The general dielectric tensor for bi-kappa magnetized plasmas

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In this paper we derive the dielectric tensor for a plasma containing particles described by an anisotropic superthermal (bi-kappa) velocity distribution function. The tensor components are written in terms of the two-variables kappa plasma special functions, recently defined by Gaelzer and Ziebell [Phys. Plasmas **23**, 022110 (2016)]. We also obtain various new mathematical properties for these functions, which are useful for the analytical treatment, numerical implementation and evaluation of the functions and, consequently, of the dielectric tensor. The formalism developed here and in the previous paper provides a mathematical framework for the study of electromagnetic waves propagating at arbitrary angles and polarizations in a superthermal plasma.

Keywords: Bi-kappa plasmas; kinetic theory; waves; methods: analytical; hypergeometric and Meijer G functions.

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I. INTRODUCTION

During the last years, a substantial portion of the space physics community has been interested in plasma environments which are not in a state of thermal equilibrium, but are instead in a turbulent state. Several of such environments can be found in a nonthermal (quasi-) stationary state. When the velocity distribution functions (VDFs) of the particles that comprise these turbulent plasmas are measured, they often display a high-energy tail that is better fitted by a power-law function of the particle's velocity, instead of the Gaussian profile found in plasmas at the thermodynamic equilibrium.

Among all possible velocity distributions with a power-law tail, the actual VDF that has been marked with a widespread application in space plasmas is the Lorentzian, or kappa, distribution (or a combination of kappas), and the number of published papers that employ the kappa velocity distribution function (κVDF) has been growing by a measurable exponential rate.[1] However, the interest on the kappa distribution is justified not only as a better curve-fitting function. A kappa function is also the velocity probability distribution that results from the maximization of the nonadditive Tsallis entropy postulate. Hence, the κVDF is also the distribution of velocities predicted by Tsallis's entropic principle for the nonthermal stationary state of a statistical system characterized by low collision rates, long-range interactions and strong correlations among the particles. For detailed discussions of the importance of kappa distributions for space plasmas and the connection with nonequilibrium statistical mechanics, the Reader is referred to Refs. 1–4. See also the Introductions of Refs. 5 and 6 for complementary discussions and references to other formulations for the κVDF.

One of the important problems related to space plasmas in which the kappa distribution has been increasingly applied concerns the excitation of temperature-anisotropy-driven instabilities that propagate in electromagnetic or electrostatic modes in a warm plasma. These instabilities (among others) play an important role on the nonlinear evolution and the steady-state of the measured VDFs. They can also lead to particle energization and acceleration and are probably related to some of the fundamental issues in space and astrophysical systems, such as the problem of the heating of the solar corona. Rather than giving here a long list of references, we suggest that the Reader consults the cited literature in our previous works.[5,6]

In the present work, we continue the development of a mathematical formulation destined

in the study of electromagnetic/electrostatic waves (and their instabilities) propagating at arbitrary angles in a warm magnetized plasma, in which the particles are described by asymmetric superthermal, or bi-kappa, VDFs. The formulation presented here employs the linear kinetic theory of plasmas and is an extension and generalization of the treatment developed in Refs. 5 and 6.

The structure of this paper is as follows. In Section II we derive the dielectric tensor for a bi-kappa plasma. The tensor components are written in terms of the kappa plasma special functions introduced and studied in our previous works. Section III contains several new developments and properties of the kappa plasma functions, destined to provide the necessary framework for the evaluation of the functions and the dielectric tensor. After the conclusions in Section IV, we have also included Appendices A, where details about the derivation of the dielectric tensor are given, and B, where additional properties of relevant special functions are derived.

II. THE DIELECTRIC TENSOR

The dielectric tensor for a bi-kappa superthermal plasma will be obtained with the use of the velocity distribution function given by

$$f_s^{(\alpha)}\left(v_\parallel, v_\perp\right) = A_s^{(\sigma_s)}\left(1 + \frac{v_\parallel^2}{\kappa_s w_{\parallel s}^2} + \frac{v_\perp^2}{\kappa_s w_{\perp s}^2}\right)^{-\sigma_s}, \tag{1}$$

which is valid for $\sigma_s > 3/2$ and where $\sigma_s = \kappa_s + \alpha_s$ and

$$A_s^{(\sigma_s)} = \frac{1}{\pi^{3/2} w_{\parallel s} w_{\perp s}^2} \frac{\kappa_s^{-3/2}\Gamma\left(\sigma_s\right)}{\Gamma\left(\sigma_s - 3/2\right)}$$

is the normalization constant. The quantities $w_{\parallel s}$ and $w_{\perp s}$ are respectively proportional to the parallel and perpendicular thermal speeds, but they can be a function of the κ parameter as well. Finally, $\Gamma\left(z\right)$ is the gamma function.

The VDF (1) is the anisotropic generalization of the isotropic $\left(w_\parallel = w_\perp = w\right)$ distribution adopted by Refs. 5 and 6. In these works, it was shown how adequate choices of the parameters α and w reproduce and formally unify seemingly different (kappa) velocity distribution functions employed in the literature. Now, in the anisotropic case, if we set $\alpha = 1$ and

$$w_{\parallel,\perp}^2 = \theta_{\parallel,\perp}^2 = \left(1 - \frac{3}{2\kappa}\right)\left(\frac{2T_{\parallel,\perp}}{m}\right),$$

ing function (1) reduces to the "bi-Lorentzian" distribution introduced by Summers & Thorne[7] (see Table I. See also Eqs. 12a,b of Ref. 4). This distribution will be named here the ST91 model and in all expressions obtained below one can simply drop the parameter α, should the ST91 model be chosen from the start.

However, the parameter α can also be useful when the function $f_s^{(\alpha)}\left(v_\parallel, v_\perp\right)$ describes (isotropic) one-particle distribution functions with an arbitrary number of degrees of freedom. If (1) describes the probability distribution function of a particle with f degrees of freedom, one can set $\kappa = \kappa_0$, where κ_0 is the *invariant* kappa parameter introduced by Livadiotis & McComas[1], $\alpha = 1 + f/2$, $w^2 = \theta^2 = 2T/m$, $v^2 = \sum_{i=1}^{f} v_i^2$, and the normalization constant is $A^{(f)} = \Gamma\left(\kappa_0 + 1 + f/2\right)\left(\pi\kappa_0\theta^2\right)^{-f/2}/\Gamma\left(\kappa_0 + 1\right)$, thereby obtaining Eq. (22c) of Ref. 1.

Particular forms of the bi-kappa VDF (1) or its bi-Maxwellian limiting case (when $\kappa_s \to \infty$) have been frequently employed in the literature in order to study temperature-anisotropy-driven instabilities that amplify parallel- or oblique-propagating eigenmodes in a magnetized plasma. Of particular importance for the present work are the effects of finite particle gyroradius (or Larmor radius) on the dispersion and amplification/damping of oblique-propagating modes. For instance, Yoon *et al.*[8] discovered the oblique Firehose instability, which is a non-propagating instability excited in a high-beta bi-Maxwellian plasma when the ions display temperature anisotropy (with $T_{\parallel i} > T_{\perp i}$) and which is continuously connected to the left-handed branch of the Alfvén waves when the ion gyroradius tends to zero. The same instability was later rediscovered by Hellinger & Matsumoto.[9]

Other studies subsequently considered the excitation of low-frequency instabilities at arbitrary angles in bi-Maxwellian plasmas for other situations, such as low-beta plasmas,[10] or with additional free energy sources such as electronic temperature anisotropy,[11] field-aligned currents,[12] loss-cones,[13] and density inhomogeneities[14] (see also reviews by Refs. 15–17).

In comparison, similar studies employing anisotropic superthermal distributions are rare. Summers *et al.*[18] obtained the first expressions for the general dielectric tensor of a bi-kappa (bi-Lorentzian) plasma. However, their final expressions are not written in a closed form, i.e., for each component of the tensor, there remains a final integral along $v_\perp$ (the perpendicular component of the particle's velocity) that should be numerically evaluated. A similar approach was later adopted by Basu,[19] Liu *et al.*[20] and Astfalk *et al.*[21]

In order to circumvent the mathematical difficulties involved in the integration along $v_\perp$, Cattaert *et al.*[22] derived the dielectric tensor and considered some simple cases of oblique waves propagating in a kappa-Maxwellian plasma. More recently, Sugiyama *et al.*[23] employed the same VDF in a first systematic study of the propagation of electromagnetic ion-cyclotron waves in the Earth's magnetosphere.

Closed-form expressions for the components of the dielectric tensor of a superthermal plasma were for the first time obtained by Gaelzer & Ziebell,[5,6] still for the particular case of isotropic $\left(w_{\parallel s} = w_{\perp s}\right)$ distributions. Here, we will obtain the dielectric tensor for the bi-kappa VDF given by (1).

The general form of the dielectric tensor can be written as[6]

$$\varepsilon_{ij}\left(\boldsymbol{k},\omega\right) = \delta_{ij} + \sum_s \chi_{ij}^{(s)}\left(\boldsymbol{k},\omega\right), \tag{2a}$$

$$\chi_{ij}^{(s)}\left(\boldsymbol{k},\omega\right) = \frac{\omega_{ps}^2}{\omega^2}\left[\sum_{n\to-\infty}^{\infty}\int d^3v \frac{v_\perp \left(\Xi_{ns}\right)_i \left(\Xi_{ns}^*\right)_j \mathcal{L}f_s}{\omega - n\Omega_s - k_\parallel v_\parallel} + \delta_{iz}\delta_{jz}\int d^3v \frac{v_\parallel}{v_\perp} L f_s\right] \tag{2b}$$

where $\chi_{ij}^{(s)}$ is the susceptibility tensor associated with particle species s, the set $\{i,j\} = \{x,y,z\}$ identifies the Cartesian (in the E^3 space) components of the tensors, with $\{\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}}\}$ being the basis in E^3, $\boldsymbol{\Xi}_{ns} = n\varrho_s^{-1}J_n\left(\varrho_s\right)\hat{\boldsymbol{x}} - iJ_n'\left(\varrho_s\right)\hat{\boldsymbol{y}} + \left(v_\parallel/v_\perp\right)J_n\left(\varrho_s\right)\hat{\boldsymbol{z}}$, where $J_n\left(z\right)$ is the Bessel function of the first kind,[24,25] $\varrho_s = k_\perp v_\perp/\Omega_s$, $Lf_s = v_\perp\partial f_s/\partial v_\parallel - v_\parallel\partial f_s/\partial v_\perp$, $\mathcal{L}f_s = \omega\partial f_s/\partial v_\perp + k_\parallel L f_s$. Also, $\omega_{ps}^2 = 4\pi n_s q_s^2/m_s$ and $\Omega_s = q_s B_0/m_s c$ are the plasma and cyclotron frequencies of species s, respectively, ω and $\boldsymbol{k} = k_\perp\hat{\boldsymbol{x}} + k_\parallel\hat{\boldsymbol{z}}$ are the wave frequency and wavenumber, $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$ $(B_0 > 0)$ is the ambient magnetic induction field and the symbols $\parallel$ $(\perp)$ denote the usual parallel (perpendicular) components of vectors/tensors, respective to $\boldsymbol{B}_0$.

Inserting the function (1) into (2b) we obtain the desired susceptibility tensor for a bi-kappa plasma. More details on the derivation of the components of χ_{ij} are given in Appendix A. Here we will presently show the final, closed-form expressions, given by

$$\chi_{xx}^{(s)} = \frac{\omega_{ps}^2}{\omega^2}\sum_{n\to-\infty}^{\infty}\frac{n^2}{\mu_s}\left[\xi_{0s}\mathcal{Z}_{n,\kappa_s}^{(\alpha_s,2)}\left(\mu_s,\xi_{ns}\right) + \frac{1}{2}A_s\partial_{\xi_{ns}}\mathcal{Z}_{n,\kappa_s}^{(\alpha_s,1)}\left(\mu_s,\xi_{ns}\right)\right] \tag{3a}$$

$$\chi_{xy}^{(s)} = i\frac{\omega_{ps}^2}{\omega^2}\sum_{n\to-\infty}^{\infty} n\left[\xi_{0s}\partial_{\mu_s}\mathcal{Z}_{n,\kappa_s}^{(\alpha_s,2)}\left(\mu_s,\xi_{ns}\right)\right.$$

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$$+\frac{1}{2}A_s\partial^2_{\mu_s,\xi_{ns}}\mathcal{Z}^{(\alpha_s,1)}_{n,\kappa_s}(\mu_s,\xi_{ns})\Big] \tag{3b}$$

$$\chi^{(s)}_{xz} = -\frac{\omega^2_{ps}}{\omega^2}\frac{w_{\parallel s}}{w_{\perp s}}\sum_{n\to-\infty}^{\infty}\frac{n\Omega_s}{k_\perp w_{\perp s}}(\xi_{0s}-A_s\xi_{ns}) \times \partial_{\xi_{ns}}\mathcal{Z}^{(\alpha_s,1)}_{n,\kappa_s}(\mu_s,\xi_{ns}) \tag{3c}$$

$$\chi^{(s)}_{yy} = \frac{\omega^2_{ps}}{\omega^2}\sum_{n\to-\infty}^{\infty}\Big[\xi_{0s}\mathcal{W}^{(\alpha_s,2)}_{n,\kappa_s}(\mu_s,\xi_{ns}) +\frac{1}{2}A_s\partial_{\xi_{ns}}\mathcal{W}^{(\alpha_s,1)}_{n,\kappa_s}(\mu_s,\xi_{ns})\Big] \tag{3d}$$

$$\chi^{(s)}_{yz} = i\frac{\omega^2_{ps}}{\omega^2}\frac{w_{\parallel s}}{w_{\perp s}}\frac{k_\perp w_{\perp s}}{2\Omega_s}\sum_{n\to-\infty}^{\infty}(\xi_{0s}-A_s\xi_{ns}) \times \partial^2_{\mu_s,\xi_{ns}}\mathcal{Z}^{(\alpha_s,1)}_{n,\kappa_s}(\mu_s,\xi_{ns}) \tag{3e}$$

$$\chi^{(s)}_{zz} = -\frac{\omega^2_{ps}}{\omega^2}\frac{w^2_{\parallel s}}{w^2_{\perp s}}\sum_{n\to-\infty}^{\infty}(\xi_{0s}-A_s\xi_{ns}) \times \xi_{ns}\partial_{\xi_{ns}}\mathcal{Z}^{(\alpha_s,1)}_{n,\kappa_s}(\mu_s,\xi_{ns}), \tag{3f}$$

where

$$\mathcal{W}^{(\alpha,\beta)}_{n,\kappa}(\mu,\xi) = \frac{n^2}{\mu}\mathcal{Z}^{(\alpha,\beta)}_{n,\kappa}(\mu,\xi) - 2\mu\mathcal{Y}^{(\alpha,\beta)}_{n,\kappa}(\mu,\xi).$$

Notice that the off-diagonal components of χ_{ij} obey the symmetry relations $\chi_{xy} = -\chi_{yx}$, $\chi_{xz} = \chi_{zx}$, and $\chi_{yz} = -\chi_{yz}$.

In (3a-f) we have defined the parameters $\mu_s = k^2_\perp\rho^2_s$, $\rho^2_s = w^2_{\perp s}/2\Omega^2_s$, and $\xi_{ns} = (\omega - n\Omega_s)/k_\parallel w_{\parallel s}$. The parameter ρ_s is the (kappa modified) gyroradius (or Larmor radius) of particle s. Hence, μ_s is the normalized gyroradius, proportional to the ratio between ρ_s and $\lambda_\perp$, the perpendicular projection of the wavelength. The magnitude of μ_s quantifies the finite Larmor radii effects on wave propagation. On the other hand, the parameter ξ_{ns} quantifies the linear wave-particle interactions in a finite-temperature plasma. Also in (3a-f), the quantity

$$A_s = 1 - \frac{w_{\perp s}}{w_{\parallel s}}$$

is the anisotropy parameter, which quantifies the effects of the VDF's departure from an isotropic distribution, due to the temperature anisotropy. The symbol $\partial^n_{z_1,\ldots,z_n} = \partial^n/(\partial z_1\cdots\partial z_n)$ is the n-th order partial derivative, relative to $z_1,\ldots,z_n$.

Finally, the functions $\mathcal{Z}^{(\alpha,\beta)}_{n,\kappa}(\mu,\xi)$ and $\mathcal{Y}^{(\alpha,\beta)}_{n,\kappa}(\mu,\xi)$ are the so-called *two-variables kappa plasma functions*. Their definitions were first given in Ref. 6 (hereafter called Paper I) and are repeated in Eqs. (20a)-(20f). Some properties and representations of $\mathcal{Z}$ and $\mathcal{Y}$

were also obtained in Paper I and several new properties and representations will be derived in Sec. III. The evaluation of the functions $\mathcal{Z}$ and $\mathcal{Y}$ is determined not only by their arguments μ (normalized gyroradius) and ξ (wave-particle resonance), but also by a set of parameters: n (harmonic number), κ (kappa index), and the pair (α, β). Parameter α is the same real number adopted for the κVDF (1). This parameter can be ignored and removed from the equations if the distribution model is fixed. On the other hand, the real parameter β is crucial for the formalism. The value of β is related to the specific dielectric tensor component, wave polarization and mathematical properties of the kappa plasma functions.

The isotropic limit of $\chi_{ij}^{(s)}$ is obtained from (3a-f) by setting $A_s = 0$ (and $w_{\perp s} = w_{\parallel s} = w_s$). In this case the susceptibility tensor for each particle species reduces to the form that can be easily gleaned from the Cartesian components of ε_{ij} presented in Appendix C of Paper I. On the other hand, the susceptibility tensor of a bi-Maxwellian plasma is also obtained from (3a-f) by the process called the *Maxwellian limit*, i.e., the result of taking the limit $\kappa_s \to \infty$, for any species s. The Maxwellian limit of $\chi_{ij}^{(s)}$ is given by Eqs. (A1a-f).

Equations (2-3) show the general form for the dielectric tensor of a bi-Kappa plasma. These expressions, along with the representations of the kappa plasma functions derived in Paper I and in Sec. III, contain sufficient information for a methodical study of the properties of wave propagation and emission/absorption in an anisotropic, superthermal plasma. Future works will implement an analysis of temperature-anisotropy-driven instabilities excited in low-frequency parallel- and oblique-propagating electromagnetic eigenmodes.

III. NEW EXPRESSIONS FOR THE KAPPA PLASMA SPECIAL FUNCTIONS

A. The superthermal plasma gyroradius function

The function $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ quantifies the physical effects on wave propagation due to the particles's finite gyroradii when their probability distribution function is described by a kappa VDF. For this reason, it was named by Paper I as the (kappa) *plasma gyroradius function* (κPGF). The basic definition of this function was given in Eq. (I.20) (i.e., Eq. 20 of Paper I) and is repeated here,

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = 2\int_0^\infty dx\, \frac{x J_n^2(yx)}{(1+x^2/\kappa)^{\lambda-1}}, \qquad \left(y^2 = 2z\right), \tag{4}$$

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where $\lambda = \kappa + \alpha + \beta$.

The Maxwellian limit of this function is the well-know representation in terms of the modified Bessel function,[24]

$$\lim_{\kappa\to\infty} \mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \mathscr{H}_n(z) = e^{-z} I_n(z). \tag{5}$$

Sections III.B and A.2 of Paper I contain several mathematical properties of $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ and most of them will not be shown here, with a few important exceptions. One of the exceptions is its general, closed-form representation in terms of the Meijer G-function, as shown in Eq. (I.22). Namely,

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{\pi^{-1/2}\kappa}{\Gamma(\lambda-1)} G_{1,3}^{2,1}\left[2\kappa z \,\middle|\, \begin{matrix} 1/2 \\ \lambda-2, n, -n \end{matrix}\right] \tag{6a}$$

$$= \frac{\pi^{-1/2}\kappa}{\Gamma(\lambda-1)} G_{3,1}^{1,2}\left[\frac{1}{2\kappa z} \,\middle|\, \begin{matrix} 3-\lambda, 1-n, 1+n \\ 1/2 \end{matrix}\right]. \tag{6b}$$

Representation (6b) was obtained employing the symmetry property of the G-function given by Eq. (I.11a).

The definition and some properties of the G-function can be found in Sec. B.2 of Paper I and in the cited literature. Some additional properties, employed in the present paper, are given in Appendix B.

Additional mathematical properties of the $\mathcal{H}$-function, that were not included in Paper I, will be presented here.

1. *Derivatives*

Equations (I.25a)-(I.25d) show recurrence relations for the $\mathcal{H}$-function that involve its first derivative and that in the Maxwellian limit reduce to the respective relations for $\mathscr{H}_n(z)$, easily obtained from the properties of the modified Bessel function.

It is also possible to obtain closed-form representations for the derivatives of $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ in any order. Applying the operator $D^k \equiv d^k/dz^k$ $(k = 0, 1, 2, \dots)$ on (6a) and employing identity (B1a), we obtain

$$\frac{\mathcal{H}_{n,\kappa}^{(\alpha,\beta)(k)}(z)}{(-z)^{-k}} = \frac{\pi^{-1/2}\kappa}{\Gamma(\lambda-1)} G_{2,4}^{3,1}\left[2\kappa z \,\middle|\, \begin{matrix} 1/2, 0 \\ k, \lambda-2, n, -n \end{matrix}\right], \tag{7a}$$

where $\mathcal{H}^{(k)} = d^k\mathcal{H}/dz^k$.

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Formula (7a) is valid for any z and k, but the value of $\mathcal{H}$ at the origin must be treated separately. Applying the operator D^k on the definition (4), we can employ the power series expansion of $J_n^2(yx)$ given by Eq. (10.8.3) of Ref. 24 in order to evaluate the integral in the limit $y \to 0$, thereby obtaining

$$\frac{\mathcal{H}_{n,\kappa}^{(\alpha,\beta)(k)}(0)}{(2k)!\kappa} = \left(\frac{-\kappa}{2}\right)^k \frac{(\lambda-2)_{-k}}{\lambda-2} \sum_{\ell=0}^{k} \frac{(-)^\ell \delta_{|n|,\ell}}{(k+\ell)!\,(k-\ell)!},$$

which is valid for $\lambda > 2+k$. Here, $\delta_{n,m}$ is the Kronecker delta and $(a)_\ell = \Gamma(a+\ell)/\Gamma(a)$ is the Pochhammer symbol. One can easily verify that the case $k=0$ reduces to Eq. (I.21).

As it happens with $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$, its derivative in any order has two different representations in terms of more usual functions, depending on whether λ is integer or not. These cases will now be addressed.

Case λ noninteger.

If $\lambda \neq 2, 3, \ldots$, then we can employ the representation of the G-function in terms of generalized hypergeometric functions, given by Eq. (I.B14). Hence, we have

$$\begin{aligned}\frac{\mathcal{H}_{n,\kappa}^{(\alpha,\beta)(k)}(z)}{(-z)^{-k}} = \frac{\pi^{-1/2}\kappa}{\Gamma(\lambda-1)}\Bigg[&\frac{\Gamma(n+2-\lambda)\,\Gamma(\lambda-3/2)}{\Gamma(\lambda-1+n)}(2-\lambda)_k\,(2\kappa z)^{\lambda-2} \\ &\times {}_2F_3\left(\begin{matrix}\lambda-3/2,\,\lambda-1\\ \lambda-1-n,\,\lambda-1+n,\,\lambda-1-k\end{matrix};2\kappa z\right) \\ +\frac{\Gamma(\lambda-2-n)\,\Gamma(n+1/2)}{\Gamma(2n+1)}&(-n)_k\,(2\kappa z)^n\,{}_2F_3\left(\begin{matrix}n+1/2,\,n+1\\ n+3-\lambda,\,2n+1,\,n+1-k\end{matrix};2\kappa z\right)\Bigg], \end{aligned} \tag{7b}$$

where ${}_2F_3\left(\cdots;z\right)$ is another hypergeometric series of class 1, discussed in Sec. B.1 of Paper I. The case $k=0$ reduces to Eq. (I.23).

Case λ integer. Now, writing $\lambda = m+2$ $(m = 0, 1, 2, \ldots)$ in (7a) and looking at the representation (B4b), we notice that if we choose $\mu = n-k$ and $\nu = n+k$ and employ the differentiation formula (I.B13a), we can write

$$\begin{aligned}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)(k)}(z) = &\frac{2\kappa\,(-z)^{-k}\,(-2\kappa z)^m}{\Gamma(m+1)} \\ &\times \frac{d^{m+k}}{dy^{m+k}}\left[y^k I_{n-k}(\sqrt{y})\,K_{n+k}(\sqrt{y})\right]\Big|_{y=2\kappa z},\end{aligned}$$

where $K_m(z)$ is the second modified Bessel function.[24] Finally, employing Leibniz formula for the derivative[26] and the identities written just before Eq. (I.24), we obtain

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)(k)}(z) = \frac{2\kappa z^k}{\Gamma(m+1)}\left(\frac{\kappa z}{2}\right)^{(m+k)/2}\sum_{s=0}^{m+k}(-)^s \times \binom{m+k}{s} K_{n-m+s}\left(\sqrt{2\kappa z}\right) I_{n-k+s}\left(\sqrt{2\kappa z}\right). \quad (7c)$$

As expected, for $k = 0$ this result reduces to (I.24).

2. *Asymptotic expansion*

The representation of the $\mathcal{H}$-function given by (6b) is formally exact for any z and the function could be formally expressed in terms of the ${}_3F_0(\cdots;z)$ hypergeometric series, after using Eq. (I.B14). However, as explained in Sec. B.1 of Paper I, the ${}_3F_0$ belongs to class 3, whose series are everywhere divergent, except at $z = 0$. Hence, the representation of $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ in terms of ${}_3F_0$ only makes sense when one is looking for an asymptotic expansion of $\mathcal{H}$, which provides a finite number of correct digits when $z \gg 1$ if only a finite numbers of terms in the series expansion is kept.

With this caveat in mind, using Eq. (I.B14) in (6b), we obtain

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{1}{\sqrt{\pi}}\frac{\kappa\Gamma(\lambda-3/2)}{\Gamma(\lambda-1)\sqrt{2\kappa z}} \times {}_3F_0\left(\begin{matrix}\lambda-3/2, 1/2+n, 1/2-n \\ -\end{matrix}; \frac{1}{2\kappa z}\right),$$

which, as explained, is only valid on the limit $z \to \infty$. Inserting the series (I.B1), we obtain the asymptotic expansion

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) \simeq \frac{1}{\sqrt{\pi}}\frac{\kappa\Gamma(\lambda-3/2)}{\Gamma(\lambda-1)\sqrt{2\kappa z}} \times \sum_{k=0}\frac{(\lambda-3/2)_k\,(1/2+n)_k\,(1/2-n)_k}{k!\,(2\kappa z)^k}. \quad (8)$$

Notice that the upper limit of the sum is absent. This upper limit must be computationally determined, taking into account the desired number of correct digits in the evaluation of $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$.

The Maxwellian limit of (8) renders

$$\mathscr{H}_n(z) \simeq \frac{1}{\sqrt{2\pi z}}\sum_{k=0}\frac{(-)^k\,\Gamma(n+k+1/2)}{\Gamma(n-k+1/2)\,k!\,(2z)^k},$$

which is exactly the asymptotic expansion of $\mathscr{H}_n(z)$ given by Eq. (8.451.5) of Ref. 27.

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3. *Sum rule*

Sum rules are useful for the numerical evaluation of special functions. If we sum (4) over all harmonic numbers and use the identity[24]

$$\sum_{n\to-\infty}^{\infty} J_n^2(z) = 1, \tag{9}$$

the remaining integral can be evaluated by the definition of the Beta function,[28] resulting

$$\sum_{n\to-\infty}^{\infty} \mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z) = \frac{\kappa}{\lambda-2}. \tag{10}$$

Several other sum rules for $\mathcal{H}$ can be found in the same fashion.

4. *The associated gyroradius function*

Among the representations for the two-variable functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$, derived in the section III C 3, the following function appears,

$$\widetilde{\mathcal{H}}_{n,k,\kappa}^{(\alpha,\beta)}(\mu) = \frac{\pi^{-1/2}\kappa}{\Gamma(\lambda-1)} G_{1,3}^{2,1}\left[2\kappa\mu \,\middle|\, \begin{matrix} 1/2-k \\ \lambda-2, n, -n \end{matrix}\right], \tag{11}$$

which is clearly related to $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$, differing by the parameter k. For this reason, it is christened here as the *associated plasma gyroradius function.*

Some properties of the $\widetilde{\mathcal{H}}$-function are now presented. A trivial property is $\widetilde{\mathcal{H}}_{n,0,\kappa}^{(\alpha,\beta)}(z) = \mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$.

Relation with $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$. The associated PGF is related to the $\mathcal{H}$-function and its derivatives. First, due to the differentiation formula (B1c), it is clear that we can write

$$\widetilde{\mathcal{H}}_{n,k,\kappa}^{(\alpha,\beta)}(\mu) = \mu^{1/2}\frac{d^k}{d\mu^k}\left[\mu^{k-1/2}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(\mu)\right].$$

Then, using Leibniz's formula and the formula for $D^m z^\gamma$ just above (B1c), we obtain

$$\widetilde{\mathcal{H}}_{n,k,\kappa}^{(\alpha,\beta)}(\mu) = \Gamma\left(k+\frac{1}{2}\right)\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{\mu^\ell \mathcal{H}_{n,\kappa}^{(\alpha,\beta)(\ell)}(\mu)}{\Gamma(\ell+1/2)}. \tag{12a}$$

The reciprocate relation is obtained starting from (7a), which is written in terms of the Mellin-Barnes integral with the help of (I.B10). Then, we have

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)(k)}(\mu) = \frac{\kappa(-\mu)^{-k}}{2\pi^{3/2}i\Gamma(\lambda-1)}$$

$$\times \int_L \frac{\Gamma(\lambda-2-s)\,\Gamma(n-s)\,\Gamma(1/2+s)}{\Gamma(n+1+s)\,(2\kappa\mu)^{-s}}\,(-s)_k\,ds.$$

On the other hand, from (11) and (I.B10) again, we have

$$\widetilde{\mathcal{H}}^{(\alpha,\beta)}_{n,k,\kappa}(\mu) = \frac{\left(2\pi^{3/2}i\right)^{-1}\kappa}{\Gamma(\lambda-1)} \times \int_L \frac{\Gamma(\lambda-2-s)\,\Gamma(n-s)\,\Gamma(1/2+s)}{\Gamma(1+n+s)\,(2\kappa\mu)^{-s}}\left(\frac{1}{2}+s\right)_k ds.$$

Then, if we employ the identity

$$(a+b)_n = \sum_{\ell=0}^{n}(-)^\ell \binom{n}{\ell}(a+\ell)_{n-\ell}\,(-b)_\ell\,,$$

we can finally write the reciprocate relation

$$\frac{\mathcal{H}^{(\alpha,\beta)(k)}_{n,\kappa}(\mu)}{\mu^{-k}} = \sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{1}{2}-k\right)_{k-\ell}\widetilde{\mathcal{H}}^{(\alpha,\beta)}_{n,\ell,\kappa}(\mu)\,. \tag{12b}$$

Representations. The computation of $\widetilde{\mathcal{H}}^{(\alpha,\beta)}_{n,k,\kappa}(\mu)$ can be carried out as follows. For non-integer λ, it is more efficient to employ identity (I.B14) and evaluate

$$\widetilde{\mathcal{H}}^{(\alpha,\beta)}_{n,k,\kappa}(z) = \frac{\pi^{-1/2}\kappa}{\Gamma(\lambda-1)}\left[\frac{\Gamma(n+2-\lambda)}{\Gamma(\lambda-1+n)}(2\kappa z)^{\lambda-2}\,h_k(z) + \frac{\Gamma(\lambda-2-n)}{\Gamma(2n+1)}(2\kappa z)^n\,g_k(z)\right], \tag{13}$$

where

$$\frac{h_k(z)}{\Gamma(\lambda-3/2+k)} = {}_1F_2\left(\begin{matrix}\lambda-3/2+k\\ \lambda-1-n,\lambda-1+n\end{matrix};2\kappa z\right)$$
$$\frac{g_k(z)}{\Gamma(n+1/2+k)} = {}_1F_2\left(\begin{matrix}n+1/2+k\\ n+3-\lambda,2n+1\end{matrix};2\kappa z\right).$$

On the other hand, for integer λ the only representation found for $\widetilde{\mathcal{H}}$ similar to (7c) contains a double sum. Consequently, it is equivalent to simply employ Eqs. (12a) and (7c).

Recurrence relation. The numerical computation of $\widetilde{\mathcal{H}}^{(\alpha,\beta)}_{n,k,\kappa}(z)$ can be carried out using either Eq. (13) or Eqs. (12a) and (7c) combined. However, since the associate function appears in series involving the parameter k, if a recurrence relation for $\widetilde{\mathcal{H}}^{(\alpha,\beta)}_{n,k,\kappa}(z) \equiv \widetilde{\mathcal{H}}[k]$ on this parameter could be found, it could substantially reduce the computational time required for the evaluation of the series.

Such recurrence relation can be found by first considering the particular case of noninteger λ, given by Eq. (13). We observe that the auxiliary functions $h_k(z)$ and $g_k(z)$ in (13) and, consequently, the function $\widetilde{\mathcal{H}}[k]$ itself, all obey the same four-term recurrence relation, which can be derived from the corresponding relation for the function ${}_1F_2(\cdots;z)$ in the upper parameter, given by Ref. 29. Namely,

$$\widetilde{\mathcal{H}}[k+3] - \left(\lambda + \frac{5}{2} + 3k\right)\widetilde{\mathcal{H}}[k+2] + \left[2\lambda - n^2 - \frac{3}{4} + 2\left(\lambda + 1 + \frac{3}{2}k\right)k - 2\kappa z\right]\widetilde{\mathcal{H}}[k+1] + \left(\lambda - \frac{3}{2} + k\right)\left(n + \frac{1}{2} + k\right)\left(n - \frac{1}{2} - k\right)\widetilde{\mathcal{H}}[k] = 0. \quad (14)$$

Although the relation (14) was derived for noninteger λ, it can be easily shown that it is indeed valid for any λ. Substituting into the functions $\widetilde{\mathcal{H}}[k]$ in (14) the definition (11) and then the corresponding representations in terms of Mellin-Barnes integrals (Eq. I.B10), one can show, by using known properties of the gamma function,[28] that the identity (14) is indeed valid for any real λ.

B. The superthermal plasma dispersion function

The superthermal (or kappa) *plasma dispersion function* (κPDF) was defined by Eq. (I.11), and several of its properties were discussed in sections III.A and A.1 of Paper I. Here, we will merely present a few additional properties, which were not included in Paper I and are important for the work at hand.

1. Representations in terms of the G-function.

Taking the representations (I.15) for $Z_\kappa^{(\alpha,\beta)}(\xi)$ and (I.B15a) for the Gauss function, we have

$$Z_\kappa^{(\alpha,\beta)}(\xi) = -\frac{\pi^{1/2}\kappa^{-\beta-1}\xi}{\Gamma(\sigma - 3/2)} G_{2,2}^{1,2}\left[\frac{\xi^2}{\kappa}\middle|\begin{matrix}0, 3/2-\lambda\\ 0, -1/2\end{matrix}\right] + \frac{i\pi^{1/2}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\left(1+\frac{\xi^2}{\kappa}\right)^{-(\lambda-1)}. \quad (15a)$$

As explained in Paper I, the Maxwellian limit of this representation reduces to the known expression of the Fried & Conte function in terms of the Kummer confluent hypergeometric series.

Another, more compact, representation is obtained if we first modify the limits of the integral in (I.11) to the interval $0 \leqslant s < \infty$, define the new integration variable $s = \sqrt{u}$ and identify the resulting integration with the identity (I.B12). Proceeding in this way, we obtain the equivalent representation

$$Z_{\kappa}^{(\alpha,\beta)}(\xi) = \frac{\pi^{-1/2}\kappa^{-\beta-1}\xi}{\Gamma(\sigma - 3/2)} G_{2,2}^{2,2}\left[-\frac{\xi^2}{\kappa}\,\middle|\,\begin{matrix}0, 3/2 - \lambda \\ 0, -1/2\end{matrix}\right]. \tag{15b}$$

Taking the limit $\kappa \to \infty$ of (15b), and identifying the result with (B4c), we obtain

$$\lim_{\kappa\to\infty} Z_{\kappa}^{(\alpha,\beta)}(\xi) = \xi U\left(\begin{matrix}1 \\ 3/2\end{matrix}; -\xi^2\right),$$

where $U(\cdots; z)$ is the Tricomi confluent hypergeometric function.[30] This is another known representation of the Fried & Conte function.[31]

2. *The associated plasma dispersion function*

The *associated plasma dispersion function*, defined by

$$\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi) \doteq \frac{\kappa^{-(k+\beta+1/2)}\Gamma(\lambda - 1)}{\sqrt{\pi}\Gamma(\sigma - 3/2)} \times \int_{-\infty}^{\infty} ds\, \frac{s^{2k}\left(1 + s^2/\kappa\right)^{-(\lambda-3/2+k)}}{s - \xi}, \tag{16}$$

is another new special function that appears in the series expansions derived in section III C 3 for the two-variables special functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$. It has the trivial property

$$\widetilde{Z}_{0,\kappa}^{(\alpha,\beta)}(\xi) = \frac{\Gamma(\lambda - 1)}{\sqrt{\kappa}\Gamma(\lambda - 3/2)} Z_{\kappa}^{(\alpha,\beta-1/2)}(\xi), \tag{17}$$

and, in the following, alternative representations for the $\widetilde{Z}$-function will be derived.

Representations. The first expression is valid when λ is half-integer ($\lambda = 5/2, 7/2, \dots$). In this case, writing $m = \lambda - 3/2 + k$ ($m = 1, 2, \dots$), all singular points in (16), at $s = \xi$ and $s = \pm i\sqrt{\kappa}$, are poles and thus we are permitted to evaluate $\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi)$ using the residue theorem, exactly as was done by Summers and Thorne.[7]

Let us consider the contour integral

$$I_B = \int_B ds\, \frac{s^{2k}\left(1 + s^2/\kappa\right)^{-(\lambda+k-3/2)}}{s - \xi},$$

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where the contour B is comprised by the semicircle in the lower-half plane of complex s (with radius $S \to \infty$), which is closed by the integration along the real line of s, deformed according to the Landau prescription (i.e., circulating around the pole at $s = \xi$ from below). See, for instance, the contour in Fig. 2 of Ref. 7, but with closing in the lower-half s-plane. Then, it is easy to show that the contribution along the semicircle of radius S vanishes as $S \to \infty$ and I_B is simply evaluated from the residue at $s = -i\sqrt{\kappa}$ as $I_B = -2\pi i \mathrm{Res}\left(-i\sqrt{\kappa}\right)$, since the pole at $s = \xi$ is always outside B.

The residue is evaluated by the usual formula for a pole of order m,[32] leading to the representation

$$\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi) = \frac{2\sqrt{\pi} i\,(-)^k\,\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)} \times \sum_{\ell=0}^{M}\sum_{r=0}^{m-1-\ell} \frac{(-2k)_\ell\,(m)_r\,(1)_{m-1-\ell-r}}{2^{m+r}\Gamma(m-\ell-r)\,\ell!r!} \times \left(1-\frac{i\xi}{\sqrt{\kappa}}\right)^{-(m-\ell-r)}, \quad (18a)$$

where $M = \min(m-1, 2k)$. One can easily verify in (18a) that for integer κ,

$$\widetilde{Z}_{0,\kappa}^{(1,3/2)}(\xi) = \frac{\Gamma(\kappa+3/2)}{\kappa^{1/2}\kappa!} Z_\kappa^*(\xi),$$

where $Z_\kappa^*(\xi)$ is given by Eq. (20) of Ref. 7.

A different expression for the $\widetilde{Z}$-function will now be obtained, which is valid for any real λ. We already know that for $k = 0$ the $\widetilde{Z}$-function is given in terms of the κPDF by (17). Now, for $k \geqslant 1$, we modify the limits of the integral in (16) to the interval $0 \leqslant s < \infty$, define the new variable $s = \sqrt{u}$ and employ identity (I.B12) in order to obtain

$$\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi) = \frac{i\kappa^{-(\beta+1/2)}\Gamma(\lambda-1)}{\sqrt{\pi}\Gamma(\sigma-3/2)\,\Gamma(\lambda+k-3/2)} \times G_{2,2}^{2,2}\left[-\frac{\xi^2}{\kappa}\,\middle|\,\begin{matrix}1/2,\,5/2-\lambda\\ k,\,1/2\end{matrix}\right], \quad (18b)$$

which is a G-function representation of the associated PDF. If we now employ formula (I.B13a), we can write

$$\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi) = \frac{i\kappa^{-(k+\beta+1/2)}\Gamma(\lambda-1)}{\sqrt{\pi}\Gamma(\sigma-3/2)\,\Gamma(\lambda+k-3/2)} \times \xi^{2k}\frac{d^k}{dz^k} G_{2,2}^{2,2}\left[z\,\middle|\,\begin{matrix}1/2,\,5/2-\lambda\\ 0,\,1/2\end{matrix}\right],$$

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where we have provisionally defined $z = -\xi^2/\kappa$. This result will now be identified with the derivatives of $Z_\kappa^{(\alpha,\beta)}(\xi)$.

If we take representation (15b) of the κPDF, evaluate the k-th derivative on ξ and employ the differentiation formula (Eq. 1.1.1.2 of Ref. 33),

$$\frac{d^k}{dz^k}\left[f\left(\sqrt{z}\right)\right] = \sum_{\ell=0}^{k-1}(-)^\ell \frac{\Gamma(k+\ell)}{\Gamma(k-\ell)\,\ell!} \times \left(2\sqrt{z}\right)^{-k-\ell} f^{(k-\ell)}\left(\sqrt{z}\right),$$

which is valid for $k \geqslant 1$, we finally obtain

$$\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi) = \frac{\kappa^{-1/2}\Gamma(\lambda-1)}{2^k\Gamma(\lambda-3/2+k)} \sum_{\ell=0}^{k-1} \frac{\Gamma(k+\ell)}{2^\ell\Gamma(k-\ell)\,\ell!} \times (-\xi)^{k-\ell} Z_\kappa^{(\alpha,\beta-1/2)(k-\ell)}(\xi). \quad (18c)$$

A final representation for $\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi)$ will be derived by returning to (16), changing the integration variable to t, defined as $s^2 = \kappa t^{-1}(1-t)$, and comparing the resulting integral with the formula (I.B5). In this way, we obtain

$$\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi) = \frac{\Gamma(\lambda-1)\,B(\lambda-1,k+1/2)}{\sqrt{\pi}\kappa^{\beta+1}\Gamma(\sigma-3/2)} \times \xi\,{}_2F_1\left(\begin{matrix}1,\lambda-1\\ \lambda-1/2+k\end{matrix};1+\frac{\xi^2}{\kappa}\right), \ (\Im\xi > 0), \quad (18d)$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function[28] and ${}_2F_1(\cdots;z)$ is the Gauss hypergeometric function[34] (see also Sec. B.1 of Paper I). It must be pointed out that the representation (18d) is only valid for the upper-half of the ξ-plane. In order to employ this expression when $\Im\xi \leqslant 0$, one must evaluate also its analytical continuation, employing the same technique applied to Eq. (I.13). The resulting expressions for the functions $\mathcal{Z}$ and $\mathcal{Y}$ are shown in Eqs. (25d) and (27c).

Recurrence relation. The representation (18d) also allowed us to obtain a recurrence relation for the associated PDF on the parameter k. Employing the shorthand notation $\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi) \equiv \widetilde{Z}[k]$, we can write

$$\widetilde{Z}[k] = \frac{[\Gamma(\lambda-1)]^2}{\sqrt{\pi}\kappa^{\beta+1}\Gamma(\sigma-3/2)}\xi z_k(\xi),$$
$$z_k \doteq \frac{\Gamma(k+1/2)}{\Gamma(\lambda-1/2+k)}\,{}_2F_1\left(\begin{matrix}1,\lambda-1\\ \lambda-1/2+k\end{matrix};1+\frac{\xi^2}{\kappa}\right).$$

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Hence, if one finds the recurrence relation for the auxiliary function $z_k(\xi)$, the same relation applies to $\widetilde{Z}[k]$.

Such a recurrence relation on the lower parameter of the Gauss function is given by Ref. 35. Consequently, we obtain

$$\left(\lambda-\frac{1}{2}+k\right)\left(1+\frac{\xi^2}{\kappa}\right)\widetilde{Z}[k+2] - \left[\left(\lambda-\frac{1}{2}+k\right)\frac{\xi^2}{\kappa}+\left(k+\frac{1}{2}\right)\left(1+\frac{\xi^2}{\kappa}\right)\right]\widetilde{Z}[k+1] + \left(k+\frac{1}{2}\right)\frac{\xi^2}{\kappa}\widetilde{Z}[k]=0. \quad (19)$$

This result can be verified by inserting the definition (16) in the place of $\widetilde{Z}[k]$ and then adequately manipulating the integrand.

The three-term recurrence relation (19) can potentially reduce the computational time for the evaluation of the functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$, discussed in the next section.

C. The two-variables kappa plasma functions

The dielectric tensor of a superthermal (kappa) plasma is written in terms of the special functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$, collectively called the *two-variables kappa plasma functions* (2VKPs), as can be verified in Eqs. (I.6a-6d), for an isotropic κVDF, or in Eqs. (3a)-(3f), for a bi-kappa distribution.

The functions $\mathcal{Z}$ and $\mathcal{Y}$ were defined in Eqs. (I.26a-26b) in terms of a single integral involving the superthermal plasma dispersion function (κPDF) $Z_{\kappa}^{(\alpha,\beta)}(\xi)$ (see Sec. III.A of Paper I). These definitions will be repeated below. We will include equivalent definitions in terms of double integrals, which will also be used in this work.

Hence, we define

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)=2\int_0^\infty dx\,\frac{xJ_n^2(\nu x)}{(1+x^2/\kappa)^{\lambda-1}}Z_{\kappa}^{(\alpha,\beta)}\left(\frac{\xi}{\sqrt{1+x^2/\kappa}}\right) \quad (20a)$$

$$=\frac{2}{\pi^{1/2}\kappa^{1/2+\beta}}\frac{\Gamma(\lambda-1)}{\Gamma(\sigma-3/2)}\int_0^\infty dx\int_{-\infty}^\infty ds\,\frac{xJ_n^2(\nu x)}{s-\xi}\left(1+\frac{x^2}{\kappa}+\frac{s^2}{\kappa}\right)^{-(\lambda-1)}, \quad (20b)$$

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)=\frac{2}{\mu}\int_0^\infty dx\,\frac{x^3J_{n-1}(\nu x)J_{n+1}(\nu x)}{(1+x^2/\kappa)^{\lambda-1}}Z_{\kappa}^{(\alpha,\beta)}\left(\frac{\xi}{\sqrt{1+x^2/\kappa}}\right) \quad (20c)$$

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$$= \frac{2}{\pi^{1/2}\kappa^{1/2+\beta}\mu}\frac{\Gamma(\lambda-1)}{\Gamma(\sigma-3/2)}\int_0^\infty dx\int_{-\infty}^\infty ds\,\frac{x^3 J_{n-1}(\nu x)\,J_{n+1}(\nu x)}{s-\xi}\left(1+\frac{x^2}{\kappa}+\frac{s^2}{\kappa}\right)^{-(\lambda-1)}, \tag{20d}$$

where $\nu^2 = 2\mu$ and, as usual, $\sigma = \kappa + \alpha$ and $\lambda = \sigma + \beta$.

Other definitions in terms of a single integral can be obtained, which are the counterparts of Eqs. (20a) and (20c). If we change the order of the integrations in (20b) and (20d) and define a new integration variable by $x = \sqrt{\chi}t$, where $\chi = 1 + s^2/\kappa$, the integral in t can be identified with (4) and we can write

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = \frac{\pi^{-1/2}}{\kappa^{\beta+1/2}}\frac{\Gamma(\lambda-1)}{\Gamma(\sigma-3/2)}\int_{-\infty}^\infty ds\,\frac{(1+s^2/\kappa)^{-(\lambda-2)}}{s-\xi}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}\left[\mu\left(1+\frac{s^2}{\kappa}\right)\right] \tag{20e}$$

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = \frac{\pi^{-1/2}}{\kappa^{\beta-1/2}}\frac{\Gamma(\lambda-2)}{\Gamma(\sigma-3/2)}\int_{-\infty}^\infty ds\,\frac{(1+s^2/\kappa)^{-(\lambda-4)}}{s-\xi}\mathcal{H}_{n,\kappa}^{(\alpha,\beta-1)\prime}\left[\mu\left(1+\frac{s^2}{\kappa}\right)\right]. \tag{20f}$$

The Maxwellian limits of the 2VKPs was already obtained in Eq. (I.7) and are

$$\begin{aligned}\lim_{\kappa\to\infty}\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) &= \mathscr{H}_n(\mu)\,Z(\xi)\\ \lim_{\kappa\to\infty}\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) &= \mathscr{H}_n'(\mu)Z(\xi),\end{aligned} \tag{21}$$

where the function $\mathscr{H}_n(\mu)$ is given by (5) and $Z(\xi)$ is the usual Fried & Conte function, given, for instance, by Eq. (I.10).

Some new properties and representations of the functions $\mathcal{Z}$ and $\mathcal{Y}$ that were not included in Paper I will now be discussed.

1. *Derivatives of* $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$

As can be seen in Eqs. (3a)-(3f), almost all tensor components are given in terms of partial derivatives of the function $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$. These derivatives can be easily computed from the direct function, if one uses relations derived from the definitions (20).

We need the partial derivatives $\partial_\xi\mathcal{Z}$, $\partial_\mu\mathcal{Z}$, and the mixed derivative $\partial^2_{\xi,\mu}\mathcal{Z}$. Applying ∂_ξ on (20a) and using Eq. (I.18a), we can identify with (4) and (20a) and write

$$\partial_\xi\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = -2\left[\frac{\Gamma(\lambda-1/2)}{\kappa^{\beta+1}\Gamma(\sigma-3/2)}\mathcal{H}_{n,\kappa}^{(\alpha,\beta+1/2)}(\mu) + \xi\mathcal{Z}_{n,\kappa}^{(\alpha,\beta+1)}(\mu,\xi)\right]. \tag{22a}$$

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Now, applying ∂_μ on (20b) and integrating by parts the x-integral, the resulting expression can be manipulated in order to provide the relation between the derivatives

$$\mu\partial_\mu \mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) - \frac{1}{2}\xi\partial_\xi \mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = (\lambda-2)\,\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) - \kappa\mathcal{Z}_{n,\kappa}^{(\alpha,\beta+1)}(\mu,\xi).$$

Hence, after inserting (22a) there results

$$\mu\partial_\mu \mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = (\lambda-2)\,\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) - \kappa\left(1+\frac{\xi^2}{\kappa}\right)\mathcal{Z}_{n,\kappa}^{(\alpha,\beta+1)}(\mu,\xi) - \frac{\Gamma(\lambda-1/2)}{\kappa^{\beta+1}\Gamma(\sigma-3/2)}\xi\mathcal{H}_{n,\kappa}^{(\alpha,\beta+1/2)}(\mu). \quad (22b)$$

Finally, the crossed derivative can be obtained from either of the results above, leading directly to

$$\partial_{\xi,\mu}^2 \mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = 2\frac{\xi}{\mu}\left[\kappa\left(1+\frac{\xi^2}{\kappa}\right)\mathcal{Z}_{n,\kappa}^{(\alpha,\beta+2)}(\mu,\xi) - (\lambda-1)\,\mathcal{Z}_{n,\kappa}^{(\alpha,\beta+1)}(\mu,\xi)\right] + \frac{2\Gamma(\lambda-1/2)\,\mu^{-1}}{\kappa^{\beta+1}\Gamma(\sigma-3/2)}\left[\left(\lambda-\frac{1}{2}\right)\left(1+\frac{\xi^2}{\kappa}\right)\mathcal{H}_{n,\kappa}^{(\alpha,\beta+3/2)}(\mu) - \left(\lambda-\frac{3}{2}\right)\mathcal{H}_{n,\kappa}^{(\alpha,\beta+1/2)}(\mu)\right]. \quad (22c)$$

2. *Values at $\xi = 0$ or $\mu = 0$*

From the definitions (4, 20b, 20d), and (I.11), we obtain the following limiting expressions,

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(0,\xi) = Z_\kappa^{(\alpha,\beta-1)}(\xi)\,\delta_{n0} \quad (23a)$$

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,0) = \frac{i\sqrt{\pi}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(\mu) \quad (23b)$$

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,0) = \frac{i\sqrt{\pi}\Gamma(\lambda-2)}{\kappa^{\beta-1/2}\Gamma(\sigma-3/2)}\mathcal{H}_{n,\kappa}^{(\alpha,\beta-1)\prime}(\mu) \quad (23c)$$

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(0,\xi) = -\left(\delta_{n,0} - \frac{1}{2}\delta_{|n|,1}\right)Z_\kappa^{(\alpha,\beta-3)}(\xi) \quad (23d)$$

$$\partial_\xi \mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(0,\xi) = Z_\kappa^{(\alpha,\beta-1)\prime}(\xi)\,\delta_{n0} \quad (23e)$$

$$\partial_\mu \mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,0) = \frac{i\sqrt{\pi}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\mathcal{H}_{n,\kappa}^{(\alpha,\beta)\prime}(\mu). \quad (23f)$$

3. *Series representations*

In Paper I, we have obtained representations for the functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ in terms of series involving the κPGF $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(\mu)$ and derivatives of the κPDF $Z_\kappa^{(\alpha,\beta)}(\xi)$.

These representations are given by Eqs. (I.28a-28b). Subsequent applications have shown that these expansions start to converge slower when $\xi_i \to -\frac{1}{2}\sqrt{\kappa}$ (ξ_i: imaginary part of ξ) and may diverge when $\xi_i \geqslant -\frac{1}{2}\sqrt{\kappa}$. Consequently, new function representations are necessary, in order to enlarge the convergence region of the expansions.

In this section, some new expansions for the 2VKPFs are derived. Some of the obtained expansions are applicable to particular regions of the functions's domain and some are valid throughout the domain. However, all representations that have been found have in common that at least one series expansion is involved, which contains at least one special function. This is due to the fact that we were not able to factor the functions in two simpler terms, i.e., $\mathcal{Z}(\mu,\xi) \neq F_1(\mu)\,F_2(\xi)$, for instance. Indeed, we believe that the functions $\mathcal{Z}(\mu,\xi)$ and $\mathcal{Y}(\mu,\xi)$ are in fact altogether non-separable.

The transcendental relation between the variables μ ($\sim w_\perp$) and ξ $\left(\sim w_\parallel\right)$ ultimately stems from the physical nature of the κVDF (1). According to the interpretation of Tsallis's entropic principle, one-particle distribution functions such as (1) describe the statistical distribution of particles in a (almost) noncollisional system, but with a strong correlation between the different degrees of freedom.[1,4] This strong correlation prevents the κVDF (1) from being separable in the different velocity components. In contrast, a physical system in thermal equilibrium has an entropy given by the Boltzmann-Gibbs statistical mechanics and is characterized by short-range Coulombian collisions and absence of correlation between the degrees of freedom. As a consequence, the equilibrium Maxwell-Boltzmann VDF is completely separable. Therefore, the non-separable nature of the functions $\mathcal{Z}(\mu,\xi)$ and $\mathcal{Y}(\mu,\xi)$ is a mathematical consequence of the strong correlation between different degrees of freedom of the particles that compose physical systems statistically described by the κVDF.

It is worth mentioning here that the nonadditive statistical mechanics also admits that particles without correlations may be statistically described by separable one-particle distribution functions.[4] This is the case of the product-bi-kappa (or product-bi-Lorentzian) VDF,[4,7] of which the kappa-Maxwellian distribution[22,36] is a particular case. For such distributions, the functions $\mathcal{Z}(\mu,\xi)$ and $\mathcal{Y}(\mu,\xi)$ result completely separable and the mathematical treatment is much simpler. Future works will also consider this possibility.

The first representation to be derived is a power series in ξ, valid when $|\xi| < \sqrt{\kappa}$. Starting from (20a), we introduce the form (I.15) for the κPDF and obtain

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$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = -\frac{4\Gamma(\lambda-1/2)\,\xi}{\kappa^{\beta+1}\Gamma(\sigma-3/2)} \times \int_0^\infty dx\,\frac{xJ_n^2(\nu x)}{(1+x^2/\kappa)^{\lambda-1/2}}\,{}_2F_1\left(\begin{matrix}1,\lambda-1/2\\3/2\end{matrix};-\frac{\xi^2/\kappa}{1+x^2/\kappa}\right) + \frac{2i\pi^{1/2}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\int_0^\infty dx\,\frac{xJ_n^2(\nu x)}{(1+\xi^2/\kappa+x^2/\kappa)^{\lambda-1}}.$$

The second integral can be evaluated. If we initially assume that ξ is real and define a new integration variable by $x=\sqrt{\psi}t$, where $\psi = 1+\xi^2/\kappa$, then we can identify the resulting integral with (4) and write

$$\int_0^\infty \frac{xJ_n^2(\nu x)\,dx}{(1+\xi^2/\kappa+x^2/\kappa)^{\lambda-1}} = \frac{\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}\left[\mu\left(1+\xi^2/\kappa\right)\right]}{2\left(1+\xi^2/\kappa\right)^{\lambda-2}}. \tag{24}$$

Identity (24) can be analytically continued to the complex plane of ξ as long as it stays within the principal branch of $\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}(z)$ (i.e., of the G-function). Since the origin is a branch point of the G-function and the infinity is an essential singularity,[37] the complex-valued $\mathcal{H}$-function in (24) has branch cuts along the lines $(-i\infty,-i\sqrt{\kappa}]$ and $[i\sqrt{\kappa},i\infty)$. Hence, we can employ result (24) when $|\xi|<\sqrt{\kappa}$.

On the other hand, if the Gauss function in the above expression for $\mathcal{Z}$ is substituted by its power series (I.B4), the series will also converge if $|\xi|<\sqrt{\kappa}$, and we are then allowed to integrate term by term and obtain

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = -\frac{2\Gamma(\lambda-1/2)\,\xi}{\kappa^{\beta+1}\Gamma(\sigma-3/2)} \times \sum_{k=0}^\infty \frac{(\lambda-1/2)_k}{(3/2)_k}\left(-\frac{\xi^2}{\kappa}\right)^k \mathcal{H}_{n,\kappa}^{(\alpha,\beta+k+1/2)}(\mu) + \frac{i\pi^{1/2}\Gamma(\lambda-1)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\frac{\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}\left[\mu\left(1+\xi^2/\kappa\right)\right]}{\left(1+\xi^2/\kappa\right)^{\lambda-2}}. \tag{25a}$$

For the next series expansions, we will consider the $\mathcal{H}$-function in (20e). Since $1+s^2/\kappa \geqslant 1$, we can use the multiplication theorem (B2) to write

$$\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}\left[\mu\left(1+\frac{s^2}{\kappa}\right)\right] = \left(1+\frac{s^2}{\kappa}\right)^{-\frac{1}{2}} \times \sum_{k=0}^\infty \frac{1}{k!}\left(\frac{s^2}{\kappa}\right)^k\left(1+\frac{s^2}{\kappa}\right)^{-k}\widetilde{\mathcal{H}}_{n,k,\kappa}^{(\alpha,\beta)}(\mu), \tag{25b}$$

In this result, the function $\widetilde{\mathcal{H}}_{n,k,\kappa}^{(\alpha,\beta)}(\mu)$ is the *associated plasma gyroradius function*, defined by (11).

In this way, the $\mathcal{Z}$-function can be written in the generic (and compact) form

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)=\sum_{k=0}^{\infty}\frac{1}{k!}\widetilde{\mathcal{H}}_{n,k,\kappa}^{(\alpha,\beta)}(\mu)\,\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi)\,, \tag{25c}$$

where, accordingly, the function $\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi)$ is the *associated plasma dispersion function*, defined by (16).

Therefore, we can evaluate the function $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ using for $\widetilde{\mathcal{H}}_{n,k,\kappa}^{(\alpha,\beta)}(\mu)$ the representations (12a) or (13), and for $\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi)$ the representations (18a-18c).

For the $\widetilde{Z}$-function, we can also employ representation (18d); however, in this case, as was then mentioned, we also need to include the analytical continuation when $\xi_i=\Im\xi\leqslant 0$. The necessary expressions can be gleaned from the discussion concerning the related continuation of Eq. (I.13). In this process, one would have to include the continuation of the Gauss function. Alternatively, one can start anew from Eq. (20b) and introduce the adequate continuation for the s-integration. In this way, one would end up with an additional term, which is proportional to Eq. (24). Proceeding in this way, the last series expansion for the $\mathcal{Z}$-function is finally

$$\begin{aligned}\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)=\sum_{k=0}^{\infty}\frac{1}{k!}\widetilde{\mathcal{H}}_{n,k,\kappa}^{(\alpha,\beta)}(\mu)\,\widetilde{Z}_{(18d)}^{(\beta)}(\xi)\\ +\frac{2\sqrt{\pi}i\Gamma(\lambda-1)\,\Theta(-\xi_i)}{\kappa^{\beta+1/2}\Gamma(\sigma-3/2)}\left(1+\frac{\xi^2}{\kappa}\right)^{-(\lambda-2)}\\ \times\mathcal{H}_{n,\kappa}^{(\alpha,\beta)}\left[\mu\left(1+\frac{\xi^2}{\kappa}\right)\right],\end{aligned} \tag{25d}$$

where we have used the shorthand notation $\widetilde{Z}_{(18d)}^{(\beta)}(\xi)\equiv\widetilde{Z}_{k,\kappa}^{(\alpha,\beta)}(\xi;\text{Eq.}18d)$. We have also employed the Heaviside function $\Theta(x)=+1$ (if $x\geqslant 0$) or $\Theta(x)=0$ (if $x<0$).

The series expansions for the function $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ follow the same methodologies and their derivations will not be repeated. The only intermediate result shown here is the identity

$$\begin{aligned}\int_0^\infty dx\frac{x^3J_{n-1}(\nu x)\,J_{n+1}(\nu x)}{(1+\xi^2/\kappa+x^2/\kappa)^{-(\lambda-1)}}=\frac{1}{2}\frac{\kappa\mu}{\lambda-2}\\ \times\left(1+\frac{\xi^2}{\kappa}\right)^{-(\lambda-4)}\mathcal{H}_{n,\kappa}^{(\alpha,\beta-1)\prime}\left[\mu\left(1+\frac{\xi^2}{\kappa}\right)\right],\end{aligned} \tag{26}$$

which is derived similarly to Eq. (24) and to which apply the same considerations about the analiticity domain.

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Without further ado, the series expansions for $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ are:

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = -\frac{2\Gamma(\lambda-3/2)\,\xi}{\kappa^{\beta}\Gamma(\sigma-3/2)}\sum_{k=0}^{\infty}\frac{(\lambda-3/2)_k}{(3/2)_k} \times \mathcal{H}_{n,\kappa}^{(\alpha,\beta+k-1/2)\prime}(\mu)\left(-\frac{\xi^2}{\kappa}\right)^k + \frac{i\pi^{1/2}\Gamma(\lambda-2)}{\kappa^{\beta-1/2}\Gamma(\sigma-3/2)} \times \frac{\mathcal{H}_{n,\kappa}^{(\alpha,\beta-1)\prime}[\mu(1+\xi^2/\kappa)]}{(1+\xi^2/\kappa)^{\lambda-4}}, \quad (27a)$$

valid for $|\xi| < \sqrt{\kappa}$,

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = \sum_{k=0}^{\infty}\frac{1}{k!}\widetilde{\mathcal{H}}_{n,k,\kappa}^{(\alpha,\beta-1)\prime}(\mu)\,\widetilde{Z}_{k,\kappa}^{(\alpha,\beta-1)}(\xi), \quad (27b)$$

valid for any ξ, and

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) = \sum_{k=0}^{\infty}\frac{1}{k!}\widetilde{\mathcal{H}}_{n,k,\kappa}^{(\alpha,\beta-1)\prime}(\mu)\,\widetilde{Z}_{(18d)}^{(\beta-1)}(\xi) + \frac{2\sqrt{\pi}i\Theta(-\xi_i)\,\Gamma(\lambda-2)}{\kappa^{\beta-1/2}\Gamma(\sigma-3/2)}\left(1+\frac{\xi^2}{\kappa}\right)^{-(\lambda-4)} \times \mathcal{H}_{n,\kappa}^{(\alpha,\beta-1)\prime}\left[\mu\left(1+\frac{\xi^2}{\kappa}\right)\right], \quad (27c)$$

also valid for any ξ.

The series expansions and the other properties derived in this section and in Paper I are sufficient to enable a computational implementation of the functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$, and hence for the evaluation of the dielectric tensor (2a) for a bi-kappa plasma.

The numerical evaluation of the series can be substantially accelerated if one also employs the recurrence relations (14) and (19). However, we must point out that so far no analysis of the stability of these relations for forward recursion has been made. It is possible that for a given set of parameters either or both relations are only stable for backward recursion, and so different strategies must be implemented.

4. *Asymptotic expansions*

Here we will derive expressions valid for either $|\xi| \gg 1$ or $\mu \gg 1$. Starting with ξ, the expansion we want to derive is not the ordinary series representation for $|\xi| > \sqrt{\kappa}$. Although such a series can be easily obtained from the expressions already shown, they would be unnecessarily complicated, as it was hinted by the derivation of the representation

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(I.16) for the κPDF. Instead, we want to derive an expansion valid for $|\xi| \gg \sqrt{\kappa}$, convenient for a fluid approximation of the dielectric tensor.

Accordingly, in the s-integrals of Eqs. (20b) and (20d) we will approximate

$$\frac{1}{s-\xi} \simeq -\frac{1}{\xi}\left(1+\frac{s}{\xi}+\frac{s^2}{\xi^2}+\cdots\right) = -\frac{1}{\xi}\sum_{\ell=0}\frac{s^\ell}{\xi^\ell},$$

i.e., we ignore the high-energy particles at the tail of the VDF and the kinetic effect of the pole at $s=\xi$. Notice also that we have not written the upper limit of the sum above, since such expansion is only meaningful for a finite number of terms. Inserting this expansion into the s-integrals, all the terms with ℓ odd vanish and the others can be easily evaluated. However, these integrals only exist if the additional condition $\lambda > k + 3/2$ ($k = 0, 1, 2, \dots$) is satisfied.

Identifying the remaining x-integrals with (4) and (26), we obtain

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) \simeq -\frac{\pi^{-1/2}\kappa^{-\beta}}{\Gamma(\sigma-3/2)}\frac{1}{\xi}\sum_{k=0}\Gamma\left(\lambda-k-\frac{3}{2}\right) \times\Gamma\left(k+\frac{1}{2}\right)\frac{\kappa^k}{\xi^{2k}}\mathcal{H}_{n,\kappa}^{(\alpha,\beta-k-1/2)}(\mu) \tag{28a}$$

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) \simeq -\frac{\pi^{-1/2}\kappa^{1-\beta}}{\Gamma(\sigma-3/2)}\frac{1}{\xi}\sum_{k=0}\Gamma\left(\lambda-k-\frac{5}{2}\right) \times\Gamma\left(k+\frac{1}{2}\right)\frac{\kappa^k}{\xi^{2k}}\mathcal{H}_{n,\kappa}^{(\alpha,\beta-k-3/2)\prime}(\mu)\,. \tag{28b}$$

Now, the large gyroradius expansion ($\mu \gg 1$) is obtained if we start from (20e, 20f) and introduce the expansion (8). The resulting integrals can be identified with the definition of the κPDF in (I.11). Hence, we obtain

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) \simeq \frac{1}{\sqrt{2\pi\mu}}\sum_{k=0}\frac{(1/2+n)_k\,(1/2-n)_k}{k!\,(2\mu)^k} \times Z_\kappa^{(\alpha,\beta+k-1/2)}(\xi) \tag{28c}$$

$$\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi) \simeq \frac{-1}{\sqrt{2\pi\mu^3}}\sum_{k=0}\frac{(1/2+n)_k\,(1/2-n)_k\,(k+1/2)}{k!\,(2\mu)^k} \times Z_\kappa^{(\alpha,\beta+k-3/2)}(\xi)\,. \tag{28d}$$

5. *A closed-form expression for* $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$

Since $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ are non-separable functions of two variables, it is a relevant question whether they can be represented by some special function discussed in the

literature. Here, we will show for $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ that indeed it can be represented in closed, compact form by the relatively newly-defined Meijer G-function of two variables, introduced in section B 2.

Returning to the definition (20b) and defining the new integration variables $x=\sqrt{\kappa u}$ and $s=\sqrt{\kappa v}$, the double integral can be written as

$$I_2=\frac{\sqrt{\kappa}}{4}\int_0^\infty du\int_0^\infty dv\,v^{-1/2}J_n^2\left(\sqrt{2\kappa\mu u}\right)\frac{(1+u+v)^{-(\lambda-1)}}{v-\xi^2/\kappa}.$$

Introducing now the function representations (B4a), (I.B15c), and (B9), and then expressing the last in terms of the double Mellin-Barnes integral (B5), one obtains

$$I_2=-\frac{\kappa^{3/2}\xi^{-2}}{4\sqrt{\pi}\Gamma(\lambda-1)}\frac{1}{(2\pi i)^2}\int_{L_s}\int_{L_t}dsdt\,\Gamma(\lambda-1-s-t)\,\Gamma(s)\,\Gamma(t)$$
$$\times\left\{\int_0^\infty du\,u^{-s}G_{1,3}^{1,1}\left[2\kappa\mu u\,\middle|\,\begin{matrix}1/2\\ n,-n,0\end{matrix}\right]\right\}\left\{\int_0^\infty dv\,v^{-t-1/2}G_{1,1}^{1,1}\left[-\frac{\kappa v}{\xi^2}\,\middle|\,\begin{matrix}0\\ 0\end{matrix}\right]\right\},$$

where we have also interchanged the order of integrations.

The u- and v-integrations can now be performed by means of the Mellin transform (B3), resulting

$$I_2=\frac{\sqrt{\kappa}\,(2\kappa\mu)^{-1}}{4\sqrt{\pi}\Gamma(\lambda-1)}\left(-\frac{\kappa}{\xi^2}\right)^{1/2}\frac{1}{(2\pi i)^2}\int_{L_s}\int_{L_t}dsdt\,\Gamma(\lambda-1-s-t)$$
$$\times\frac{\Gamma(-1/2+s)\,\Gamma(n+1-s)}{\Gamma(n+s)}\Gamma(t)\,\Gamma\left(\frac{1}{2}+t\right)\Gamma\left(\frac{1}{2}-t\right)\left[(2\kappa\mu)^{-1}\right]^{-s}\left(-\frac{\xi^2}{\kappa}\right)^{-t}. \quad (29)$$

This result can be compared with (B5), in which case we obtain finally

$$\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)=-\frac{\pi^{-1}\kappa^{1-\beta}}{\Gamma(\sigma-3/2)\,\xi}G_{1,0:2,1:1,2}^{0,1:1,1:2,1}\left[\begin{matrix}(2\kappa\mu)^{-1}\\ -\xi^2/\kappa\end{matrix}\,\middle|\,\begin{matrix}7/2-\lambda:1-n,1+n:1\\ -:1/2:1/2,1\end{matrix}\right]. \quad (30)$$

The final expression for the $\mathcal{Z}$-function in (30) was obtained after employing also the translation property (B7).

The Maxwellian limit of (30) can be obtained. Expressing again the $G^{(2)}$-function in (30) in terms of the definition (B5), and applying the limit $\kappa\to\infty$ on the resulting expression, one can evaluate the limit using Stirling's formula.[28] As a result, the s- and t-integrations factor out, and the remaining integrals can be identified with G-functions from the definition (I.B10), which in turn can be identified with representations (I.B15d) and (B4c). After employing properties (I.B11a), one finally obtains

$$\lim_{\kappa\to\infty}\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)=e^{-\mu}I_n(\mu)\,\xi U\left(\begin{matrix}1\\ 3/2\end{matrix};-\xi^2\right)=\mathscr{H}_n(\mu)\,Z(\xi),$$

as expected.

Formula (30) is the more compact representation of the function $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ that we have obtained. However, despite of being a closed-form for $\mathcal{Z}$, this representation is not yet very useful, since there is no known computational implementation that evaluates the $G^{(2)}$-function, contrary to the one-variable G, which is implemented by some Computer Algebra Software and also by the `python` library mpmath.[38] Nevertheless, we find it important to include the derivation of formula (30) in order to stress the necessity of further development on the numerical evaluation of these special functions and also to present to the plasma physics community the techniques involved with Meijer's G- and $G^{(2)}$-functions and Mellin-Barnes integrals in general, since we believe that as more complex aspects of the physics of plasmas are considered, such as more general VDFs and dusty plasmas, for instance, the techniques employed in this work and in Paper I have the potential to provide mathematical answers to the challenges that will appear.

IV. CONCLUSIONS

In this paper we have presented two major developments for the study of waves with arbitrary frequency and direction of propagation in anisotropic superthermal plasmas. First, we have derived the dielectric tensor of a bi-kappa plasma. This tensor will be employed in future studies concerning wave propagation and amplification/damping in anisotropic superthermal plasmas.

The tensor components were written in terms of the kappa plasma special functions, which must be numerically evaluated for practical applications. To this end, we have derived in the present paper (and in Paper I) several mathematical properties and representations for these functions. With the development presented here and in Paper I, we believe that all the necessary framework for a systematic study of electromagnetic/electrostatic waves propagating at arbitrary angles in a bi-kappa plasma has been obtained. In future studies we will apply this formalism to specific problems concerning temperature-driven-instabilities in kappa plasmas.

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Appendix A: Derivation of the susceptibility tensor

The derivation of $\chi_{ij}^{(s)}$ for a bi-kappa plasma (or for any VDF, for that matter) is simplified if one observes that all tensor components have common factors. First, inserting the function (1) into the tensor (2b), all components contain the derivatives Lf_s and $\mathcal{L}f_s$. Using these derivatives, one can proceed with the evaluation of the integrals. Using a cylindrical coordinate system and defining the nondimensional integration variables $t = v_\perp/w_{\perp s}$ and $u = v_\parallel/w_{\parallel s}$, one obtains, after some straightforward algebra, the unified form

$$\chi_{ij}^{(s)} = 2\frac{\omega_{ps}^2}{\omega^2}\frac{\sigma_s g\left(\kappa_s,\alpha_s\right)}{\pi^{1/2}\kappa_s}\sum_{n\to-\infty}^{\infty} \times \int_0^\infty dt \int_{-\infty}^\infty du\, I_{ij,n}^{(s)}\left(1+\frac{u^2}{\kappa_s}+\frac{t^2}{\kappa_s}\right)^{-\sigma_s-1},$$

where

$$I_{ij,n}^{(s)} = \left(\xi_{0s} - A_s u\right) J_{ij,n}^{(s)}, \qquad I_{iz,n}^{(s)} = \left(\xi_{0s} - A_s \xi_{ns}\right) K_{ij,n}^{(s)},$$

$$\begin{aligned}
J_{xx,n}^{(s)} &= \frac{n^2}{\mu_s}\frac{tJ_n^2\left(\nu_s t\right)}{u-\xi_{ns}} \\
J_{xy,n}^{(s)} &= \sqrt{2}i\frac{n}{\mu_s}\frac{t^2 J_n\left(\nu_s t\right)J_n'\left(\nu_s t\right)}{u-\xi_{ns}} \\
J_{yy,s}^{(s)} &= 2\frac{\frac{n^2}{2\mu_s}tJ_n^2\left(\nu_s t\right) - t^3 J_{n-1}\left(\nu_s t\right)J_{n+1}\left(\nu_s t\right)}{u-\xi_{ns}} \\
K_{xz,n}^{(s)} &= \sqrt{2}\frac{w_{\parallel s}}{w_{\perp s}}\frac{n}{\mu_s}\frac{tuJ_n^2\left(\nu_s t\right)}{u-\xi_{ns}} \\
K_{yz,n}^{(s)} &= -2i\frac{w_{\parallel s}}{w_{\perp s}}\frac{t^2 uJ_n\left(\nu_s t\right)J_n'\left(\nu_s t\right)}{u-\xi_{ns}} \\
K_{zz,n}^{(s)} &= 2\frac{w_{\parallel s}^2}{w_{\perp s}^2}\xi_{ns}\frac{tuJ_n^2\left(\nu_s t\right)}{u-\xi_{ns}},
\end{aligned}$$

with $g(\kappa_s,\alpha_s)=\kappa_s^{-3/2}\Gamma(\sigma_s)/\Gamma(\sigma_s-3/2)$, $\nu_s=k_\perp w_{\perp s}/\Omega_s$, and where the anisotropy parameter $A_s=1-w_{\perp s}^2/w_{\parallel s}^2$ appears for the first time. These results were obtained using the identity (9) and the recurrence relations of the Bessel functions.

The remaining integrals in the Js and Ks can now be identified with the definitions of the two-variables kappa plasma functions $\mathcal{Z}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and $\mathcal{Y}_{n,\kappa}^{(\alpha,\beta)}(\mu,\xi)$ and their derivatives, given by Eqs. (20) and (22). In this way, one arrives at the final expressions shown in Eqs. (3a-3f).

The Maxwellian limit of the partial susceptibility tensor is obtained by the process $\kappa_s\to\infty$. Upon applying this limit, one must replace $w_{\parallel(\perp)}\to v_{T\parallel(\perp)}=\sqrt{2T_{\parallel(\perp)}/m}$ and the kappa plasma functions are replaced by their limiting representations (21). In this way, one arrives at

$$\chi_{xx}^{(s)}=\frac{\omega_{ps}^2}{\omega^2}\sum_{n\to-\infty}^{\infty}\frac{n^2}{\mu_s}\mathscr{H}_n(\mu_s)\left[\xi_{0s}Z(\xi_{ns})+\frac{1}{2}A_sZ'(\xi_{ns})\right] \tag{A1a}$$

$$\chi_{xy}^{(s)}=i\frac{\omega_{ps}^2}{\omega^2}\sum_{n\to-\infty}^{\infty}n\mathscr{H}_n'(\mu_s)\left[\xi_{0s}Z(\xi_{ns})+\frac{1}{2}A_sZ'(\xi_{ns})\right] \tag{A1b}$$

$$\chi_{xz}^{(s)}=-\frac{\omega_{ps}^2}{\omega^2}\frac{v_{T\parallel s}}{v_{T\perp s}}\sum_{n\to-\infty}^{\infty}\frac{n\Omega_s}{k_\perp v_{T\perp s}}(\xi_{0s}-A_s\xi_{ns})\times\mathscr{H}_n(\mu_s)Z'(\xi_{ns}) \tag{A1c}$$

$$\chi_{yy}^{(s)}=\frac{\omega_{ps}^2}{\omega^2}\sum_{n\to-\infty}^{\infty}\left[\frac{n^2}{\mu_s}\mathscr{H}_n(\mu_s)-2\mu_s\mathscr{H}_n'(\mu_s)\right]\times\left[\xi_{0s}Z(\xi_{ns})+\frac{1}{2}A_sZ'(\xi_{ns})\right] \tag{A1d}$$

$$\chi_{yz}^{(s)}=i\frac{\omega_{ps}^2}{\omega^2}\frac{v_{T\parallel s}}{v_{T\perp s}}\frac{k_\perp v_{T\perp s}}{2\Omega_s}\sum_{n\to-\infty}^{\infty}(\xi_{0s}-A_s\xi_{ns})\times\mathscr{H}_n'(\mu_s)Z'(\xi_{ns}) \tag{A1e}$$

$$\chi_{zz}^{(s)}=-\frac{\omega_{ps}^2}{\omega^2}\frac{v_{T\parallel s}^2}{v_{T\perp s}^2}\sum_{n\to-\infty}^{\infty}(\xi_{0s}-A_s\xi_{ns})\times\xi_{ns}\mathscr{H}_n(\mu_s)Z'(\xi_{ns}). \tag{A1f}$$

These results agree with expressions that can be found in the literature. See, e.g., Eq. (20) of Ref. 22.

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Appendix B: The one- and two-variables Meijer G-functions

1. The G-function

The definition and some properties of the G-function are given in Sec. B.2 of Paper I. All identities shown there and in the following can be found in Refs. 37 and 39, except when explicitly mentioned.

a. Derivatives

We have

$$\frac{d^k}{dz^k} G_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right.\right] = (-z)^{-k} G_{p+1,q+1}^{m+1,n}\left[z\left|\begin{matrix}(a_p),0\\k,(b_q)\end{matrix}\right.\right]. \tag{B1a}$$

We will now derive a formula that is not usually found in the literature. If $n \geqslant 1$, we can employ the definition of the G-function in terms of a Mellin-Barnes integral, given by (I.B10), and evaluate, for $k = 0, 1, 2, \ldots$,

$$\begin{aligned}\frac{d^k}{dz^k}\left\{z^{k-a_1} G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1,\ldots,a_p\\(b_q)\end{matrix}\right.\right]\right\}\\ = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s)\prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+s)\prod_{j=n+1}^p \Gamma(a_j-s)} \frac{\Gamma(1-a_1+k+s)}{\Gamma(1-a_1+s)} z^{-a_1+s} ds,\end{aligned} \tag{B1b}$$

since $D^m z^\gamma = \Gamma(\gamma+1)\, z^{\gamma-m}/\Gamma(\gamma+1-m)$. Consequently, we obtain the differentiation formula

$$\frac{d^k}{dz^k}\left\{z^{k-a_1} G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1,\ldots,a_p\\(b_q)\end{matrix}\right.\right]\right\} = z^{-a_1} G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1-k,\ldots,a_p\\(b_q)\end{matrix}\right.\right] \quad (n \geqslant 1). \tag{B1c}$$

b. Multiplication theorems

If $\Re w > 1/2$ and $n > 0$,

$$\begin{aligned}G_{p,q}^{m,n}\left[zw\left|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right.\right] = w^{a_1-1}\\ \times \sum_{k=0}^{\infty} \frac{(1-1/w)^k}{k!} G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1-k,a_2,\ldots,a_p\\(b_q)\end{matrix}\right.\right].\end{aligned} \tag{B2}$$

c. *Mellin transform*

The Mellin transform of the G-function is

$$\int_0^\infty y^{s-1} G_{p,q}^{m,n}\left[\eta y \middle| \begin{matrix}(a_p)\\(b_q)\end{matrix}\right] dy = \frac{\prod_{j=1}^m \Gamma(b_j+s)\prod_{j=1}^n \Gamma(1-a_j-s)\,\eta^{-s}}{\prod_{j=m+1}^q \Gamma(1-b_j-s)\prod_{j=n+1}^p \Gamma(a_j+s)}. \tag{B3}$$

d. *Function representations*

A short list of function representations is:

$$(1+x)^{-\rho} = \frac{1}{\Gamma(\rho)} G_{1,1}^{1,1}\left[x \middle| \begin{matrix}1-\rho\\0\end{matrix}\right] \tag{B4a}$$

$$\frac{I_\mu(\sqrt{z})\,K_\nu(\sqrt{z})}{(2\sqrt{\pi})^{-1}} = G_{2,4}^{2,2}\left[z \middle| \begin{matrix}0,\,1/2\\ \frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, -\frac{\mu-\nu}{2}, -\frac{\mu+\nu}{2}\end{matrix}\right] \tag{B4b}$$

$$\frac{\Gamma(a)\,U\left(\begin{smallmatrix}a\\b\end{smallmatrix};z\right)}{[\Gamma(a-b+1)]^{-1}} = G_{1,2}^{2,1}\left[z \middle| \begin{matrix}1-a\\0,\,1-b\end{matrix}\right]. \tag{B4c}$$

2. The two-variables Meijer function

The logical extension of Meijer's G-function for two variables was first proposed by Agarwal[40] in 1965. Subsequent publications proposed slightly different definitions for the same extension.[41–43] In this work, we will adopt the definition by Hai and Yakubovich (Eq. 13.1 of Ref. 43),

$$G_{p_1,q_1:p_2,q_2:p_3,q_3}^{m_1,n_1:m_2,n_2:m_3,n_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}\left(a_{p_1}^{(1)}\right):\left(a_{p_2}^{(2)}\right):\left(a_{p_3}^{(3)}\right)\\ \left(b_{q_1}^{(1)}\right):\left(b_{q_2}^{(2)}\right):\left(b_{q_3}^{(3)}\right)\end{matrix}\right] = \frac{1}{(2\pi i)^2}\int_{L_s}\int_{L_t}\Psi_1(s+t)\,\Psi_2(s)\,\Psi_3(t)\,x^{-s}y^{-t}dsdt, \tag{B5}$$

where, for $k=1,2,3$,

$$\Psi_k(r) = \frac{\prod_{j=1}^{m_k}\Gamma\left(b_j^{(k)}+r\right)\prod_{j=1}^{n_k}\Gamma\left(1-a_j^{(k)}-r\right)}{\prod_{j=n_k+1}^{p_k}\Gamma\left(a_j^{(k)}+r\right)\prod_{j=m_k+1}^{q_k}\Gamma\left(1-b_j^{(k)}-r\right)}.$$

The Reader is referred to Sec. II.13 of Ref. 43 for explanation on the notation and a discussion on the general conditions on the validity of (B5). Whenever convenient and unambiguous, we will refer to the two-variables Meijer function as the $G^{(2)}$-function.

We list some elementary properties of the $G^{(2)}$-function, some of which are employed in this work. The symmetry property

$$G^{m_1,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}\left(a^{(1)}_{p_1}\right):\left(a^{(2)}_{p_2}\right):\left(a^{(3)}_{p_3}\right)\\ \left(b^{(1)}_{q_1}\right):\left(b^{(2)}_{q_2}\right):\left(b^{(3)}_{q_3}\right)\end{matrix}\right] = G^{n_1,m_1:n_2,m_2:n_3,m_3}_{q_1,p_1:q_2,p_2:q_3,p_3}\left[\begin{matrix}x^{-1}\\y^{-1}\end{matrix}\middle|\begin{matrix}1-\left(b^{(1)}_{q_1}\right):1-\left(b^{(2)}_{q_2}\right):1-\left(b^{(3)}_{q_3}\right)\\ 1-\left(a^{(1)}_{p_1}\right):1-\left(a^{(2)}_{p_2}\right):1-\left(a^{(3)}_{p_3}\right)\end{matrix}\right], \tag{B6}$$

and the translation property

$$x^\alpha y^\beta G^{m_1,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}\left(a^{(1)}_{p_1}\right):\left(a^{(2)}_{p_2}\right):\left(a^{(3)}_{p_3}\right)\\ \left(b^{(1)}_{q_1}\right):\left(b^{(2)}_{q_2}\right):\left(b^{(3)}_{q_3}\right)\end{matrix}\right] = G^{m_1,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}\left(a^{(1)}_{p_1}\right)+\alpha+\beta:\left(a^{(2)}_{p_2}\right)+\alpha:\left(a^{(3)}_{p_3}\right)+\beta\\ \left(b^{(1)}_{q_1}\right)+\alpha+\beta:\left(b^{(2)}_{q_2}\right)+\alpha:\left(b^{(3)}_{q_3}\right)+\beta\end{matrix}\right]. \tag{B7}$$

A product of two G-functions can be written as a single $G^{(2)}$-function as

$$G^{0,0:m_2,n_2:m_3,n_3}_{0,0:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}-:\left(a^{(2)}_{p_2}\right):\left(a^{(3)}_{p_3}\right)\\ -:\left(b^{(2)}_{q_2}\right):\left(b^{(3)}_{q_3}\right)\end{matrix}\right] = G^{m_2,n_2}_{p_2,q_2}\left[x\middle|\begin{matrix}\left(a^{(2)}_{p_2}\right)\\ \left(b^{(2)}_{q_2}\right)\end{matrix}\right] G^{m_3,n_3}_{p_3,q_3}\left[y\middle|\begin{matrix}\left(a^{(3)}_{p_3}\right)\\ \left(b^{(3)}_{q_3}\right)\end{matrix}\right]. \tag{B8}$$

We will also use the function representation

$$(1+x+y)^{-\alpha} = \frac{1}{\Gamma(\alpha)} G^{1,0:0,1:0,1}_{0,1:1,0:1,0}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}-:1:1\\ \alpha:-:-\end{matrix}\right]. \tag{B9}$$

Properties (B6)-(B8) can be inferred from the definition (B5). The identity (B9) is given in Sec. II.13 of Ref. 43.

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