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The dispersion relations of dispersive Alfvén waves in superthermal plasmas

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Abstract The effects of velocity distribution functions (VDFs) that exhibit a power law dependence on the high-energy tail have been the subject of intense research by the space plasma community. Such functions, known as *superthermal* or *kappa* distributions, have been found to provide a better fitting to the VDF measured by several spacecraft in the plasma environment of the solar wind. In the literature, the general treatment for waves excited by (bi-)Maxwellian plasmas is well established. However, for kappa distributions, either isotropic or anisotropic, the wave characteristics have been studied mostly for the limiting cases of purely parallel or perpendicular propagation. Contributions for the general case of obliquely propagating waves have been scarcely reported so far. In this work we introduce a mathematical formalism that provides expressions for the dielectric tensor components and subsequent dispersion relations for oblique propagating dispersive Alfvén waves (DAWs) resulting from a kappa VDF. We employ an isotropic distribution, but the methods used here can be easily applied to more general anisotropic distributions, such as the bi-kappa or product-bi-kappa. The effect of the kappa index and thermal corrections on the dispersion relations of DAW is discussed.

1. Introduction

Great interest and effort have been dedicated in recent years on the study of the properties of plasmas composed of particle species described by the so-called superthermal or kappa velocity distribution functions. These velocity distribution functions (VDFs) distinguish themselves from the usual Maxwell-Boltzmann distribution by the existence of a tail (the high-velocity portion of the VDF) that decays according to a power law dependence on the velocity, instead of the Gaussian profile characteristic of the Maxwellian distribution.

Several space plasma environments, such as planetary magnetospheres, the solar corona, or the solar wind, are composed of particle species whose VDFs are much better described by kappa or by combinations of kappa distributions, instead of any possible combination of Maxwellian distributions. As a consequence, the physical processes that occur inside these environments are also strongly influenced by the particular profile of the superthermal distributions and they can behave quite differently from what would be expected from a quasi-thermal plasma, i.e., modeled by Maxwell-Boltzmann VDF.

Due to their adequacy to model space plasma environments, the number of published communications employing kappa distributions has been growing exponentially in the last years. An assessment on the importance and impact of the subject on the field of plasma physics has been recently made by *Livadiotis and McComas* [2009, 2011, 2013].

From a conceptual point of view, the determination of the physical postulates and mechanisms responsible for the ubiquitous nonthermal kappa distributions in space plasmas has demanded (and received) intense theoretical work. One of the earliest theories in this regard was proposed by *Scudder and Olbert* [1979], who showed that during the expansion of the solar wind inside the heliosphere the combined effects of inhomogeneity with the reduced Coulomb collisions is able to bifurcate the electron VDF into a combination of thermal and superthermal populations.

More recently, other theories have been put forth, which assume that the observed VDFs are the result of dynamical processes occurring in essentially noncollisional systems characterized by long-range correlations and in quasi-equilibrium with a turbulent wave field. The scope of these formulations ranges from revisions and extensions of fundamental postulates of statistical mechanics to sophisticated analyses of the nonlinear and turbulent evolution of nonequilibrium plasmas. One such interpretation of the origin of kappa VDF is based on the principle of nonadditive entropy proposed by *Tsallis* [1988, 2009]. According to

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this proposal, many-particle physical systems that evolve subjected to long-scale correlations and nonlinear effects can reach a quasi-stationary state in which its VDF cannot be described by a Maxwell-Boltzmann distribution, characteristic of systems governed by short-range (mostly) binary collisions. The consequences and implications of Tsallis' entropic principle on space plasma physics is thoroughly discussed by *Livadiotis and McComas* [2009, 2011, 2013].

An alternative approach has been followed by R. Treumann and collaborators. Based on the assumption that a turbulent system with long-scale correlations can reach a transient quasi-equilibrium state different from a thermal collisional system, an heuristic expression for the collision integral resulting from the Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy of kinetic equations was proposed [*Treumann*, 1999a, 1999b; *Treumann et al.*, 2004]. The collision integral thus obtained enabled the authors to formulate a thermodynamic framework for an equilibrium turbulent system, providing quantities such as the entropy, number density, internal energy, temperature, and, in particular, a particle distribution function with the observed characteristics of space VDFs. More recently, *Treumann and Jaroschek* [2008] reinterpreted the formulation using Gibbsian theory and proposed a new expression for the probability distribution of physical states assuming that any two subsystems are not statistically independent but instead correlated. The generalized expression for the Gibbs function has a Lorentzian form, with the κ index serving as an ordering parameter. Among the consequences, the resulting entropy is again nonadditive.

A detail that is common to the formulations cited above is that the value of the κ (or q) index is a given quantity to be determined either from observational data or from a specific physical mechanism. An example of the latter is the work by *Hasegawa et al.* [1985], where the particle distribution function is determined from a Fokker-Planck equation that models a plasma in a thermal bath with a turbulent, superthermal Coulomb field of longitudinal waves. The equilibrium solution provided both a kappa-like VDF and the value of κ in terms of the longitudinal dielectric constant.

A more recent theory has been proposed [see *Yoon et al.*, 2012; *Yoon*, 2014, and references therein], which employs the Klimontovich-Dupree version of the weak turbulence theory in order to predict the κ-electron VDF as a consequence of the self-consistent turbulent equilibrium between particles and the electrostatic Langmuir field. According to the theory, a kappa-like VDF arises as one of the possible asymptotic, steady state ($\partial/\partial t \to 0$) solutions of the particle and wave kinetic equations when in the latter only the spontaneous and induced emission processes are retained. As a bonus, the superthermal spectral intensity of Langmuir waves is also obtained. Then, by balancing the spontaneous and induced scattering terms in the wave kinetic equation, the value of the κ index is determined. Finally, *Yoon* [2014] discusses the total spectral intensity of Langmuir waves, including the contribution of the nonresonant quasi-thermal oscillations as well as from the eigenmode.

Notwithstanding the mentioned nonthermal equilibrium formalisms for space plasmas, their observed VDF is almost never found in a true state of quasi-equilibrium. The observed distributions display an abundance of nonthermal features such as beams, temperature anisotropies (relative to the ambient magnetic field), and loss cones, and the effects of these sources of free energy must be somehow accounted for by a statistical framework of turbulent systems.

One important example is the temperature anisotropy displayed by solar wind protons and electrons on the vicinity of the Earth's foreshock [*Marsch*, 2006]. The proton VDF show nonthermal features such as a very anisotropic core, an extended high-energy tail, and a beam population, aligned to the local magnetic field and separated from the core by speeds on the order of the Alfvén speed. The existence of these nonthermal characteristics implies that the VDF contains a large amount of free energy that can be used to excite the Alfvén waves present in the solar wind. Conversely, the wave-particle interaction is important to determine the shape of the VDF, as in the case of obliquely propagating dispersive Alfvén waves (DAWs), which are relevant for the particle acceleration processes in the Earth's magnetosphere.

In the literature, the general treatment for waves excited by (bi-)Maxwellian plasmas is well established. However, for kappa distributions, either isotropic or anisotropic, the wave characteristics have been studied mostly for the limiting cases of purely parallel or perpendicular propagation. For an account about the state of the art on the subject, as well as for a comprehensive list of publications, the reader may refer to *Lazar and Poedts* [2014], *Lazar et al.* [2012], and *Pierrard and Lazar* [2010].

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Contributions for the general case of obliquely propagating waves have been scarcely reported so far. A general formalism was proposed by *Summers et al.* [1994] for a particular model of kappa VDF. In their work, however, the results presented were mostly obtained by numerical integration. A similar approach was more recently adopted by *Basu* [2009] and *Liu et al.* [2014] in order to obtain the dispersion relations of dispersive Alfvén waves. As we shall verify in the following sections, these authors obtained their results from power series expansions that do not consider all possible mathematical properties of the integrals involved in the derivation of the dielectric tensor components for a superthermal VDF.

A general treatment for electrostatic waves in a superthermal plasma was also obtained by *Mace* [2003] by means of a Gordeyev integral technique. However, applications of this formalism for nonparallel propagation have been restricted to the perpendicular direction [see *Nsengiyumva et al.*, 2013, and references therein].

The absence of a general treatment prevents a complete analysis of the wave-particle interaction in superthermal plasmas, since some instabilities, such as the firehose instability, can operate simultaneously both in the parallel and oblique directions. In this work, we obtain expressions for the dielectric tensor components and subsequent dispersion relations for oblique DAW resulting from a kappa VDF. We employ an isotropic distribution, but the methods used here can be easily applied to more general anisotropic distributions, such as the bi-kappa or product-bi-kappa. The effect of the kappa index and thermal corrections on the dispersion relations of the obliquely propagating DAW is discussed.

The plan of the paper is as follows. In section 2 the model velocity distribution function employed in the present work is introduced and its relation to some commonly used VDFs is established. Section 3 presents some important mathematical properties of the special functions that appear in the derivation of the dielectric tensor for a superthermal plasma. These properties are then employed in section 4 where the dispersion equation and dispersion relations of dispersive Alfvén waves propagating in kappa plasmas are obtained. In section 5 some numerical results are presented and commented upon, and finally, in section 6 we make our final observations.

2. The Distribution Function

We will adopt the following model for an isotropic superthermal (or kappa) velocity distribution function (VDF),

$$f_{w,\alpha}(\boldsymbol{v}) = \frac{1}{(\pi\kappa w^2)^{3/2}} \frac{\Gamma(\sigma)}{\Gamma(\sigma - 3/2)} \left(1 + \frac{v^2}{\kappa w^2}\right)^{-\sigma}. \tag{1}$$

In (1), the vector $\boldsymbol{v}$ is the particle's velocity, $\sigma = \kappa + \alpha$ ($\sigma > 3/2$), where κ is the kappa index, α is a free parameter, and $w = w(\kappa)$ is another parameter with the same physical dimension and meaning as the particle's thermal velocity. This parameter can depend on the κ index, with the proviso that it reduces to the Maxwellian thermal velocity on the limit $\kappa \to \infty$, as we explain below. Also, $\Gamma(z)$ is the gamma function [*Askey and Roy*, 2010].

Using Stirling's formula [*Askey and Roy*, 2010] and the exponential limit

$$\lim_{\kappa\to\infty} \left(1 + \frac{y^2}{\kappa}\right)^{-\kappa} = e^{-y^2}, \tag{2}$$

the reader can easily demonstrate that on the limit $\kappa \to \infty$ the function $f_{w,\alpha}(\boldsymbol{v})$ reduces to the well-known Maxwell-Boltzmann distribution of velocities, i.e.,

$$\lim_{\kappa\to\infty} f_{w,\alpha}(\boldsymbol{v}) = f_M(\boldsymbol{v}) = \frac{e^{-v^2/v_T^2}}{\pi^{3/2} v_T^3}, \tag{3}$$

where $v_T^2 = 2T/m$ and m are the particle's thermal velocity and its mass, respectively, and T is the classical thermodynamic temperature, defined in the Boltzmann-Gibbs (BG) statistical mechanics. This limiting process will be referred to in this work as the Maxwellian limit. The above result also implies that the parameter w should be such that $w \to v_T$ on the Maxwellian limit.

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Another important parameter frequently considered on discussions about kappa plasmas is the second moment $\left\langle v^2 \right\rangle$ of the distribution, which is a measure of the velocity dispersion of the particles and can be used to define the kinetic temperature T_K as

$$T_K = \frac{1}{3} m \left\langle v^2 \right\rangle = \frac{1}{3} m \int d^3v\, v^2 f_{w,\alpha}(\mathbf{v}).$$

Using the distribution (1), we find that T_K is given by

$$T_K = \frac{\kappa}{\sigma - 5/2} \left(\frac{1}{2} m w^2 \right), \left(\sigma > \frac{5}{2} \right), \tag{4}$$

which reduces to the usual result

$$T_K = \frac{1}{2} m v_T^2 = T$$

on the Maxwellian limit. One should notice the more restrictive condition applied to σ, needed so that the measure of kinetic temperature is meaningful.

The particular form of the VDF given by (1) was chosen in order to reproduce most of the models employed recently. One only needs to set adequate values and expressions for the parameters α and w.

For instance, the by far most frequently adopted model was defined by *Summers and Thorne* [1991], hereafter called the ST91 model. The distribution (1) reduces to the ST91 model by choosing

$$\alpha = 1 \text{ and } w^2 = \frac{\kappa - 3/2}{\kappa} v_\kappa^2, \text{where } v_\kappa^2 = \frac{2T_\kappa}{m}$$

and where T_κ is called the physical temperature. Notice that for this model, $\kappa > 3/2$. The ST91 model is a development of the original power law VDF proposed by *Vasyliunas* [1968] in order to model the low-energy electrons observed in the magnetosphere.

One interesting result that is obtained with the ST91 model is given by the definition (4) for T_K. In this case, $T_K = T_\kappa$, i.e., the kinetic temperature is the same as the parameter T_κ. This is one of the arguments that have been put forth recently [*Livadiotis and McComas*, 2009, 2011, 2013] proposing the ST91 model as the correct one to describe the distribution of velocities for superthermal plasmas.

Another model adopted on the literature was proposed by *Leubner* [2002] and will be called the Le02 model. It is reproduced by (1) after choosing

$$\alpha = 0 \text{ and } w^2 = v_T^2 = \frac{2T}{m}.$$

For the Le02 model, the kinetic temperature, given by (4), becomes

$$T_K = \frac{\kappa}{\kappa - 5/2} T, \left(\kappa > \frac{5}{2} \right),$$

and thus, it does not equal the physical temperature, which in this model is precisely the BG definition. The reader is referred to *Leubner* [2002, 2004] for a discussion on the physical differences between the ST91 and Le02 models.

Recently, a debate ensued on the correct form for the VDF from the statistical mechanics point of view [*Hellberg et al.*, 2009; *Hau et al.*, 2009; *Livadiotis and McComas*, 2009, 2011]. In this work, we will not contribute for this debate, since the focus here is on the proposal for a mathematical formulation destined to describe the propagation and the linear wave-particle interactions in superthermal plasmas. We will only point out here that if the Le02 model is modified by choosing $\alpha = 5/2$ in (1), in which case $\kappa > 0$, then it would also predict $T_K = T$, that is, the kinetic temperature would equal the physical temperature.

Finally, a third model was recently employed by *Benson et al.* [2013] in order to study the VDF of electrons in the low-latitude region of the magnetotail. Their model is reproduced by (1) by setting $\alpha = 1$ and $w = v_T$, that is, by using the BG temperature in order to measure the thermal spread of velocities. Accordingly, the electronic kinetic temperature evaluated by the authors is $T_K = \kappa T/(\kappa - 3/2)$.

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3. Special Functions for Kappa Plasmas

In this section, we will define and present some mathematical properties and discussion about two special functions that appear on the study of wave propagation in superthermal plasmas. The first function to be considered is the kappa generalization of the plasma gyroradius function, which appears for nonparallel propagation of waves in thermal plasmas, when Larmor radius effects are important. The second function is a generalization of the modified plasma dispersion function proposed by *Summers and Thorne* [1991] and *Mace and Hellberg* [1995], which reduces to the well-known plasma dispersion function (or Fried and Conte function) on the Maxwellian limit.

3.1. The Kappa Gyroradius Function

When considering electrostatic and/or electromagnetic waves propagating in thermal plasmas, one invariably needs to evaluate the components of the dielectric or susceptibility tensor in order to describe the waves' propagation characteristics and their interaction with the plasma particles. When the propagation direction is nonparallel, finite gyroradius effects need to be included in the description of the dielectric tensor components.

Using the $f_{w,\alpha}(\boldsymbol{v})$ distribution, we found that the gyroradius effects can be described by the following special function:

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = 2\frac{\kappa^{-\beta-3/2}\Gamma(\lambda)}{\Gamma(\sigma-3/2)}\int_0^\infty dx\,\frac{xJ_n^2(zx)}{(1+x^2/\kappa)^\lambda}, \tag{5}$$

where κ is the usual κ index, $\sigma = \kappa + \alpha$ as in (1), $\lambda = \sigma + \beta$, with β being a free parameter, and $J_n(y)$ is the Bessel function of the first kind and of order n [*Olver and Maximon*, 2010]. Here $n = 0, \pm1, \pm2, \ldots$ is the harmonic number.

Using Stirling's formula and identity (2), we obtain the Maxwellian limit

$$\lim_{\kappa\to\infty}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = 2\int_0^\infty dx\,xJ_n^2(zx)e^{-x^2} = \mathscr{H}_n(\mu), \tag{6}$$

where $\mu = z^2/2$ and $\mathscr{H}_n(y) = e^{-y}I_n(y)$, with $I_n(y)$ being the modified Bessel function of the first kind [*Olver and Maximon*, 2010]. This is a well-known result for thermal waves in Maxwellian plasmas, and the reader can refer to *Brambilla* [1998] for a detailed derivation.

The parameter z in (5) is proportional to the particle's gyroradius, and when this is small, one would naturally wish to expand the Bessel function in a power series, in order to compute the small Larmor radius contributions to wave propagation.

This task can be accomplished using the identity [*Olver and Maximon*, 2010]

$$\begin{aligned} J_n^2(y) &= \left(\frac{y}{2}\right)^{2|n|}\sum_{k=0}^\infty \frac{(-)^k(2|n|+k+1)_k(y/2)^{2k}}{k!\left[\Gamma(|n|+k+1)\right]^2} \\ &\simeq \frac{1}{(|n|!)^2}\left(\frac{y}{2}\right)^{2|n|}, \end{aligned}$$

where the last result is the lowest-order term of the series. Inserting this expression into (5), one readily obtains, to lowest order,

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) \simeq \frac{\kappa^{-\beta-3/2}\kappa^{|n|+1}\Gamma(\lambda-|n|-1)}{\Gamma(\sigma-3/2)|n|!}\left(\frac{z}{2}\right)^{2n}.$$

Higher-order terms could be included from the series above, with the outcome that the term of order $m > 0$ will be proportional to $\Gamma(\lambda-|n|-m-1)$.

Even though such procedure could be thought justifiable for the case of very small Larmor radius, the obtained result would be incorrect for the following reasons:

1. The integral containing the lowest-order term only exists if $\lambda > |n|+1$. Higher-order terms will impose an even more restrictive condition on the minimal value of the κ index, and this condition ties κ_{min} to the harmonic number. This is an undesirable restriction on the allowable values of the κ index, since it has been measured to be as low as $\kappa \simeq 2$ both in the solar wind and in the magnetosphere [*Maksimovic et al.*, 1997; *Štverák et al.*, 2009; *Benson et al.*, 2013].

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2. The result implies that $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ could always be represented by a power series, which is not true. As we shall see below, for certain values of the λ index the function has a logarithmic behavior.
3. Even when $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ is expressible as a power series, the lowest-order terms contain a contribution proportional to a power that is not an integer, in general; these terms will also be lacking from the simplifying approach described above.

On the other hand, the asymptotic behavior of the Bessel functions for large argument is given by [*Olver and Maximon*, 2010].

$$J_n^2(y) \sim \frac{2}{\pi y}\cos^2\left(y - \frac{1}{2}n\pi - \frac{1}{4}\pi\right) \leqslant \frac{2}{\pi y}.$$

In this limit, the behavior of $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ would be obtained from (5) as

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) \sim 4\frac{\kappa^{-\beta-3/2}\Gamma(\lambda)}{\Gamma(\sigma-3/2)\,\pi z}\int^{\infty}\frac{\mathrm{d}x}{(1+x^2/\kappa)^{\lambda}} = 2\frac{\kappa^{-\beta-1}\Gamma(\lambda-1/2)}{\Gamma(\sigma-3/2)\sqrt{\pi}z},$$

which is only subjected to the condition $\lambda > 1/2$, independent of n.

Hence, although, in general, the integral in (5) will not converge if one employs a power series expansion for the Bessel function, when its closed form is considered, the integral always exists for the physically relevant values of the λ index. Therefore, in order to obtain the correct contribution due to the particle's finite gyroradius, one must first evaluate the integral in (5) in closed form and then subsequently perform a small Larmor radius approximation.

First of all, we observe that for $z = 0$ it is possible to perform the integration in (5), in which case we obtain

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(0) = \frac{\Gamma(\lambda-1)\,\delta_{n0}}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}. \tag{7}$$

Now for $z \neq 0$, $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ can be represented in closed form in terms of the Meijer G function, discussed in section A2. First, after using the identity (A13), expression (5) can be written as

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{\Gamma(\lambda)\kappa^{\lambda-\beta-3/2}}{\sqrt{\pi}\Gamma(\sigma-3/2)}\int_0^{\infty}\mathrm{d}y\,(\kappa+y)^{-\lambda}\,G_{1,3}^{1,1}\left[z^2y\,\middle|\,\begin{matrix}1/2\\ n,-n,0\end{matrix}\right].$$

Then, using the identities (A12) and (A10), we obtain the representation

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{1}{\sqrt{\pi}}\frac{\kappa^{-\beta-1/2}}{\Gamma(\sigma-3/2)}G_{1,3}^{2,1}\left[\kappa z^2\,\middle|\,\begin{matrix}1/2\\ \lambda-1,n,-n\end{matrix}\right], \tag{8}$$

which is valid for $\lambda > 1/2$ and $\sigma > 3/2$.

As discussed in section A2, the Meijer G function displays two main types of behavior. When $\lambda \neq 1, 2, \dots$, the function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ belongs to the hypergeometric regime, in which case the identity (A9) shows that

$$\begin{aligned}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) =& \frac{\sqrt{\pi}\kappa^{-\beta-1/2}\left(\kappa z^2\right)^{-1/2}}{\sin\left[(\lambda-n)\pi\right]\Gamma(\sigma-3/2)}\left[\frac{\Gamma(\lambda-1/2)\left(\kappa z^2\right)^{\lambda-1/2}}{\Gamma(\lambda-n)\Gamma(\lambda+n)}\,{}_1F_2\left(\begin{matrix}\lambda-1/2\\ \lambda-n,\lambda+n\end{matrix};\kappa z^2\right)\right.\\ &\left.-\frac{\Gamma(n+1/2)\left(\kappa z^2\right)^{n+1/2}}{\Gamma(n+2-\lambda)\Gamma(2n+1)}\,{}_1F_2\left(\begin{matrix}n+1/2\\ n+2-\lambda,2n+1\end{matrix};\kappa z^2\right)\right],\end{aligned} \tag{9}$$

where ${}_1F_2(\cdots)$ is the hypergeometric series defined by (A2).

From the representation (9) and (A2), one obtains the following power series expansion for $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$,

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{1}{\sqrt{\pi}}\frac{\kappa^{-\beta-1/2}}{\Gamma(\sigma-3/2)}\left[\left(\kappa z^2\right)^n\mathcal{S}_1\left(\kappa z^2\right) + \left(\kappa z^2\right)^{\lambda-1}\mathcal{S}_2\left(\kappa z^2\right)\right], \tag{10}$$

$$\mathcal{S}_1(z) = \sum_{k=0}^{\infty}\frac{\Gamma(\lambda-n-1)\Gamma(n+1/2+k)}{(n+2-\lambda)_k\Gamma(2n+1+k)}\frac{z^k}{k!}$$

$$\mathcal{S}_2(z) = \sum_{k=0}^{\infty}\frac{\Gamma(n+1-\lambda)\Gamma(\lambda-1/2+k)}{\Gamma(\lambda+n+k)(\lambda-n)_k}\frac{z^k}{k!}.$$

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Expression (10) shows that the expansion for $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ is given by the combination of two power series, one of which $\left(\mathcal{S}_2\right)$ is proportional to $\left(\kappa z^2\right)^{(\lambda-1)}$, hence a noninteger power term, in general, and which can be of the same order as the other one when $\lambda \simeq n+1$.

Now if $\lambda = 1, 2, \dots$, expression (9) no longer can be trivially employed. However, it can be shown that the singularity that arises from $\sin\left[(\lambda-n)\pi\right]$ is canceled out by the summation of the ${}_1F_2(\cdots)$ functions. Instead of presenting here the laborious algebra involved in the demonstration, we will instead return to (8) and initially consider the case when $\lambda = 1$ and identify from (A14) the representation

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{2\kappa^{-\beta-1/2}}{\Gamma(\sigma-3/2)} I_n\left(\sqrt{\kappa}z\right) K_n\left(\sqrt{\kappa}z\right),$$

where $\kappa+\alpha+\beta = 1$ and $K_n(y)$ is the modified Bessel function of the second kind.

Using now formula (A11), one can easily verify that for $\lambda = 1, 2, \dots$,

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{2(-)^{\lambda-1}\left(\kappa z^2\right)^{\lambda-1}}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)} \frac{\mathrm{d}^{\lambda-1}}{\mathrm{d}y^{\lambda-1}}\left[I_n\left(\sqrt{y}\right)K_n\left(\sqrt{y}\right)\right]\Big|_{y=\kappa z^2}.$$

A closed-form expression for the multiple derivatives will be presented in a future publication.

Since [*Olver and Maximon*, 2010]

$$K_n(y) \propto (-)^{n+1} I_n(y) \ln\left(\frac{y}{2}\right),$$

we observe that when λ is integer, the function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}$ enters the logarithmic regime, as previously mentioned, and can no longer be expressed as a simple power series.

Figure 1 shows some plots of the function $\mathcal{H}_{n,\kappa}^{(0,0)}(z)$ given by the representation (8) versus z for some values of the harmonic number n and κ index. The dashed curves are plots of $\mathscr{H}_n(\mu)$ for the respective values of n. We observe that the function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ indeed converges to $\mathscr{H}_n(\mu)$ as κ grows and that for small values of κ the $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ function is substantially different from the Maxwellian limit. The difference is, in fact, proportionally larger for small values of z, i.e., for small gyroradius. Consequently, one can expect that the gyroradius effect on wave propagation in superthermal plasmas will be quite different from the effect it has on Maxwellian plasmas.

Figure 2 shows the function $\mathcal{H}_{n,\kappa}^{(0,0)}(z)$ versus the κ index for some values of n and z. These plots intend to show that although $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ can be either in the logarithmic or in the hypergeometric regimes, depending on whether λ is integer or not, there are no singularities involved in the transition between regimes. The function is continuous throughout the regions $\lambda > 3/2$ and $z \geqslant 0$.

3.2. The Kappa Plasma Dispersion Function

The other special function involved in the derivation of the dielectric tensor of a thermal plasma depends on an integral which takes into account the wave-particle resonance condition and whose evaluation depends on the ratio of the phase velocity of the wave and the thermal velocities of the particles. This function is usually called the plasma dispersion function (PDF). For a detailed derivation of the dielectric tensor for a thermal Maxwellian plasma and for some mathematical properties of the ensuing plasma dispersion function, the reader is again referred to *Brambilla* [1998].

For a superthermal plasma, the correspondent PDF depends on the particular profile adopted by the distribution function and on the wave polarization. In this work, we can unify the treatment for any wave polarization and VDF model given by (1) with the following modified plasma dispersion function, hereafter called the κPDF:

$$Z_\kappa^{(\alpha,\beta)}(\zeta) = \frac{\kappa^{-\beta-1/2}}{\sqrt{\pi}} \frac{\Gamma(\lambda-1)}{\Gamma(\sigma-3/2)} \int_{-\infty}^{\infty} \mathrm{d}s \frac{\left(1+s^2/\kappa\right)^{-(\lambda-1)}}{s-\zeta}, \tag{11}$$

where κ is the usual κ index, $\sigma = \kappa+\alpha$ as in (1), $\lambda = \sigma+\beta$, with β being a free parameter, and where ζ is the argument usually given by the ratio of the phase and thermal velocities. The definition (11) is valid for $\lambda > 1$ and $\zeta_i > 0$, where we have written $\zeta = \zeta_r + i\zeta_i$. The PDF defined by (11) must be analytically continued to the region $\zeta_i \leq 0$ according to the Landau prescription [*Brambilla*, 1998]. It is also important to mention that the function $\left(1+s^2/\kappa\right)^{-(\lambda-1)}$ has two branch points at $s = \pm i\sqrt{\kappa}$, whenever λ is not integer.

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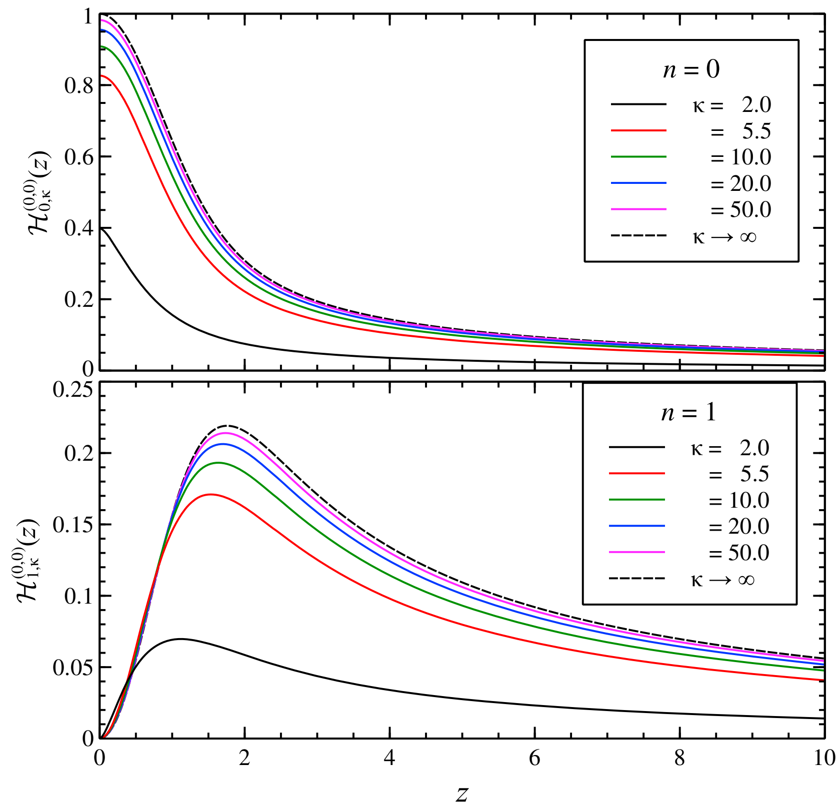

Figure 1. Plots of $\mathcal{H}^{(0,0)}_{n,\kappa}(z)$ given by (8) versus z for $n = 0$ and $n = 1$ and for several values of the κ index. The dashed curves ($\kappa \to \infty$) are the plots of $\mathscr{H}_n(\mu)$.

As expected, the function $Z^{(\alpha,\beta)}_\kappa(\zeta)$ reduces to the usual plasma dispersion (or Fried and Conte) function on the Maxwellian limit,

$$\lim_{\kappa\to\infty} Z^{(\alpha,\beta)}_\kappa(\zeta) = Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-s^2} ds}{s-\zeta}, \tag{12}$$

if one employs the exponential limit (2).

The function (11) is a variation of the modified PDF defined in *Gaelzer et al.* [2010] and can reproduce most of the superthermal PDFs found in the literature. For instance, by choosing $\alpha = \beta = 1$ one obtains the modified PDF studied by *Summers and Thorne* [1991] and *Mace and Hellberg* [1995]. This function usually occurs when one is considering longitudinal waves that propagate parallel to the ambient magnetic field.

On the other hand, if we set $\alpha = 1$ and $\beta = 0$, we obtain the $Z_{\kappa M}(\xi)$ function proposed by *Hellberg and Mace* [2002] for electromagnetic waves propagating in a kappa-Maxwellian plasma or the $Z^0_\kappa(g)$ obtained by *Lazar and Poedts* [2009] when studying the propagation of circularly polarized waves and the temperature anisotropy-driven instabilities that result from a bi-kappa VDF.

The κPDF given by (11) can be represented in terms of the Gauss hypergeometric function, discussed on section A. The derivation can be carried out using either the residue theorem, as was originally performed by *Mace and Hellberg* [1995], or by changing the integration variable and then identifying the resulting integral as one of the representations of the Gauss function, as was done by *Gaelzer et al.* [2010].

Employing again the latter method, we obtain

$$Z^{(\alpha,\beta)}_\kappa(\zeta) = \frac{\kappa^{-\beta-1}\Gamma(\lambda-1/2)\,\zeta}{(\lambda-1)\,\Gamma(\sigma-3/2)} F\left(\begin{matrix}1, \lambda-1/2 \\ \lambda\end{matrix}; 1+\frac{\zeta^2}{\kappa}\right). \tag{13}$$

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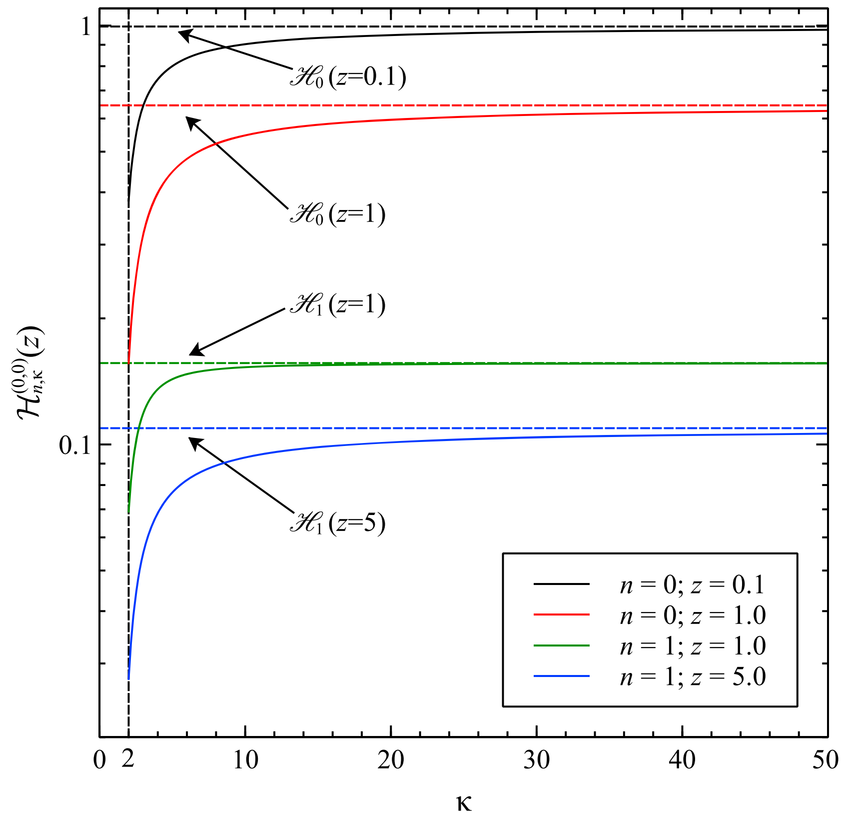

Figure 2. Plots of $\mathcal{H}_{n,\kappa}^{(0,0)}(z)$ versus κ for $n = 0$ and $n = 1$ and some values of z. The dashed lines correspond to the respective values of $\mathscr{H}_n(\mu)$.

Due to the branch cut of the Gauss function, this representation is only valid for $\zeta_i \neq 0$. The branch line is crossed on the limit $\zeta_i \to 0$, and in this case the above representation needs to be analytically continued. This task can be accomplished by using the continuation formula for the Gauss function.

Alternatively, a different representation is obtained from one of the several transformation formulae for the $F(z)$ function. By using the quadratic transformation (A4) one obtains

$$Z_\kappa^{(\alpha,\beta)}(\zeta) = i\frac{\kappa^{-\beta-1/2}\Gamma(\lambda-1/2)}{(\lambda-1)\Gamma(\sigma-3/2)}F\left[\begin{matrix}1, 2(\lambda-1)\\ \lambda\end{matrix}; \frac{1}{2}\left(1+\frac{i\zeta}{\sqrt{\kappa}}\right)\right], \tag{14}$$

which reduces to the result first obtained by *Mace and Hellberg* [1995]. The advantage of this representation is that the branch cut has moved to the strip $-i\sqrt{\kappa} \leq \zeta < -i\infty$, making it ideal for most of the applications in plasma physics. It is important to point out here that the transformation described above must be performed in such a way as to be consistent with the Landau prescription. Hence, one must always take the root given by $\sqrt{-\zeta^2} = -i\zeta$.

In spite of the advantages of this representation, it is still not the best suited for the derivation of power series and asymptotic expansions, which are needed for the applications we have in mind. Therefore, we return to (13) and apply the transformation (A5), resulting in

$$\begin{aligned} Z_\kappa^{(\alpha,\beta)}(\zeta) = &-\frac{2\Gamma(\lambda-1/2)\zeta}{\kappa^{\beta+1}\Gamma(\sigma-3/2)}F\left(\begin{matrix}1, \lambda-1/2\\ 3/2\end{matrix}; -\frac{\zeta^2}{\kappa}\right) \\ &+\frac{i\sqrt{\pi}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\left(1+\frac{\zeta^2}{\kappa}\right)^{-(\lambda-1)}, \end{aligned} \tag{15}$$

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where we have also used (A7). The representation (15) is the generalization of a result first derived by *Mace and Hellberg* [1995] and has the same analytical properties of (14). The advantage of (15) is that by means of definition (A3), one obtains the power series expansion

$$Z_{\kappa}^{(\alpha,\beta)}(\zeta) = -\frac{\sqrt{\pi}\kappa^{-\beta-1}\zeta}{\Gamma(\sigma-3/2)}\sum_{k=0}^{\infty}\frac{\Gamma(\lambda+k-1/2)}{\Gamma(k+3/2)}\left(-\frac{\zeta^2}{\kappa}\right)^k + \frac{i\sqrt{\pi}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\left(1+\frac{\zeta^2}{\kappa}\right)^{-(\lambda-1)}, \tag{16}$$

which converges for $\left|\zeta^2/\kappa\right| < 1$.

Alternatively, if one applies transformation (A6) on (15), the following asymptotic expansion results:

$$Z_{\kappa}^{(\alpha,\beta)}(\zeta) = -\frac{\Gamma(\lambda-3/2)}{\kappa^{\beta}\Gamma(\sigma-3/2)}\frac{1}{\zeta}\sum_{k=0}^{\infty}\frac{(1/2)_k}{(5/2-\lambda)_k}\left(-\frac{\kappa}{\zeta^2}\right)^k + \frac{\pi^{1/2}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\left[i-\tan(\lambda\pi)\right]\left(1+\frac{\zeta^2}{\kappa}\right)^{-(\lambda-1)}. \tag{17}$$

Notice that (17) is not valid when λ is half integer. In this case, one must employ a special transformation which effectively removes the singularity. Details on this procedure can be found in *Daalhuis* [2010] and will not be presented here. Suffice it to say that in this case the asymptotic representation of the κPDF is no longer a simple power series but contains a logarithmic term too. In this respect the asymptotic representation for $Z_{\kappa}^{(\alpha,\beta)}(\zeta)$ possesses the same duality of hypergeometric/logarithmic regimes as the function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$, discussed in the previous section.

Finally, after deriving (11) and integrating by parts, we obtain the expression for the derivative of the κPDF given by

$$\frac{d}{d\zeta}Z_{\kappa}^{(\alpha,\beta)}(\zeta) = -2\left[\frac{\Gamma(\lambda-1/2)}{\kappa^{\beta+1}\Gamma(\sigma-3/2)} + \zeta Z_{\kappa}^{(\alpha,\beta+1)}(\zeta)\right]. \tag{18}$$

One can easily verify the known Maxwellian limit $Z'(\zeta) = -2(1+\zeta Z)$.

We have also obtained expressions for the recurrence relation of the n-order derivative of $Z_{\kappa}^{(\alpha,\beta)}(\zeta)$ as well as explicit expressions in terms of the Gauss function. These results will be shown in a future publication.

4. The Dispersion Relations for DAW

Dispersive Alfvén waves (DAWs) consist in a modification of the usual shear Alfvén waves in the case of near-perpendicular propagation, through the coupling with the magnetoacoustic mode in the case when the perpendicular component of the wave number is much greater than the parallel component [*Cramer*, 2001].

In the very low frequency regime, where the usual MHD treatment predicts that the shear Alfvén waves are essentially nondispersive, a kinetic description shows that the inclusion of finite gyroradius effects in a small-beta plasma has the outcome of inducing a highly dispersive behavior on the shear mode. Moreover, the effects of the electrons on the dielectric tensor have the consequence of forking the dispersive surface of the mode in two distinct branches, namely, the kinetic Alfvén waves (KAWs), that occur when the electron beta parameter is moderately low $\left(m_e/m_i \ll \beta_e < 1\right)$, in which case the dispersion is affected by the thermal pressure of the electrons, and the inertial Alfvén waves (IAWs), when $\beta_e \ll m_e/m_i$ and where the dispersion is essentially determined by the electron's finite mass. One important difference between both dispersive modes is that they have opposing perpendicular components of the group velocity. Hence, the refractive characteristics are quite different for each wave.

Both kinetic and inertial waves can be observed in the same general environment, but usually in different regions of the same environment, depending on the variation of the physical parameters. For instance, along a given auroral field line, the IAWs are thought to exist in the region just above the ionosphere, whereas the KAW should be present at higher altitudes. In either case, the relatively large parallel component of the wave's electric field should play an important role on particle acceleration [*Lysak and Lotko*, 1996].

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In this section, we will discuss the modification induced by the superthermal tail of the distribution function on the usual dispersion relations of both KAW and IAW, as well as on the general dispersion equation. The modifications of the dispersive characteristics of parallel-propagating waves due to non-Maxwellian velocity spread have been discussed at length in the literature for quite some time already [*Thorne and Summers*, 1991; *Mace and Hellberg*, 1995; *Hellberg et al.*, 2005; *Lazar et al.*, 2008; *Mace and Hellberg*, 2009; *Mace and Sydora*, 2010; *Lazar and Poedts*, 2009; *Lazar et al.*, 2011a, 2011b, 2012; *Lazar and Poedts*, 2014].

On the other hand, for nonparallel propagation, studies of wave propagation in superthermal plasmas have been usually restricted to perpendicular propagation for isotropic κ-VDF [*Mace*, 2003, 2004; *Viñas et al.*, 2005] or, in the case of oblique propagation, for the kappa-Maxwellian (κM) VDF model [*Hellberg and Mace*, 2002; *Hellberg et al.*, 2005; *Cattaert et al.*, 2007] where the dependence of the VDF on the parallel and perpendicular components of the particle velocity factor out, with the parallel part modeled as a κ-VDF and the perpendicular as a Maxwellian (i.e., Gaussian).

The κM model is physically sound when the parallel velocity spread is much larger than in the perpendicular direction and when the last one can be reasonably modeled by a Maxwellian distribution. However, for a more general situation there are already some models of separable distribution functions that are nonthermal in both directions, such as the product-bi-kappa (PBK) VDF [*Summers and Thorne*, 1991]. In this case, as we shall see, a mathematical treatment that provides closed-form analytical expressions for the components of the dielectric tensor for isotropic or anisotropic κ-VDFs is still lacking.

In order to introduce such treatment, we will here derive an equation dispersion for DAW in an isotropic kappa plasma and present the ensuing dispersion relations for both the KAW and IAW branches of the dispersive Alfvén waves.

We will adopt the same procedure originally proposed by *Lysak and Lotko* [1996] for the derivation of the DAW dispersion equation. In their work, the authors assumed low-frequency waves $\left(\omega \ll \Omega_i\right)$, where ω is the wave's angular frequency and $\Omega_a = q_a B_0/m_a c$ is the (angular) cyclotron frequency of the particle species a, propagating in the quasi-perpendicular direction in such a way that both conditions $\left(k_\parallel \ll k_\perp\right)$, where $k_{\parallel,\perp}$ is the parallel (perpendicular) component of the wave vector and $k_\parallel v_{Ta} \ll \left|\Omega_a\right|$ apply. It was also assumed that $\beta_e < 1$, where $\beta_a = 8\pi n_a T_a/B_0^2$ is the beta parameter of species a with number density n_a, but all other values are possible, including the limiting cases $m_e/m_i \ll \beta_e < 1$ and $\beta_e \ll m_e/m_i$.

If the conditions above apply, including others we will discuss afterward, *Lysak and Lotko* [1996] showed that the dispersion equation for DAW propagating in an electron-ion plasma is given by

$$\det\begin{pmatrix} \varepsilon_{xx} - N_\parallel^2 & N_\perp N_\parallel \\ N_\perp N_\parallel & \varepsilon_{zz} - N_\perp^2 \end{pmatrix} = 0, \tag{19}$$

where $N_{\parallel,\perp} = k_{\parallel,\perp} c/\omega$ are the components of the refractive index and ε_{xx} and ε_{zz} are components of the dielectric tensor given by

$$\varepsilon_{xx} \approx \sum_{a=e,i} \frac{\omega_{pa}^2}{\omega^2} \sum_{n\to-\infty}^{\infty} \int d^3v \frac{v_\perp \left[nJ_n\left(r_a\right)/r_a\right]^2 \mathcal{L} f_{a0}}{\omega - n\Omega_a - k_\parallel v_\parallel} \tag{20a}$$

$$\varepsilon_{zz} \approx \sum_{a=e,i} \frac{\omega_{pa}^2}{\omega^2} \sum_{n\to-\infty}^{\infty} \int d^3v \frac{v_\parallel^2}{v_\perp} \frac{J_n^2\left(r_a\right) \mathcal{L} f_{a0}}{\omega - n\Omega_a - k_\parallel v_\parallel}, \tag{20b}$$

where $\omega_{pa}^2 = 4\pi n_a q_a^2/m_a$ is the plasma frequency, $f_{a0}(v)$ is a (isotropic) general VDF, $\mathcal{L} f_{a0} = \omega \partial f_{a0}/\partial v_\perp$, and $r_a = k_\perp v_\perp/\Omega_a$. Notice that the unity was already neglected, as per the usual procedure for very low frequency waves [see *Lysak and Lotko*, 1996, for justification].

Using then the $f_M(v)$ distribution given by (3), *Lysak and Lotko* [1996] obtained the following dispersion equation for dispersive Alfvén waves,

$$\frac{\omega^2}{k_\parallel^2 v_A^2} = \frac{\mu_i}{1 - \mathscr{H}_0\left(\mu_i\right)} + \frac{k_\perp^2 \rho_s^2}{\mathscr{H}_0\left(\mu_e\right)\left[1 + \xi_{0e} Z\left(\xi_{0e}\right)\right]}, \tag{21}$$

where $v_A^2 = B_0^2/4\pi n_i m_i$ is the squared Alfvén speed, $\mu_a = k_\perp^2 \rho_a^2$, where $\rho_a^2 = v_{Ta}^2/2\Omega_a^2$ is the squared gyroradius of species a, $\rho_s^2 = m_e v_{Te}^2/2 m_i \Omega_i^2$ is the squared ion-acoustic gyroradius, $\xi_{na} = \left(\omega - n\Omega_a\right)/k_\parallel v_{Ta}$, and $\mathscr{H}_n\left(\mu_a\right)$

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and $Z\left(\xi_{na}\right)$ are respectively the gyroradius function, given by (6), and the plasma dispersion function, given by (12).

Considering both the kinetic limit of the DAW, when $m_e/m_i \ll \beta_e < 1$ (KAW), and the inertial limit, when $\beta_e \ll m_e/m_i$ (IAW), *Lysak and Lotko* [1996] derived the following dispersion relations from (21):

$$\frac{\omega^2}{k_\parallel^2 v_A^2} \approx 1 + k_\perp^2\left(\rho_s^2 + \frac{3}{4}\rho_i^2\right) \quad \text{(KAW)} \tag{22a}$$

$$\frac{\omega^2}{k_\parallel^2 v_A^2} \approx \frac{\mu_i}{1 - \mathscr{H}_0\left(\mu_i\right)}\left(1 + \frac{k_\perp^2 c^2}{\omega_{pe}^2}\right)^{-1} \quad \text{(IAW).} \tag{22b}$$

We now set forth on the same path followed by the above mentioned authors, except that we will employ the κ-VDF given by (1), aiming at the generalizations of equations (21), (22a), and (22b). By doing so, we also intend to present a mathematical treatment that can be readily applied to the general problem of wave propagation in superthermal plasmas.

Introducing (1) into the dielectric tensor components (20a) and (20b) and performing the $v_\parallel$ integration, we are able to write the remaining integrals as

$$\varepsilon_{xx} = 4\sum_a \frac{\omega_{pa}^2}{\omega^2}\xi_{0a}\sum_{n\to-\infty}^{\infty}\frac{n^2}{\nu_a^2}\int_0^\infty \mathrm{d}x\,\frac{xJ_n^2\left(\nu_a x\right)}{\left(1+x^2/\kappa_a\right)^{\sigma_a+1}}Z_{\kappa_a}^{(\alpha_a,2)}\left(\frac{\xi_{na}}{\sqrt{1+x^2/\kappa_a}}\right)$$

$$\varepsilon_{zz} = -2\sum_a \frac{\omega_{pa}^2}{\omega^2}\xi_{0a}\sum_{n\to-\infty}^{\infty}\xi_{na}\int_0^\infty \mathrm{d}x\,\frac{xJ_n^2\left(\nu_a x\right)}{\left(1+x^2/\kappa_a\right)^{\sigma_a+1/2}}Z_{\kappa_a}^{(\alpha_a,1)\prime}\left(\frac{\xi_{na}}{\sqrt{1+x^2/\kappa_a}}\right),$$

where $\nu_a^2 = 2\mu_a$, $\mu_a = k_\perp^2\rho_a^2$ as before, but now $\rho_a^2 = w_a^2/2\Omega_a^2$ is the (squared) superthermal gyroradius, which depends on the κ index. Moreover, $\xi_{na} = \left(\omega - n\Omega_a\right)/k_\parallel w_a$ also becomes κ dependent and $Z_\kappa^{(\alpha,\beta)}(\zeta)$ is the κPDF given by (11). We also have introduced the notation $Z_\kappa^{(\alpha,\beta)\prime}(\zeta) = \mathrm{d}Z_\kappa^{(\alpha,\beta)}/\mathrm{d}\zeta$.

The remaining integrals in the expressions for ε_{xx} and ε_{zz} are difficult to evaluate analytically. The argument of the κPDF depends on $x \propto v_\perp$ due to the fact that a kappa or a Bi-kappa VDF are not separable on $v_\parallel$ and $v_\perp$ as is the case with the Maxwellian or even with the κM and PBK distributions.

Expressions similar to the above were already obtained by a general treatment of obliquely propagating waves in a Lorentzian plasma by *Summers et al.* [1994]. In their work, however, the results presented were obtained by numerical integration and analytical expressions were obtained from the power series expansion of the Bessel functions.

The same approach was adopted on the more recent works by *Basu* [2009] and *Liu et al.* [2014] in order to obtain expressions for the dispersion relations of the DAW. These authors were able to provide expressions for both the kinetic and inertial Alfvén waves, after expanding the Bessel functions in the case of small Larmor radius and using either the power series or the asymptotic expansions for $Z_\kappa^{(\alpha,\beta)}$. The problem with using a power series expansion for the Bessel functions was discussed in section 3.1. If one employs this method, the convergence of the integrals will demand a constraint between the λ (or κ) index and the harmonic number n, which is undesirable. Moreover, the transition between the hypergeometric and logarithmic regimes of the result is not clear.

We sustain, therefore, that the integrals above must be evaluated using the Bessel functions in a closed form first, and only then small gyroradius approximations can be made. Of course, one could always evaluate the integrals numerically, but an analytical expression will always provide more information on the behavior and mathematical properties of the dielectric tensor components.

Returning to the evaluation of the ε_{xx} and ε_{zz} components of the dielectric tensor, since we are assuming low-frequency waves, $\left|\xi_{na}\right| \gg 1$ for $n \neq 0$ and we will employ the asymptotic limit of $Z_\kappa^{(\alpha,\beta)}$ given by (17) in these terms. It must be emphasized here that our formulation does not depend on this approximation. The asymptotic form of the $Z_\kappa^{(\alpha,\beta)}$ function is adopted in this work only in order to simplify the final expressions. A general formulation for obliquely propagating waves, where no approximations are made, will be presented in a future publication. For this approximation to be rigorously valid, one should assume that σ is in the vicinity of an integer greater than 2. This is not a too stringent restriction, however. If λ is half integer, one must employ the logarithmic expression for the asymptotic expansion of $Z_\kappa^{(\alpha,\beta)}$, but it can be shown

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that the first term containing the logarithmic contribution is always much smaller than the approximations adopted above.

On the other hand, the term with $n = 0$ should be kept without approximations. However, since $\left|\xi_{0i}\right| = |\omega|/k_\parallel w_i \simeq \beta_i^{-1/2} > 1$, we will also employ the asymptotic limit for the ions and only keep the full thermal effect for the electrons. Conversely, the temperature of the ions will be important for the gyroradius, which will be considered finite, whereas the electronic Larmor radius will be assumed very small.

Using then the asymptotic expressions and also taking into account that $\omega_{pe}\Omega_i \ll \omega_{pi}\Omega_e$, $k_\parallel^2\rho_a^2 \ll 1$ and $c_s^2/v_A^2 = 2T_e/m_i v_A^2 = \beta_e < 1$, we obtain the expressions below for ε_{xx} and ε_{zz}. We will not present this derivation in greater detail, since it closely follows the procedure adopted by *Lysak and Lotko* [1996]. We only mention that we have employed the identity [*Olver and Maximon*, 2010]

$$\sum_{n\to-\infty}^{\infty} J_n^2(z) = 1 \Longrightarrow \sum_{n=1}^{\infty} J_n^2(z) = \frac{1}{2}\left[1 - J_0^2(z)\right].$$

Hence, we obtain

$$\varepsilon_{xx} \approx \frac{c^2}{v_A^2}\frac{1_i - \mathcal{H}_{0,\kappa_i}^{(\alpha_i,1/2)}\left(\nu_i\right)}{\mu_i} \tag{23a}$$

$$\varepsilon_{zz} \approx -\frac{1}{k_\parallel^2\lambda_{De}^2}\mathcal{Z}_{\kappa_e}\left(\nu_e,\xi_{0e}\right), \tag{23b}$$

where $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(\nu)$ is the gyroradius function defined by (5), $\lambda_{Da}^2 = w_a^2/2\omega_{pa}^2$ is the squared Debye length, $1_a \doteq \left(\sigma_a - 3/2\right)/\kappa_a$ and $\mathcal{Z}_\kappa(\nu,\xi)$ is a new special function defined as

$$\mathcal{Z}_\kappa(\nu,\xi) \doteq \int_0^\infty \mathrm{d}x\,\frac{xJ_0^2(\nu x)}{(1+x^2/\kappa)^{\sigma+1/2}}Z_\kappa^{(\alpha,1)\prime}\left(\frac{\xi}{\sqrt{1+x^2/\kappa}}\right). \tag{24}$$

The reader can verify that on the Maxwellian limit our expressions reduce to the dielectric tensor components inside equation (5) of *Lysak and Lotko* [1996]. It is also noteworthy to point out that the general expressions from our formulation, without any approximation, would have a structure similar to (24) but with contributions from any harmonic number n.

Inserting now (23a) and (23b) into (19), we obtain the dispersion equation for the superthermal dispersive Alfvén waves (κDAW),

$$\frac{\omega^2}{k_\parallel^2 v_A^2} = \frac{\mu_i}{1_i - \mathcal{H}_{0,\kappa_i}^{(\alpha_i,1/2)}} - \frac{k_\perp^2\rho_s^2}{\mathcal{Z}_{\kappa_e}\left(\nu_e,\xi_{0e}\right)}, \tag{25}$$

with the ion-acoustic gyroradius now given by $\rho_s^2 = m_e w_e^2/2m_i\Omega_i^2$. This result reduces to (21) on the Maxwellian limit.

A closed-form analytical solution to the integral in $\mathcal{Z}_\kappa(\nu,\xi)$ can be obtained by first employing the formula for the derivative given by (18). Then, if ξ always remains inside the region on the complex plane that defines the principal branch of $Z_\kappa^{(\alpha,\beta)}$, as the integration in (24) is carried out, its argument will trace a path on the complex plane that lies along a strip between ξ_{0e} and the origin. Hence, for every point along the integration path there is always a finite neighborhood around $\zeta(x) = \xi/\sqrt{1+x^2/\kappa}$ where the function $Z_\kappa^{(\alpha,\beta)}$ is analytic and the superposition of all these regions define adequate analytic continuations which ensure that $Z_\kappa^{(\alpha,\beta)}$ is analytic also inside a finite region containing the strip. Therefore, for any x, we can always expand $Z_\kappa^{(\alpha,2)}\left(\xi/\sqrt{1+x^2/\kappa}\right)$ in a Taylor series around ξ_{0e}, obtaining then

$$Z_\kappa^{(\alpha,2)}\left(\frac{\xi}{\sqrt{1+x^2/\kappa}}\right) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-)^\ell\binom{k}{\ell}\frac{(-\xi)^k}{k!}Z_\kappa^{(\alpha,2)(k)}(\xi)\left(1+\frac{x^2}{\kappa}\right)^{-\ell/2},$$

where $Z_\kappa^{(\alpha,\beta)(k)}(\zeta) = \mathrm{d}^k Z_\kappa^{(\alpha,\beta)}/\mathrm{d}\zeta^k$. This series is assured to converge thanks to the Taylor theorem. As mentioned above, we have obtained representations for the derivatives of $Z_\kappa^{(\alpha,\beta)}$ in any order, but the explicit expressions will be presented elsewhere.

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Inserting the above series in (24), we then obtain the analytical expression for $\mathcal{Z}_{\kappa_e}\left(\nu_e, \xi_{0e}\right)$ given by

$$\mathcal{Z}_{\kappa_e}\left(\nu_e, \xi_{0e}\right) = -\mathcal{H}_{0,\kappa_e}^{(\alpha_e,1/2)}\left(\nu_e\right) + \kappa_e^{5/2}\Gamma\left(\sigma_e - \frac{3}{2}\right) \times \sum_{k=0}^{\infty} \frac{\left(-\xi_{0e}\right)^{k+1}}{k!} Z_{\kappa_e}^{(\alpha_e,2)(k)}\left(\xi_{0e}\right) \sum_{\ell=0}^{k} (-)^{\ell} \binom{k}{\ell} \frac{\kappa_e^{\ell/2}\mathcal{H}_{0,\kappa_e}^{(\alpha_e,\ell/2+1)}\left(\nu_e\right)}{\Gamma\left(\sigma_e + 1 + \ell/2\right)}. \quad (26)$$

Expression (26), although exact, is still complicated due to all the sums involved. We have also obtained more compact representations for $\mathcal{Z}_{\kappa}(\nu, \xi)$ in terms of generalized hypergeometric functions of two variables, but these expressions will not be shown here.

Instead of employing the full series expansion in (26), we will make a further approximation and consider only the term with $k = 0$. Thus, the form we will henceforth adopt for $\mathcal{Z}_{\kappa_e}\left(\nu_e, \xi_{0e}\right)$ is

$$\mathcal{Z}_{\kappa_e}\left(\nu_e, \xi_{0e}\right) \approx -\mathcal{H}_{0,\kappa_e}^{(\alpha_e,1/2)}\left(\nu_e\right) - \frac{\kappa_e^{5/2}\Gamma\left(\sigma_e - 3/2\right)}{\Gamma\left(\sigma_e + 1\right)}\mathcal{H}_{0,\kappa_e}^{(\alpha_e,1)}\left(\nu_e\right)\xi_{0e} Z_{\kappa_e}^{(\alpha_e,2)}\left(\xi_{0e}\right). \quad (27)$$

This approximation still retains all pertinent physical effects due to finite gyroradius, thermal pressure, wave-particle interaction, and superthermal particles.

From the dispersion equation (25) and the $\mathcal{Z}_{\kappa}$ function given by (27), we are now able to derive the dispersion relations for DAW in superthermal plasmas.

4.1. Superthermal Kinetic Alfvén Waves

Assuming hot electrons $\left(\xi_{0e} \ll 1\right)$ with small Larmor radius $\left(\nu_e \ll 1\right)$ in expression (27), in the second term we can keep the lowest-order contributions from (7) and (16) and approximate

$$\begin{aligned}\mathcal{Z}_{\kappa_e}\left(\nu_e, \xi_{0e}\right) &\approx -\mathcal{H}_{0,\kappa_e}^{(\alpha_e,1/2)}\left(\nu_e\right) + 2\frac{\Gamma\left(\sigma_e + 3/2\right)}{\kappa_e^{1/2}\Gamma\left(\sigma_e + 1\right)}\mathcal{H}_{0,\kappa_e}^{(\alpha_e,1)}\left(\nu_e\right)\xi_{0e}^2 \\ &\approx -\mathcal{H}_{0,\kappa_e}^{(\alpha_e,1/2)}\left(\nu_e\right).\end{aligned}$$

Inserting this result into (25), we obtain the following dispersion relation for kinetic Alfvén waves propagating in superthermal plasmas (κKAW),

$$\frac{\omega^2}{k_\parallel^2 v_A^2} = \frac{\mu_i}{1_i - \mathcal{H}_{0,\kappa_i}^{(\alpha_i,1/2)}} + \frac{k_\perp^2\rho_s^2}{\mathcal{H}_{0,\kappa_e}^{(\alpha_e,1/2)}}. \quad (28)$$

Notice that the right-hand side no longer depends on $k_\parallel$.

A more usual form for this result can be obtained if we employ the expansion (10) and remember that $\nu_i \gg \nu_e$, in order to write down

$$\begin{aligned}\mathcal{H}_{0,\kappa_i}^{(\alpha_i,1/2)}\left(\nu_i\right) &\approx 1_i - \mu_i + \frac{3}{4}\frac{\kappa_i}{\sigma_i - 5/2}\mu_i^2 + \gamma_i\mu_i^{\sigma_i - 1/2} \\ \mathcal{H}_{0,\kappa_e}^{(\alpha_e,1/2)}\left(\nu_e\right) &\approx 1_e,\end{aligned}$$

where

$$\gamma_i \doteq \frac{\Gamma\left(1/2 - \sigma_i\right)\Gamma\left(\sigma_i\right)\left(2\kappa_i\right)^{\sigma_i - 1/2}}{\sqrt{\pi}\kappa_i\Gamma\left(\sigma_i - 3/2\right)\Gamma\left(\sigma_i + 1/2\right)}.$$

This result is valid as long as σ_i is not too close to a half integer. Therefore, we finally obtain the κKAW dispersion relation:

$$\frac{\omega^2}{k_\parallel^2 v_A^2} \approx 1 + k_\perp^2\left(\frac{\kappa_e}{\sigma_e - 3/2}\rho_s^2 + \frac{3}{4}\frac{\kappa_i}{\sigma_i - 5/2}\rho_i^2\right) + \gamma_i\left(k_\perp^2\rho_i^2\right)^{\sigma_i - 3/2}. \quad (29)$$

Comparing this result with (22a), we observe the following differences between the usual KAW dispersion relation and expression (29). First, we notice that the κ indices of both electrons and ions contribute, which

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means that the wave dispersion will vary in different ways if electrons and ions have different degrees of departure from the thermodynamic equilibrium. Moreover, there appears a new term on the dispersion relation, which does not exist for a Maxwellian plasma. For small values of the κ index, we have $\sigma_i - 3/2 \gtrsim 1$, but we also observe that $|\gamma_i| > 1$, which means that for a superthermal plasma this last term can be of the same order as the other terms. Notice also that this term, in general, contains a noninteger power index, as we have mentioned in the discussion in section 3.1.

However, the final profile of the dispersion relation also depends on the specific model employed, since ρ_s and ρ_i depend on the parameter w_a, which depends on the adopted model. As a practical example, for the ST91 model, $\sigma_a = \kappa_a + 1$, $w_a^2 = (\kappa_a - 3/2)\, v_{\kappa a}^2/\kappa_a$, and $\kappa_a > 3/2$. Therefore, (29) becomes

$$\frac{\omega^2}{k_\parallel^2 v_A^2} \approx 1 + \left(\frac{\kappa_e - 3/2}{\kappa_e - 1/2} + \frac{3}{4}\frac{T_{\kappa i}}{T_{\kappa e}}\right)\frac{v_{\kappa e}^2}{2v_A^2}\frac{k_\perp^2 c^2}{\omega_{pe}^2} + \gamma_i \left(\frac{\kappa_i - 3/2}{\kappa_i}\frac{T_{\kappa i}}{T_{\kappa e}}\frac{v_{\kappa e}^2}{2v_A^2}\frac{k_\perp^2 c^2}{\omega_{pe}^2}\right)^{\kappa_i - 1/2},$$

with

$$\gamma_i \doteq \frac{\Gamma(-\kappa_i - 1/2)\,\Gamma(\kappa_i)\,(2\kappa_i)^{\kappa_i + 1/2}}{\sqrt{\pi}\,\Gamma(\kappa_i - 1/2)\,\Gamma(\kappa_i + 3/2)}.$$

The traditional term only depends on κ_e, but the new term is strongly influenced by κ_i. Since for KAW, $v_{\kappa e}^2/v_A^2 = m_i\beta_e/m_e \gg 1$ and $k_\perp c/\omega_{pe} \simeq 1$, if $\kappa_i = 2$, there results $\gamma_i = -5.79$, and thus, the new term can really become as important as the usual term.

4.2. Superthermal Inertial Alfvén Waves

On the other limit, if the electrons are cold, $|\xi_{0e}| \gg 1$ and $\nu_e \ll 1$, and so we can approximate

$$\begin{aligned}
Z_\kappa^{(\alpha,2)}(\xi_{0e}) &\approx -\frac{\Gamma(\sigma_e + 1/2)}{\kappa_e^2\Gamma(\sigma_e - 3/2)}\frac{1}{\xi_{0e}}\left[1 + \frac{1}{2}\frac{\Gamma(\sigma_e - 1/2)}{\Gamma(\sigma_e + 1/2)}\frac{\kappa_e}{\xi_{0e}^2}\right],\\
\mathcal{H}_{0,\kappa_e}^{(\alpha_e,1/2)}(\nu_e) &\approx \frac{\Gamma(\sigma_e - 1/2)}{\kappa_e\Gamma(\sigma_e - 3/2)}\\
\mathcal{H}_{0,\kappa_e}^{(\alpha_e,1)}(\nu_e) &\approx \frac{\Gamma(\sigma_e)}{\kappa_e^{3/2}\Gamma(\sigma_e - 3/2)},
\end{aligned}$$

resulting for (27),

$$\mathcal{Z}_{\kappa_e}(\xi_{0e}, \nu_e) \approx -\frac{1_e}{2\sigma_e} + \frac{\sigma_e - 3/2}{2\sigma_e}\frac{k_\parallel^2 w_e^2}{\omega^2}.$$

Therefore, from (21) we obtain the dispersion relation for inertial Alfvén waves in a superthermal plasma (κIAW),

$$\frac{\omega^2}{k_\parallel^2 v_A^2} \approx \frac{\mu_i}{1_i - \mathcal{H}_{0,\kappa_i}^{(\alpha_i,1/2)}}\left(1 + \frac{\sigma_e}{\sigma_e - 3/2}\frac{k_\perp^2 c^2}{\omega_{pe}^2}\right)^{-1}. \tag{30}$$

Even if the ion gyroradius is very small, we obtain

$$\frac{\omega^2}{k_\parallel^2 v_A^2} \approx \left(1 + \frac{\sigma_e}{\sigma_e - 3/2}\frac{k_\perp^2 c^2}{\omega_{pe}^2}\right)^{-1}$$

and the dispersion relation will still depend on κ_e. If $k_\perp\rho_i \simeq 1$, then the κIAW dispersion relation will also depend on the γ_i parameter.

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5. Numerical Results

Some results from the dispersion relations obtained in the preceding section will be presented here.

In order to obtain the desired results, we first need to adopt a particular model for the distribution function. So we will consider once again the ST91 model, for which $\alpha_a = 1$ and $w_a^2 = \left(\kappa_a - 3/2\right) v_{\kappa a}^2/\kappa_a$.

The kinetic and inertial branches of the dispersive Alfvén waves are distinguished by the parameter

$$r_v^2 = \frac{v_{\kappa e}^2}{2v_A^2} = \frac{1}{2}\frac{m_i}{m_e}\beta_e,$$

in such a way that when $r_v \ll 1$, the dispersion occurs along the IAW branch and when $1 \ll r_v < m_i/2m_e$, we have the KAW. This distinction is also important because the approximate expressions that were obtained for the dispersion relations of both branches do not compare on the same footing with the numerical solution of the dispersion equation. This happens due to the fact that the ion gyroradius also depends on the r_v parameter,

$$\rho_i^2 = \frac{\kappa_i - 3/2}{\kappa_i}\frac{T_{\kappa i}}{T_{\kappa e}}\frac{r_v c^2}{\omega_{pe}^2}.$$

Hence, the assumption of small ion gyroradius $\left(k_\perp \rho_i \ll 1\right)$ for an arbitrary value of $k_\perp$ is reasonable for IAW but, in general, is not valid for KAW.

We will consider first the inertial waves. The dispersion relation for κIAW with the ST91 model is given, from (30), by

$$\frac{\omega^2}{k_\parallel^2 v_A^2} = \frac{\mu_i}{1_i - \mathcal{H}_{0,\kappa_i}^{(1,1/2)}}\left(1 + \frac{\kappa_e + 1}{\kappa_e - 3/2}\frac{k_\perp^2 c^2}{\omega_{pe}^2}\right)^{-1}, \tag{31}$$

where

$$\mu_i = \frac{\kappa_i - 3/2}{\kappa_i}\frac{T_{\kappa i}}{T_{\kappa e}} r_v^2 \frac{k_\perp^2 c^2}{\omega_{pe}^2}.$$

The results are shown in Figure 3, where we have chosen $r_v^2 = 0.1$ and $T_{\kappa i} = T_{\kappa e}$. In this figure, we show the dispersion relations both for the inertial and κ-inertial Alfvén waves. Figure 3a displays $\mathrm{Re}\left[\left(\omega/k_\parallel v_a\right)^2\right]$ and Figure 3b displays $\mathrm{Im}\left[\left(\omega/k_\parallel v_a\right)^2\right]$, both as functions of $k_\perp c/\omega_{pe}$ in the interval [0.1, 5].

The dotted line in Figure 3a (marked IAW) is the IAW dispersion given by (22b), whereas the numerical solution of the dispersion equation (21), for DAW in a Maxwellian plasma, is the dashed curve (marked Maxwell). We first point out in Figure 3a that even for a Maxwellian plasma there is a visible difference between the approximate dispersion relation (22b) and the real part of the numerical solution of equation (21). This difference will be even greater for the kinetic waves.

In the same panel (Figure 3a), one can observe also the κIAW dispersion relation given by (31) for some values of $\kappa_e = \kappa_i$ (long-dashed curves) and the corresponding solutions of the equation (25) (real parts, continuous curves). One can clearly see how the κIAW distinguishes itself from its IAW counterpart, both in the approximate and in the "exact" numerical solution, as the κ index decreases. However, the difference is more pronounced with the numerical solutions.

Figure 3b shows the dependence of the damping coefficient, evaluated as $\mathrm{Im}\,\omega^2$, as a function of $k_\perp$. Again, the dashed curve marked "Maxwell" is the solution of the equation (21), whereas the continuous curves are solutions of (25) for the same values of κ index adopted in Figure 3a. The behavior of the damping coefficient with κ varies with the perpendicular wave number. For large perpendicular wavelength $\left(k_\perp c/\omega_{pe} \lesssim 1.6\right)$, the absorption rate slightly increases with κ, whereas for small wavelength $\left(k_\perp c/\omega_{pe} \gtrsim 1.6\right)$ one can observe exactly the opposite effect.

The results shown in Figure 3 are by no means an exhaustive study about the effect of superthermal particles on the usual dispersion relation (and their damping rate) of the inertial Alfvén waves. The main objective here was to show a practical application of the formalism developed in the preceding sections, destined to the full description of oblique propagating waves in superthermal plasmas.

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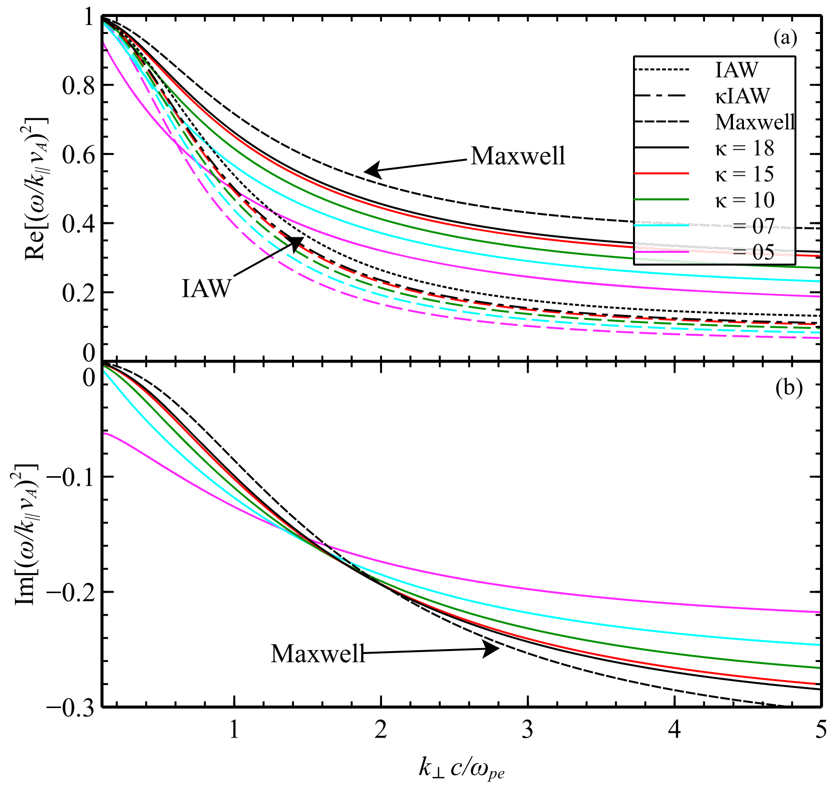

Figure 3. Dispersion relations for inertial and κ-inertial Alfvén waves for $r_v^2 = 0.1$ and $T_i = T_e$. (a) Real part of the solution of (25). (b) Imaginary part of the solution of (25).

Considering now the kinetic Alfvén waves, one has to be even more careful analyzing the validity of the approximate dispersion relations. Since now $r_v \gg 1$, the argument of the gyroradius function $\left(\mu_i = k_\perp^2 \rho_i^2\right)$ soon becomes large $\left(\mu_i \gtrsim 1\right)$ for the same range of perpendicular wave numbers employed in the previous case. However, for the same range of values one can still consider $k_\perp \rho_e \ll 1$. For this reason, the usual dispersion relation for KAW given by (22a) should be compared with the more exact expression

$$\frac{\omega^2}{k_\parallel^2 v_A^2} = \frac{\mu_i}{1 - \mathscr{H}_0\left(\mu_i\right)} + k_\perp^2 \rho_s^2, \tag{32}$$

for all the considered values of $k_\perp$. Both dispersion relations (22a) and (32) will be shown below for the same physical parameters.

On the same vein, if one compares the dispersion relations for κKAW obtained from either expressions (28) and (29), where the former is the more exact, one will also find marked differences between them. In fact, the discrepancy is much more pronounced for κ-kinetic waves than for the KAW.

These observations are exemplified by Figure 4, where we have adopted $r_v^2 = 2$ and $T_i = T_e$. The curves marked M-A1 (dotted) and M-A2 (dashed) are, respectively, plots of expressions (22a) and (32) as functions of $k_\perp$. The difference between both approximations may seem small, but the reader must notice that the scale is logarithmic.

In the same figure, we included plots of dispersion relations (29) (κ-A1, dash dotted) and (28) (κ-A2, continuous) for the κ-kinetic waves for some values of $\kappa_e = \kappa_i = \kappa$. One can clearly see that the approximation κ-A1 (expression (29)) is only valid for a very restricted range of $k_\perp$ values. Recall that approximation κ-A2 (expression (28)) is more exact than approximation κ-A1. In fact, the latter can even change sign, depending on the values of κ and $k_\perp$. This odd behavior is a mere consequence of the varying relative importance between the consecutive terms of the two series that appear in the expansion of the gyrofunction $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ given by (10). When $\kappa \to \infty$, the second series $\left(\mathcal{S}_2\right)$ vanishes, and only $\mathcal{S}_1$ remains, which is why the KAW dispersion relation (22a) does not display the same behavior. However, when κ is small, both $\mathcal{S}_1$ and $\mathcal{S}_2$ series can be

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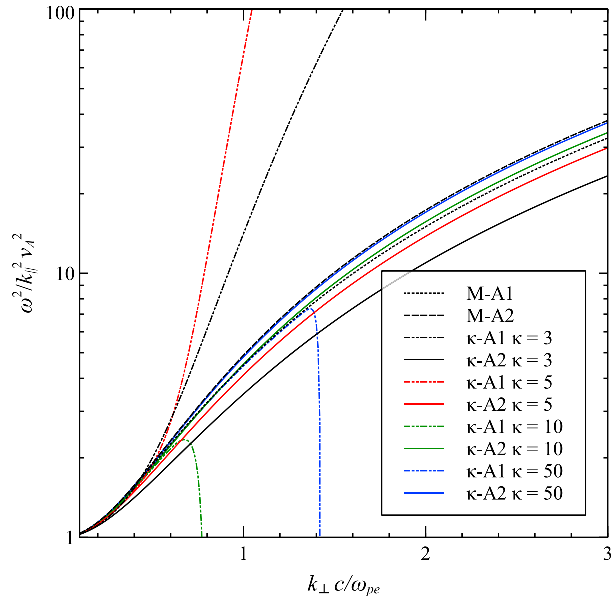

Figure 4. Plots of the dispersion relations of KAW (M-A1 and M-A2) and κKAW (κ-A1 and κ-A2) for $r_v^2 = 2$, $T_i = T_e$, and some values of $\kappa_e = \kappa_i = \kappa$.

of the same order and then one has to exert an extra care on the assessment of the validity of the approximations employed. Therefore, the first conclusion one must reach is that one should only employ approximation κ-A2 (28) to study the κ-kinetic Alfvén waves.

Figure 4 also shows that for small κ index ($\kappa = 3$), the dispersions of KAW and κKAW at $k_\perp c/\omega_{pe} = 3$ differ by around 40%, whereas when $\kappa = 50$, the κKAW are indistinguishable from the KAW.

The numerical solutions of the dispersion equations (21) and (25), where the full thermal effects due to the electrons are maintained, are shown in Figure 5, for the same plasma parameters as in Figure 4. In Figure 5a, the M-A2 dispersion relation for KAW presented in the preceding figure is repeated, in order to compare it with the real part of the solution of (21), marked with the label Maxwell. One can clearly observe that when full thermal

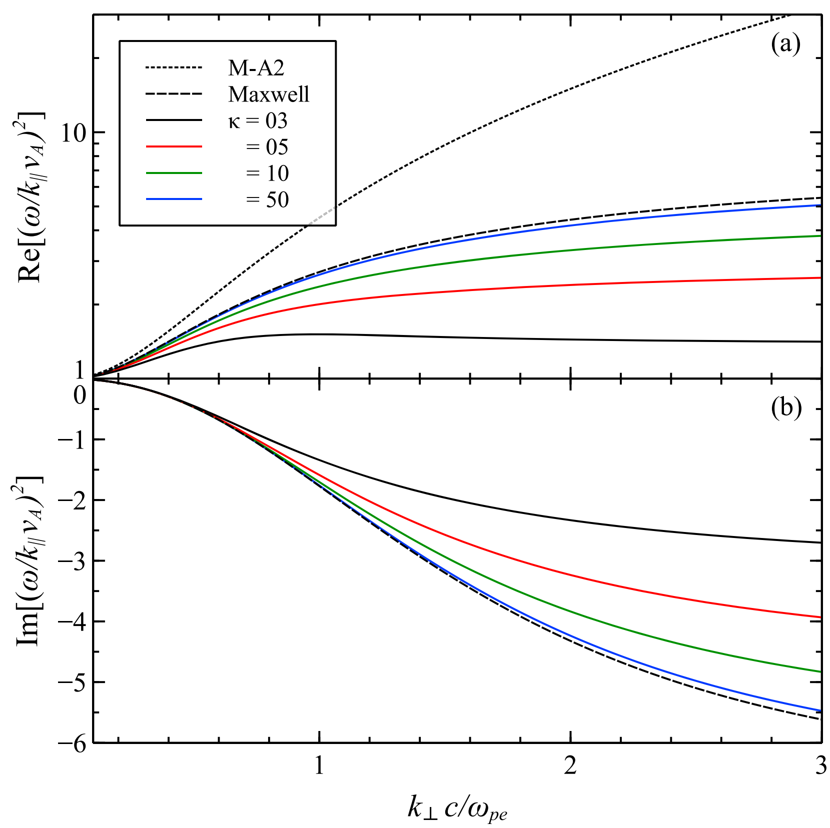

Figure 5. Plots of the (a) real and (b) imaginary parts of the solution of the dispersion equations (21) and (25) for $r_v^2 = 2$, $T_i = T_e$, and some values of $\kappa_e = \kappa_i = \kappa$.

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effects from electrons are maintained in the solution, even in Maxwellian plasmas the approximate dispersion relation (32) differs substantially from the full solution. Its validity, in this case, is restricted to $k_\perp c/\omega_{pe} \lesssim 1$.

The solid curves in Figure 5a show the real parts of the solutions of the κDAW dispersion equation (25). Clearly, when κ is small ($\kappa = 3$), there is a substantial difference in the dispersion of κ-kinetic waves as compared with their Maxwellian counterpart when $k_\perp$ is large. Once again, when κ increases, the difference goes to naught, with the case $\kappa = 50$ already very close to the Maxwellian limit.

Figure 5b shows that the damping coefficient of κKAW, differently to what happens with the κIAW, always decreases with the κ index, with $|\mathrm{Im}\,\omega|_{\max}$ resulting for $\kappa \to \infty$ for all perpendicular wave numbers.

As we have pointed out above, the results shown by Figures 4 and 5 are by no means an exhaustive study about the effect of superthermal particles on the usual dispersion relation (and their damping rate) of the kinetic Alfvén waves. Future communications will endeavor more comprehensive analyses, with the inclusion of anisotropy features in the VDF, in which case one will be able to analyze the effect of the superthermal tails on obliquely propagating instabilities.

6. Conclusions

We have presented a formalism destined to the general description of obliquely propagating waves in superthermal (kappa) plasmas. The details presented in this work were tailored out for the case of the quasi-perpendicular, low-frequency modes called, in general, dispersive Alfvén waves, which are essentially a modification of the usual shear Alfvén waves with the inclusion of the effects of finite ion gyroradius and finite electron mass and thermal pressure terms. These modes are usually denominated inertial or kinetic Alfvén waves, depending on whether the electron inertia or the temperature, respectively, is the dominant effect on wave dispersion.

The main objective of the present work was on the presentation of our formalism, which was then applied to the mentioned dispersive modes. We have shown that the superthermal tails of the kappa distribution functions can affect a visible modification to the usual dielectric tensor components, to the DAW dispersion relations, as well as to their respective damping coefficients. Among the discussed mathematical properties regarding the dielectric tensor, we have shown that for the case of nonparallel waves, not only a generalized plasma dispersion function is necessary but also a superthermal generalization of the gyroradius function appears. Interestingly, the new kappa gyroradius function is shown to display an ambivalent behavior, belonging to either a hypergeometric or a logarithmic class, depending on whether the κ index is integer or not. Moreover, we have shown that even in the hypergeometric regime, the gyroradius function for finite κ is always expressed by a combination of two power series, one of which is proportional to a noninteger power of the particle's gyroradius. These characteristics are completely absent in the Maxwellian gyroradius function, and the numerical results have shown that as a consequence the derivation of approximate expressions for the dispersion relations must be carried out taking these mathematical characteristics into account.

Beyond the practical application presented here, our formalism can be readily extended to more general cases. The adopted VDF is able to reproduce several distributions employed in the literature, such as the ST91 and Le02 models, for instance, which are all examples of isotropic, equilibrium distributions in a canonical ensemble. On the other hand, a straightforward generalization of our model VDF for a grand canonical ensemble is possible, by introducing the chemical potential and by the adequate definition of the constant *w*, thereby reproducing, e.g., the generalized Lorentzian distribution derived by *Treumann* [1999a].

Additionally, although the mathematical expressions and results presented here were adequate for the particular dispersive Alfvén modes, our formalism can be straightforwardly extended to the general case with arbitrary propagation angle, frequency range, and plasma species and temperatures. The formalism can also be applied to other isotropic or to anisotropic (kappa) velocity distribution functions, such as the bi-kappa or product-bi-kappa functions, in which case one will be able to study the generation and propagation of kinetic and reactive instabilities in a superthermal plasma as well. The necessary details for this general treatment will be divulged in future publications.

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Appendix A: Generalized Hypergeometric Functions

This section presents some mathematical properties of the hypergeometric functions that appear in our work.

A1. The Generalized Hypergeometric Series

The generalized hypergeometric function is defined by the power series *Askey and Daalhuis* [2010]

$$ {}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix};z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}, \tag{A1} $$

where $p,q=0,1,2,\dots$ and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

The power series in ${}_pF_q(\cdots)$ is defined as long as $b_j \neq 0,-1,-2,\dots$ $(j=1,\dots,q)$, and it can be divided in three classes, depending on whether $p \le q$, $p=q+1$ or $p \ge q+2$. In the present work, there appear functions belonging to the first two cases, which are briefly presented below.

A1.1. Case $p \le q$: The ${}_1F_2$ Function

The ${}_1F_2(\cdots)$ function is defined from (A1) as

$$ \begin{aligned} {}_1F_2\left(\begin{matrix} a \\ b,c \end{matrix};z\right) &= \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k(c)_k}\frac{z^k}{k!} \\ &= 1 + \frac{a}{bc}z + \frac{a}{bc}\frac{(a+1)}{(b+1)(c+1)}\frac{z^2}{2} + \cdots . \end{aligned} \tag{A2} $$

For fixed a, b, and c parameters, the series (A2) defines an entire function of z, as long as $b,c \neq 0,-1,-2\dots$. If $a=-m$ $(m=0,1,\dots)$, the ${}_1F_2(\cdots)$ reduces to a polynomial of degree m.

A1.2. Case $p = q + 1$: The ${}_2F_1$ Function

The so-called Gauss hypergeometric function is defined by the Gauss series given from (A1) as [*Daalhuis*, 2010]

$$ \begin{aligned} {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) \doteq F\left(\begin{matrix} a,b \\ c \end{matrix};z\right) &= \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \\ &= 1 + \frac{ab}{c}z + \frac{ab}{c}\frac{(a+1)(b+1)}{(c+1)}\frac{z^2}{2} + \cdots . \end{aligned} \tag{A3} $$

The series (A3) converges inside the unit circle $|z|<1$ and has conditional convergence along $|z|=1$. As a consequence, the series has to be analytically continued to the region $|z|>1$. Since the point $z=1$ is a branch point and the function also has a singularity at $|z| \to \infty$, the branch line runs in the interval $1 \le z < \infty$, defining the principal branch of $F(z)$ as the region $|\arg(1-z)| \le \pi$.

A1.2.1. Transformation Formulae

Among the existing linear and quadratic transformation formulas, we have employed the following below:

$$ F\left(\begin{matrix} a,b \\ a+b-1/2 \end{matrix};z\right) = (1-z)^{-1/2} F\left(\begin{matrix} 2a-1,2b-1 \\ a+b-1/2 \end{matrix};\frac{1}{2}-\frac{1}{2}\sqrt{1-z}\right) \tag{A4} $$

$$ \begin{aligned} F\left(\begin{matrix} a,b \\ c \end{matrix};z\right) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F\left(\begin{matrix} a,b \\ a+b-c+1 \end{matrix};1-z\right) \\ &+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F\left(\begin{matrix} c-a,c-b \\ c-a-b+1 \end{matrix};1-z\right) \quad (|\arg(1-z)|<\pi) \end{aligned} \tag{A5} $$

$$ \begin{aligned} &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} F\left(\begin{matrix} a,1-c+a \\ 1-b+a \end{matrix};\frac{1}{z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} F\left(\begin{matrix} b,1-c+b \\ 1-a+b \end{matrix};\frac{1}{z}\right) \quad (|\arg(-z)|<\pi) \end{aligned} \tag{A6} $$

A1.2.2. Particular Cases

$$ F(a,b;b;z) = {}_1F_0(a;-;z) = (1-z)^{-a}. \tag{A7} $$

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A2. The Meijer *G* Function

The Meijer *G* function is that function whose Mellin transform [*Paris and Kaminski*, 2001] can be expressed as a ratio of certain products of gamma functions. Explicitly,

$$G_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(b_j-s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-s\right)}z^s\mathrm{d}s. \tag{A8}$$

In (A8), $0\le m\le q$ and $0\le n\le p$. If $m+1>q$ or $n+1>p$, the product is replaced by one. The notation is such that $\left(a_p\right)\doteq a_1,a_2,\dots,a_p$ and $\left(b_q\right)\doteq b_1,b_2,\dots,b_q$. It is assumed that the parameters $\left(a_p\right)$ and $\left(b_q\right)$ are such that no pole of $\Gamma\left(b_j-s\right)$ $(j=1,\dots,m)$ coincides with any pole of $\Gamma\left(1-a_k+s\right)$ $(k=1,\dots,n)$. That is, $\left(a_k-b_j\right)$ is not a positive integer. It is also assumed that $z\neq 0$, since the origin is a branch point.

The integral in (A8) is called a Mellin-Barnes integral, and the integration contour L corresponds to that of the inverse Mellin transform but is deformed in such a way that the poles of $\Gamma\left(b_j-s\right)$ $(j=1,\dots,m)$ lie to the right of the integration path and the poles of $\Gamma\left(1-a_j+s\right)$ $(j=1,\dots,n)$ lie to the left of the path. A detailed account on all possible integration contours and properties of the *G* function is given by *Luke* [1975].

The *G* function, although little known in the plasma physics community, has remarkable properties, some of which are presented here. In particular, it can display two different types of behavior, or regimes, depending on the parameters. If no two of the b_h $(h=1,\dots,m)$ parameters differ by an integer, all poles are simple and the *G* function can be expressed in terms of the hypergeometric series (A1) as

$$\begin{aligned}G_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right.\right]=&\sum_{h=1}^{m}\frac{\prod_{j=1}^{m}\Gamma\left(b_j-b_h\right)^*\prod_{j=1}^{n}\Gamma\left(1+b_h-a_j\right)}{\prod_{j=m+1}^{q}\Gamma\left(1+b_h-b_j\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-b_h\right)}\\&\times z^{b_h}\,{}_pF_{q-1}\left(\begin{matrix}1+b_h-\left(a_p\right)\\1+b_h-\left(b_q\right)^*\end{matrix};(-)^{p-m-n}z\right),\end{aligned} \tag{A9}$$

expression which is valid for $p<q$ or $p=q$ and $|z|<1$. The notation $\Gamma\left(b_j-b_h\right)^*$ means that this term is absent when $h=j$.

If any pair of values of b_h differs by an integer, then the *G* function can no longer be expressed simply as the combination of hypergeometric functions given by (A9). In this case, the poles of the $\Gamma(z)$ functions have to be canceled out by a limiting process and the resulting representation of the *G* function contains a logarithmic term. The final expression is rather large and will not be presented here, since for our applications we will always be able to express the logarithmic regime for the *G* function in a compact closed form. The reader is referred to *Luke* [1975] for a detailed account on the derivation.

A2.1. Elementary Properties

$$G_{p,q}^{m,n}\left[z\left|\begin{matrix}\alpha,a_2,\dots a_n,a_{n+1},\dots,a_p\\b_1,\dots,b_m,b_{m+1},\dots,b_{q-1},\alpha\end{matrix}\right.\right]=G_{p-1,q-1}^{m,n-1}\left[z\left|\begin{matrix}a_2,\dots,a_p\\b_1,\dots,b_{q-1}\end{matrix}\right.\right],\ (n,p,q\ge 1) \tag{A10}$$

$$\frac{d^k}{dz^k}\left\{z^{-b_1}G_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right.\right]\right\}=(-)^k z^{-b_1-k}G_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p)\\b_1+k,b_2,\dots,b_q\end{matrix}\right.\right]\ (m\ge 1). \tag{A11}$$

A2.2. Integrals Containing the *G* Function

Several integrals that contain the Meijer function can be expressed by the same function.

$$\int_0^\infty dy\,y^{\alpha-1}\left(y+\beta\right)^{-\sigma}G_{p,q}^{m,n}\left[zy\left|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right.\right]=\frac{\beta^{\alpha-\sigma}}{\Gamma(\sigma)}G_{p+1,q+1}^{m+1,n+1}\left[\beta z\left|\begin{matrix}1-\alpha,(a_p)\\\sigma-\alpha,(b_q)\end{matrix}\right.\right]. \tag{A12}$$

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A2.3. Representations of Special Functions

All special functions can be represented by the G function. A few examples are listed below [*Luke*, 1975; *Prudnikov et al.*, 1990].

$$J_\nu^2\left(\sqrt{z}\right) = \frac{1}{\sqrt{\pi}} G_{1,3}^{1,1}\left[z \middle| \begin{matrix} 1/2 \\ \nu, -\nu, 0 \end{matrix}\right] \tag{A13}$$

$$I_\nu\left(\sqrt{z}\right) K_\nu\left(\sqrt{z}\right) = \frac{1}{2\sqrt{\pi}} G_{1,3}^{2,1}\left[z \middle| \begin{matrix} 1/2 \\ 0, \nu, -\nu \end{matrix}\right]. \tag{A14}$$

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