} + \sqrt{1 + \xi_0^2} \omega_{\text{pe}}^2 + \omega_{\text{pi}}^2 \right).
\]  

For small values of the wavenumber \( k \), this expression can be approximated as:

\[
\omega_{\text{approx}}^2 \approx \frac{\omega_{\text{pe}}^2}{1 + \xi_0^2} + \omega_{\text{pi}}^2 + \xi_0^2 \frac{\omega_{\text{pe}}^2 + c^2k^2\xi_0^2}{1 + \xi_0^2} \approx \omega_{\text{pi}}^2 + C_0^2 k^2.
\]  

This is visibly a parabolic function in the wavenumber \( k \), which is analogous to the known textbook functional form for (classical) Langmuir waves, viz. \( \omega^2 = \omega_{\text{pe}}^2 + c_0^2k^2 \), where \( c_0 = (k_B T_e/m_e) \) [30], where both frequency cutoff and characteristic speed are hereby modified due to quantum relativistic corrections, incorporated in our model.

Note that the frequency cutoff \( \omega_0 \) satisfies Eq. (13) above, as expected. Furthermore, the characteristic speed \( C_0 \):

\[
C_0 = \left( \frac{c_0^2}{1 + \xi_0^2} \right)^{1/2} \left( \frac{\omega_{\text{pe}}^2}{\omega_{\text{pe}}^2 + \sqrt{1 + \xi_0^2} \omega_{\text{pi}}^2} \right)^{1/2} c
\]

\[
= \left( \frac{\xi_0^2}{1 + \xi_0^2} \right)^{1/2} \left( \frac{1}{1 + \frac{m_i}{m_e}} \sqrt{1 + \xi_0^2} \right)^{1/2} c
\]

(22)

behaves as

\[
C_0 \approx \left( \frac{1}{1 + \frac{m_i}{m_e}} \right)^{1/2} \xi_0 \sim \frac{h \nu_0}{4m_e} = \frac{p_{F,0}}{m_e},
\]

in the small density (\( \xi_0 \ll 1 \)) limit. This expression coincides with the pseudo-"sound" speed defined above. On the other hand, for large values of the density (viz. \( 1 \ll \xi_0 \ll m_i/m_e \)), the characteristic speed \( C_0 \) approaches \( c \), as inferred from (22).

It should be noted here, for rigor, that our electrostatic model breaks down for ultrahigh-densities, where quantum electrodynamic effects (neglected here) are dominant.

### VI. QUANTUM-RELATIVISTIC ION-ACOUSTIC WAVES

The conventional method to describe ion-acoustic waves is by neglecting the electron inertia from the outset. In the classical picture, one considers an ion fluid, surrounded by an electron cloud which, given the large mass disparity between electrons and ions, can be considered to be at thermal equilibrium, viz. \( n_e = n_{e,0} \exp(\nu_0/(k_B T_e)) \). The classical, non-relativistic dispersion relation thus obtained reads

\[
\omega^2 = \omega_{\text{pi}}^2 \left( k^2 + \lambda_{\text{scr}}^2 \right).
\]  

(23)

where \( k_D^2 = e^2 n_0/(\epsilon_0 k_B T_e) \) is the inverse Debye length (square) [30].

In our case, this physical situation can be reproduced by neglecting the electron inertia, i.e. by ignoring the convective term in the left-hand of the momentum equation (11) for the electron fluid. It is a matter of straightforward algebra [31] to obtain the "ion-acoustic" wave dispersion relation:

\[
\omega_{\text{IA}}^2 = \omega_{\text{pi}}^2 \frac{k^2}{k^2 + \lambda_{\text{scr}}^2},
\]

(24)

where we have defined the charge screening length as \( \lambda_{\text{scr}} = \sqrt{\frac{2\alpha_0 E_{F,e}}{n_e e^2}} \), \( \xi_0^2 = \frac{\omega_{\text{pe}}^2}{\omega_{\text{pi}}^2} \), \( \sqrt{1 + \xi_0^2} \) and \( \frac{\epsilon_0 \nu_0^2}{4m_e} \) (the classical, i.e. non-relativistic definition of the Fermi energy \( E_{F,e} = p_{F,e}^2/2m_e \) was adopted here). The last relation for the ion-acoustic frequency coincides with the one derived in Ref. [92].

It may be appropriate to compare the ion-acoustic dispersion relation (13) to the approximate relation (14) obtained earlier. The two expressions are close for small equilibrium density, but begin to diverge as a higher density results in a larger relativistic electron mass, as expressed via the factor \( m_e \sqrt{1 + \xi_0^2} \). This relativistic mass may not, properly speaking, compare to the ion mass for reasonable densities (\( \nu_0 \lesssim 10^{13} \text{m}^{-1} \text{in 1D} \)), but it certainly entails a definite variation: see Fig. 3.

We have derived three versions of the low-frequency (ion-acoustic) mode, namely: (i) the lower root \( \omega_- \) of the exact relation (3), deriving from the two-fluid model, (ii) the approximate form \( \omega_{\text{approx}} \) given by (13), and (iii) \( \omega_{\text{IA}} \), given by (23), which was derived from the ion-fluid model (neglecting electron inertia). These three curves are shown in Fig. 3, for an indicative density.
value \( n_0 = 10^{11} \text{m}^{-3} \) (roughly equal to the cubic root of a density characteristic of the interior of a white dwarf star \([53]\)). We see that the exact dispersion relation and its approximation are almost identical, while \( \omega_{IA} \) lies just above the other two: taking electron inertia into account therefore slightly increases the ion-acoustic frequency.

\[
\omega^2 = \omega_{p,e}^2 + 3v_{th}^2 k^2, \tag{26}
\]
onc

upon formally substituting the thermal velocity with \( v_0 \).

It appears imposed to compare the latter dispersion relation with the approximate solution \([22]\) of the exact dispersion relation \([8]\): In fact, \([22]\) approaches \([22]\) as one cancels the ion plasma frequency or, alternatively, as the equilibrium density increases, since the static ion assumption gains validity as electron Fermi energy increases.

Summarizing our findings for the high-frequency (Langmuir-like) mode, we have derived three versions of the dispersion relation, namely: (i) the upper branch \( \omega_+ \) of the exact relation \([8]\), deriving from the two-fluid model, (ii) the approximate form \( \omega_{+, \text{approx}} \) given by \([21]\), and (iii) \( \omega_{EP} \), given by \([24]\), which was derived from the electron-fluid model (considering an infinite ion inertia). These three curves are shown in Fig. 3 (for an indicative density value \( n_0 = 10^{11} \text{m}^{-3} \), roughly equal to the cubic root of a density characteristic of the interior of a white dwarf star \([53]\)). We see that the exact dispersion relation and its approximation are almost identical, while \( \omega_{EP} \) lies below the other two: taking finite ion inertia into account therefore slightly increases the electron-plasma frequency.

VII. QUANTUM-RELATIVISTIC ELECTRON PLASMA (LANGMUIR) WAVES

Let us now consider a different physical limit. Suppose that the ions are so slow, relative to the electrons, as to be effectively static. It is then safe to neglect the ion-fluid equations \([11, 2]\), assuming that \( n_i = \text{constant} \). It is straightforward, upon linearization, to find the electron plasma dispersion relation:

\[
\omega_{EP}^2 = \omega_{p,e}^2 + \omega_{\text{rel}}^2 k^2,
\]

\[
\equiv \omega_{p,e}^2 \sqrt{1 + \frac{2}{\omega_{\text{rel}}^2} + \frac{1}{2} v_0^2 k^2.} \tag{25}
\]

One easily recognizes in the first term in the RHS the expression recovered from the relativistic Langmuir frequency \([4] \) or \([10]\) above, upon setting the ion mass to infinity. Furthermore, the characteristic speed \( V_0 = \left( \frac{\omega_{\text{rel}}^2}{1 + \xi_0} \right)^{1/2} c = \left( \frac{(\hbar n_i/4m_e c)^2}{1 + (\hbar n_i/4m_e c)^2} \right)^{1/2} c = \left[ \frac{c^2}{1 + (\hbar n_i/4m_e c)^2} \right]^{1/2} \)

defined above is related to the Fermi speed (the characteristic speed \( c_0 \) was defined above). It is straightforward to show that \([23]\) above is exactly recovered from the exact 2-fluid dispersion relation \([12]\) above – recalling definition \([11]\) – upon setting \( \omega_{p,i} \) to zero (i.e., in the infinite ion mass limit), as intuitively expected. This mirrors the structure of the classical, non-relativistic equivalent:

where the static ion assumption gains validity as electron Fermi energy increases.

VIII. DISCUSSION

We have derived the new dispersion relation \([8]\) for ultradense (quantum) plasmas modeled by introducing a relativistic multi-fluid model, taking into account the finite inertia of both electron and ion fluids. In other words, no simplifying hypothesis was adopted, e.g. of a vanishing electron inertia (for ionic excitations) or, reversely, of infinite inertia (stationary state) for ions (as regards high-frequency electron plasma waves). This new dispersion relation, bearing the form of a quartic polynomial, reduces to the approximate dispersion relations \([22]\) and \([24]\), which agree with expressions earlier derived for ion-acoustic and electron plasma waves, respectively \([13, 17]\). This agreement was confirmed numerically for both low- and high-frequency branches. This agreement is also reflected in the phase speed and group velocity, as shown in Figures 1 and 9. In the former (Fig. 9), the exact relation \([8]\), and the approximations \([13]\) and \([24]\) are seen to be practically indistinguishable. In a similar manner, in Fig. 1, the exact relation \([8]\), and the approximations \([13]\) and \([24]\) also overlap almost perfectly. Recall that for a modulated envelope wave, the
carrier waves move at phase speed while the wave envelope moves at the group speed.

FIG. 4: Top panel: The electron plasma (Langmuir) mode is plotted here, as obtained from three different functional relations: the exact (two-fluid) dispersion relation (1) (black, dashed), the approximation (6) to this (green) and the dispersion relation (7) (red, bottom curve) derived from the electron-fluid model (i.e. for an infinite ion inertia). Bottom panel: closeup view for small $k$. We have taken $n_0 = 10^{13} \text{m}^{-1}$ for the plots, as in Fig. 2.

It was shown earlier that the relativistically-degenerate system only begins to distinguish itself from the non-relativistic version at high density values ($n_0 \gtrsim 10^{11} \text{m}^{-1}$); recall the dependence of the dispersion characteristics on the density $n_0$ via the variable $\xi_0$; also, note Fig. 2. To investigate the dependence of the propagation characteristics on the electron (number) density, we have depicted the group- and phase speed of the lower (ion-acoustic) and upper (optical-like, electron plasma) modes, for various values of the number density, in Figures 5 and 8, respectively. We see that both group- and phase speed values differ substantially between successive values of the density, and in fact increase for higher density value (this is true for both dispersion modes, i.e. in both of the latter figures).

FIG. 5: The phase speed $v_{ph} = \omega/k$ and the group velocity $v_g = d\omega/dk$ for the Langmuir dispersion curve. Note that both curves tend to a constant asymptotic value for large wavenumber (short wavelength) values.

FIG. 6: The phase speed $v_{ph} = \omega/k$ and group velocity $v_g = d\omega/dk$ for the Langmuir dispersion curve. Note that both curves tend to a constant asymptotic value for large wavenumber (short wavelength) values.

FIG. 7: The group velocity $v_g$ and the phase speed $v_{ph}$ of the low-frequency (ion-acoustic) dispersion mode is shown, for different values of the electron number density. The thick (red), thin (green) and dashed (black) curves are as described in the inset label. For each pair of same color/style, the top curve corresponds to the phase speed, while the bottom one to the group velocity.

FIG. 8: The group velocity $v_g$ and the phase speed $v_{ph}$ of the high-frequency (electron plasma) dispersion mode is shown, for different values of the electron number density. The thick (red), thin (green) and dashed (black) curves are as described in the inset label. For each pair of same color/style, the top curve corresponds to the phase speed, while the bottom one to the group velocity.
The density dependence discussed above is intuitively expected, if one recalls the expression for the Fermi energy: \( E_F = m_e c^2 \left( \sqrt{1 + \frac{p_F^2}{m_e^2 c^2}} - 1 \right) \), to be compared to the non-relativistic working expression \( E_F = \frac{p_F^2}{2 m_e} \). A comparison of the resulting numerical value of the Fermi energy, which also influences the “Debye” screening length \( \lambda_D = \sqrt{\frac{2 e^2 n_F}{\epsilon \varepsilon_0}} \), is given in Fig. 3. We see that deviation from the non-relativistic regime requires high density values. For low densities, in other words, the non-relativistic formulation is sufficient.

The relativistically-degenerate theory is valid under certain assumptions. The temperature and density must be of such a magnitude that the relativistic Fermi energy exceeds both the non-relativistic Fermi energy and the thermal energy. Figures 9 and 10 show how the exact dispersion relation (8) compares with the classical, non-relativistic equivalent under conditions which might be found in the interior of a white dwarf (\( n_{LD} = 10^{11} m^{-1}, T = 10^7 K \)). In the above plots, the relativistic quantum relations approach the non-relativistic classical equivalents as the density is reduced (for a given temperature). Using a simple, heuristic argument, one may approximate the density value for which the two descriptions are approximately equal by depicting the boundary line \( (E_F = k_B T_e) \) – see Fig. 12 – which delimits the range of temperatures and densities for which the quantum framework is valid (i.e., in the grey area in this plot).

It may be noted that, according to the general theory of 1D Coulomb systems (22), the “graininess” parameter in our case is defined as \( q = 1 / (\omega_0 \lambda_D) \), which is much smaller than unity as can be seen from Figure 11. Therefore, the systems treated here are characterized by weak interactions (collisions), so that the relativistic Vlasov equation is applicable. As shown in Section IV,
the fluid-theoretical results for linear waves are equivalent to the obtained from kinetic-theoretical considerations (provided that the imaginary part of the dielectric function is negligible). In the high frequency limit, this amounts to a sufficiently large phase velocity to avoid wave-particle resonances, which is obeyed throughout.

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