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On the dimensionally correct kinetic theory of turbulence for parallel propagation

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Yoon and Fang [Phys. Plasmas **15**, 122312 (2008)] formulated a second-order nonlinear kinetic theory that describes the turbulence propagating in directions parallel/anti-parallel to the ambient magnetic field. Their theory also includes discrete-particle effects, or the effects due to spontaneously emitted thermal fluctuations. However, terms associated with the spontaneous fluctuations in particle and wave kinetic equations in their theory contain proper dimensionality only for an artificial one-dimensional situation. The present paper extends the analysis and re-derives the dimensionally correct kinetic equations for three-dimensional case. The new formalism properly describes the effects of spontaneous fluctuations emitted in three-dimensional space, while the collectively emitted turbulence propagates predominantly in directions parallel/anti-parallel to the ambient magnetic field. As a first step, the present investigation focuses on linear wave-particle interaction terms only. A subsequent paper will include the dimensionally correct nonlinear wave-particle interaction terms. *© 2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4916054]

I. INTRODUCTION

The turbulence in plasmas is important for charged-particle energization and acceleration. For large spatial- and/or slow temporal-scale dynamics, the macroscopic (i.e., magnetohydrodynamic or MHD for short) theory is often employed for studying turbulent phenomena.[1,2] However, when the spatial or temporal scales of the plasma perturbation become comparable to the spatio-temporal scales of the thermal motions of individual particle, then kinetic effects such as the Landau and cyclotron wave-particle interaction and finite Larmor radius effects across the magnetic field must be included in the physical formalism. If one is interested in relatively low-amplitude perturbations and for initial response of the plasma when there exist free energies, then often linear kinetic treatment may be appropriate, but when the amplitude becomes even moderately high, or when one is interested in the long-time behavior of the plasma, nonlinear and/or turbulent kinetic effects become necessary for a comprehensive description of the system.

At present, kinetic turbulence theory is available largely for plasmas in field-free environment only.[3–12] Drift kinetic and gyrokinetic theories are reduced turbulence theories, which are applicable for magnetized plasmas,[13–15] but in these theories wave-particle interaction by the cyclotron resonance is not present, since in these theories physical quantities are averaged over the particle gyration motion about the ambient magnetic field. Fully kinetic turbulence theories for magnetized plasmas that include cyclotron resonance are generally not yet available in the literature, and it is a major undeveloped research area in plasma physics.

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This does not mean that no attempts have been made to extend the standard weak turbulence theory for unmagnetized plasmas to include the effects of ambient magnetic field, however. In fact, the essential formal groundwork was already established in the 1960s and 1970s.[8,10,12,16,17] The progress along this line up to the decade of 1970s and 1980s can be found in the monograph by Tsytovich[10]—pp. 129–137 therein—or in the monograph by Sitenko[12]—see pp. 52–54, where a brief summary of nonlinear response tensors for magnetized plasmas can be found without detailed derivation. However, the basic formalisms found in these references are either highly abstract in that often the formal governing equations are expressed in terms of nonlinear susceptibility tensors, which do not immediately lend themselves to practical and concrete numerical computations,[10,12,16,17] or are expressed in terms of highly simplified forms.[8,16]

Nevertheless, some sporadic attempts have been made in the literature to apply such formalisms to partially address various nonlinear phenomena in magnetized plasmas. Specifically, Melrose and his collaborators employed the semi-classical method originally developed by Tsytovich to discuss Langmuir wave and plasma radiation interacting with upper-hybrid waves.[18–20] Luo and Melrose[21] also applied the same methodology to derive the wave kinetic equation for low-frequency Alfvén turbulence. Recently, Nerisyan and Matevosyan[22] made use of nonlinear susceptibilities for cold, magnetic plasmas first derived by Pustovalov and Silin[17] and also found in the monograph by Sitenko,[12] in order to discuss the scattering and transformation of waves on heavy particles in magnetized plasmas.

In spite of these efforts, however, in order for the weak turbulence theory for magnetized plasmas to be elevated to the level of its counterpart for unmagnetized plasmas, one must further develop the existing formalism to a stage where the underlying equations can be readily solved by computational means. This cannot be accomplished unless the foundational nonlinear response functions—which are expressed in terms of formal momentum integrals over the derivatives of the particle distribution functions, and also involve multiple infinite series expansions of Bessel functions, etc.—are simplified and approximated to mathematical forms that lend themselves to actual numerical and analytical treatment. Moreover, these nonlinear response tensors are of third- and fourth-order ranks, and unless the matrix operations involving such quantities are carried out in detail, which are formidable, the theory remains formal with little practical use.

The relatively recent papers by one of the present authors[23–25] took a first step toward a comprehensive kinetic theory of turbulence in magnetized plasmas in which the resultant governing equations are sufficiently concrete so as to allow direct numerical solutions. However, owing to the complicated nature of the problem, these references considered electromagnetic turbulence propagating parallel to the ambient magnetic field. Moreover, in these works, the effects of spontaneously emitted thermal fluctuations, which cannot be discussed unless one incorporates the effects resulting from the discrete-particle nature into the theoretical formalism, were not included. In a subsequent paper, however, Yoon and Fang[26] extended the earlier analysis to include the discrete-particle effects via the Klimontovich formalism, in contrast to the Vlasov formalism employed in Refs. 23–25. The authors of Ref. 26 then applied their theory to discuss the proton heating by parallel Alfvén-wave cascade.[27]

In should be noted that the above preliminary works[23–27] are limited in that the turbulence was assumed to propagate strictly along the direction parallel or anti-parallel to the ambient magnetic field. The more general theory that does not impose such an assumption is available only in a formal expression, which is not very practical. We are working on the more general problem, which can be viewed as a continuation of the efforts by earlier pioneers. In the mean time, the purpose of the present paper is to revisit the paper by Yoon and Fang[26] and address a shortcoming therein.

Specifically, while Ref. 26 basically succeeded in adding the essential physics of the discrete particle nature of individual plasma particles, the principal outcome enjoys correct dimensionality only for an artificial world of one-dimensional (1D) space. This is because the assumption of one-dimensionality was made at the outset. The incorrect dimensionality affects those terms that stem from spontaneous thermal fluctuations. The basic formalism developed in Ref. 26 is still useful, however, since it can be employed to interpret computer simulation experiment in 1D space, for instance. Nevertheless, if one is interested in a real three-dimensional (3D) situation where the electromagnetic fluctuations are emitted in all directions while the collective perturbations are propagating in predominantly parallel directions, then the legitimate procedure is to integrate the spontaneously emitted fluctuations over perpendicular spatial variables, or in spectral space, over the perpendicular wave numbers.

The present paper outlines the above-mentioned procedure and reformulates the particle and wave kinetic equations. As a first step, however, we only focus on the linear wave-particle interaction terms. The nonlinear wave-particle interaction terms are also affected by the dimensionality associated with the discrete-particle terms. A subsequent paper will address this issue.

Before we go into the details of the present findings, it is perhaps suitable for us to explain the applicability of the present approach of assuming quasi-parallel propagation. In Refs. 23–27, our original attempts were focused on applying the formalism to low-frequency turbulence. However, it is well known that for low-frequency turbulence with characteristic frequency ranging from Alfvénic domain, to ion-cyclotron frequency, and to lower-hybrid frequency range, the turbulence is characterized by perpendicular wave number cascade, such that the assumption of quasi-parallel propagation is rather limited.

However, in a recent observation, Lacombe *et al.*[28] unequivocally identified the quasi-parallel propagation of whistler turbulence in the solar wind. On the basis of such a finding, a recent paper posits that the whistler fluctuations propagating in parallel and anti-parallel directions with respect to the solar wind magnetic field may be responsible for maintaining a superthermal population of solar wind electrons called the halo.[29] Note that when the whistler waves are in the intermediate frequency range, $\omega \sim \omega_{LH} \ll |\Omega_e|$, where $\omega_{LH} = eB_0/(\sqrt{m_e m_i}c)$ is the lower hybrid frequency and $|\Omega_e| = eB_0/m_e c$ is the electron cyclotron frequency, e, B_0, m_e, m_i, and c being the unit electric charge, the ambient magnetic field intensity, electron and proton mass, and the speed of light *in vacuo*, it is known from both simulations[30–33] and theory[34–36] that perpendicular cascade is important so that quasi-parallel assumption becomes invalid for long time scale.

For whistler turbulence with characteristic frequency sufficiently higher than the lower-hybrid wave modes, on the other hand, it is expected that essentially quasi-parallel characteristics will be preserved even during the nonlinear stage since perpendicular cascade to lower-hybrid or kinetic Alfvén modes may become energetically limited. In fact, a recent three-dimensional particle-in-cell simulation of short-wavelength (and thus, high-frequency) whistler anisotropy instability by Gary *et al.*[37] provides evidence for such a feature. It is in this context that the present work has its significance, and in fact, the recent paper by Kim *et al.*[29] builds their theory of solar wind electron model on the basis of the findings from the present paper as well as the observation by Lacombe *et al.*[28]

The organization of the present paper is as follows: Section II discusses the theoretical development. Specifically, Sec. II A outlines the derivation of wave kinetic equation, while Sec. II B discusses the formulation of particle kinetic equation. Section III summarizes the present findings, and two Appendixes present auxiliary discussions.

II. DIMENSIONALLY CORRECT WAVE AND PARTICLE KINETIC EQUATIONS

As in Ref. 26, a convenient starting point would be the set of Klimontovich-Maxwell system of equations for a plasma immersed in a constant magnetic field

$$\frac{\partial f_a}{\partial t} = -\frac{e_a}{m_a}\frac{\partial}{\partial v_i}\int d\mathbf{k}\int d\omega\left[\left(1-\frac{\mathbf{k}\cdot\mathbf{v}}{\omega}\right)\delta_{ij}+k_i v_j\right] \times \langle \delta E^j_{-\mathbf{k},-\omega}\delta N^a_{\mathbf{k},\omega}\rangle, \tag{1}$$

$$\left[\delta_{ij}-\frac{c^2k^2}{\omega^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\right]\delta E^j_{\mathbf{k},\omega} = -\frac{4\pi i}{\omega}\sum_a e_a\int d\mathbf{v}v_i\delta N^a_{\mathbf{k},\omega}, \tag{2}$$

$$\left(\Omega_a\frac{\partial}{\partial\varphi}+i(\omega-\mathbf{k}\cdot\mathbf{v})\right)\left(\delta N^a_{\mathbf{k},\omega}-\delta N^{a0}_{\mathbf{k},\omega}\right) = \frac{e_a}{m_a}\,\delta E^i_{\mathbf{k},\omega}\left[\left(1-\frac{\mathbf{k}\cdot\mathbf{v}}{\omega}\right)\delta_{ij}+v_ik_j\right]\frac{\partial f_a}{\partial v_j} + \frac{e_a}{m_a}\frac{\partial}{\partial v_j}\int d\mathbf{k}'\int d\omega'\left[\left(1-\frac{\mathbf{k}'\cdot\mathbf{v}}{\omega'}\right)\delta_{ij}+v_ik'_j\right] \times\left[\delta E^i_{\mathbf{k}',\omega'}\delta N^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'}-\langle\delta E^i_{\mathbf{k}',\omega'}\delta N^a_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\rangle\right], \tag{3}$$

where f_a is the averaged one-particle distribution function for species a ($a = i, e$ stands for ions and electrons, respectively), which is assumed to be gyrotropic (that is, f_a is not a function of velocity-space gyro-angle φ); $\delta N^a_{\mathbf{k},\omega}$ is the perturbed phase space density distribution in spectral representation; and $\delta\mathbf{E}_{\mathbf{k},\omega}$ and $\delta\mathbf{B}_{\mathbf{k},\omega} = c\mathbf{k}\times\delta\mathbf{E}_{\mathbf{k},\omega}/\omega$ designate the perturbed electric and magnetic field vectors, respectively. The quantity $\Omega_a = e_aB/m_ac$ is the cyclotron (or gyro-) frequency for particle species a and φ stands for the gyrophase angle. In the above, e_a, m_a, c, and B denote unit electric charge and mass for particle species a, the speed of light *in vacuo*, and the intensity of the ambient magnetic field $\mathbf{B}$, respectively. Equation (1) represents the formal particle kinetic equation, Eq. (2) corresponds to the wave equation, and Eq. (3) stands for nonlinear equation for the perturbation. In what follows, we shall ignore nonlinear terms in the equation for the perturbed phase space distribution.

In Eq. (3), the "source fluctuation" $\delta N^{a0}_{\mathbf{k},\omega}$ signifies the phase space density fluctuation that arises from the discrete particle nature of the plasma particles. If we ignore this quantity, then the above formalism reduces to that of the purely collisionless Vlasov theory. References 23–25 adopt such a "Vlasov approximation." As already noted in Ref. 26, as well as in other standard literature, we are not interested in $\delta N^{a0}_{\mathbf{k},\omega}$ *per se*, but rather, we are only interested in the ensemble average of the products of two entities, which in the spectral representation is given by

$$\langle\delta N^0_a(\mathbf{v})\,\delta N^0_b(\mathbf{v}')\rangle_{\mathbf{k},\omega} = (2\pi)^{-4}\delta_{ab}\,2\,\mathrm{Re}\int_0^\infty d\tau\, e^{i\mathbf{k}\cdot[\mathbf{r}(\tau)-\mathbf{r}]+i\omega\tau-\Delta\tau}\,\delta[\mathbf{v}(\tau)-\mathbf{v}']f_a(\mathbf{v}), \tag{4}$$

where $\Delta\to 0^+$, and

$$\mathbf{r}(\tau)-\mathbf{r} = -\frac{v_\perp}{\Omega_a}\left[\sin(\varphi+\Omega_a\tau)-\sin\varphi\right]\mathbf{e}_1 + \frac{v_\perp}{\Omega_a}\left[\cos(\varphi+\Omega_a\tau)-\cos\varphi\right]\mathbf{e}_2 - v_z\tau\,\mathbf{e}_3,$$
$$\mathbf{v}(\tau) = v_\perp\cos(\varphi+\Omega_a\tau)\,\mathbf{e}_1 + v_\perp\sin(\varphi+\Omega_a\tau)\,\mathbf{e}_2 + v_z\,\mathbf{e}_3 \tag{5}$$

represent the unperturbed particle orbits. Here, the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are defined by

$$\mathbf{e}_1 = \frac{(\mathbf{B}\times\mathbf{k})\times\mathbf{B}}{|(\mathbf{B}\times\mathbf{k})\times\mathbf{B}|},\quad \mathbf{e}_2 = \frac{\mathbf{B}\times\mathbf{k}}{|\mathbf{B}\times\mathbf{k}|},\quad \mathbf{e}_3 = \frac{(\mathbf{B}\cdot\mathbf{k})\mathbf{k}}{|(\mathbf{B}\cdot\mathbf{k})\mathbf{k}|}. \tag{6}$$

For $\mathbf{B}$ directed along z axis, we simply have $\mathbf{e}_1 = \hat{\mathbf{x}}$, $\mathbf{e}_2 = \hat{\mathbf{y}}$, and $\mathbf{e}_3 = \hat{\mathbf{z}}$. One can easily show that the formal expression (4) can be generally expressed as

$$\langle\delta N^0_a(\mathbf{v})\,\delta N^0_b(\mathbf{v}')\rangle_{\mathbf{k},\omega} = (2\pi)^{-3}\delta_{ab}\,2\,\mathrm{Re}\int_0^\infty d\tau\int_0^\infty dk_\perp k_\perp \times\sum_{n=-\infty}^{\infty}J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\delta\left(\omega-k_\parallel v_\parallel-n\Omega_a\right) \times\delta\left[\mathbf{v}(\tau)-\mathbf{v}'\right]f_a(\mathbf{v}). \tag{7}$$

As in Ref. 26 we now confine ourselves to the case of one-dimensional situation where the collective electromagnetic perturbations have predominantly parallel propagation direction. This amounts to an assumption that physical quantities vary primarily along z axis, parallel to the ambient magnetic field. In Ref. 26, we assumed that $\mathbf{k}$ vector has nonzero component only along z axis. However, such an assumption must be carefully examined when the $\mathbf{k}$ vector appears within the definition for the source fluctuation (4). Unlike the spectral wave amplitude, $\langle\delta E_i\,\delta E_j\rangle_{\mathbf{k},\omega}$, which can legitimately taken to be one dimensional by simply assuming the form

$$\langle\delta E_i\,\delta E_j\rangle_{\mathbf{k},\omega} = \langle\delta E_i\,\delta E_j\rangle_{k_\parallel,\omega}\,\frac{\delta(k_\perp)}{2\pi k_\perp}, \tag{8}$$

since the waves can indeed be excited along the ambient magnetic field, depending on the nature of the problem and the available form of the free energy source, the $\mathbf{k}$ vector associated with $\langle\delta N^0_a(\mathbf{v})\,\delta N^0_b(\mathbf{v}')\rangle_{\mathbf{k},\omega}$ cannot artificially taken to be 1D, unless one is working in the 1D world, as in the 1D computer simulation setting. This is because the source fluctuation is related to the spontaneous emission by thermal motion of the charged particles. Since the thermal motion is supposed to be random in all directions, the associated spectrum of the source fluctuation must not favor any particular direction either. The proper way to deal with the predominantly 1D situation is to integrate over $k_\perp$ associated with $\langle\delta N^0_a(\mathbf{v})\,\delta N^0_b(\mathbf{v}')\rangle_{\mathbf{k},\omega}$.

Reference 26, however, did not consider the finite $k_\perp$ associated with the source fluctuation, but instead, the 1D limit was imposed at the outset. If we focus on the linear equation for the perturbation, then such a simplification led to

$$\left(\Omega_a\frac{\partial}{\partial\varphi}+i\left(\omega-k_\parallel v_\parallel\right)\right)\left(\delta N^a_{k_\parallel,\omega}-\delta N^{a0}_{k_\parallel,\omega}\right) = \frac{e_a}{m_a}\,\delta E^{\parallel}_{k_\parallel,\omega}\frac{\partial f_a}{\partial v_\parallel}+\frac{e_a}{m_a\omega}\left(\delta E^x_{k_\parallel,\omega}\cos\varphi+\delta E^y_{k_\parallel,\omega}\sin\varphi\right) \times\left(\left(\omega-k_\parallel v_\parallel\right)\frac{\partial f_a}{\partial v_\perp}+k_\parallel v_\perp\frac{\partial f_a}{\partial v_\parallel}\right). \tag{9}$$

Of course, in reality, the perturbed phase space density associated with the source fluctuation in the above, namely, $\delta N^{a0}_{k_\parallel,\omega}$, should also depend on $k_\perp$ as well. For the moment, however, let us recapitulate the procedure as found in Ref. 26, and we will discuss the rectification thereof at the end of our discourse.

The solution to Eq. (9) is straightforward

$$\delta N^a_{k_\parallel,\omega} - \delta N^{a0}_{k_\parallel,\omega} = -\frac{ie_a}{m_a}\frac{\delta E^\parallel_{k_\parallel,\omega}}{\omega - k_\parallel v_\parallel}\frac{\partial f_a}{\partial v_\parallel} - \frac{ie_a}{m_a\omega}\sum_{+,-}\frac{\delta E^\pm_{k_\parallel,\omega} e^{\pm i\varphi}}{\omega - k_\parallel v_\parallel \pm \Omega_a} \times \left((\omega - k_\parallel v_\parallel)\frac{\partial f_a}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_a}{\partial v_\parallel}\right), \quad (10)$$

where

$$\delta E^\pm_{k_\parallel,\omega} = \frac{\delta E^x_{k_\parallel,\omega} \mp i\delta E^y_{k_\parallel,\omega}}{2}. \quad (11)$$

Inserting Eq. (10) to the wave equation (2) by assuming $k_\perp \to 0$, namely,

$$\delta E^\parallel_{k_\parallel,\omega} = -\frac{4\pi i}{\omega}\sum_a e_a \int d\mathbf{v}\, v_\parallel\, \delta N^a_{k_\parallel,\omega},$$
$$(\omega^2 - c^2 k_\parallel^2)\,\delta E^\pm_{k_\parallel,\omega} = -2\pi i\omega \sum_a e_a \int d\mathbf{v}\, v_\perp\, e^{\mp i\varphi}\, \delta N^a_{k_\parallel,\omega}, \quad (12)$$

we obtain

$$\epsilon_\parallel(k_\parallel,\omega)\,\delta E^\parallel_{k_\parallel,\omega} = -\frac{4\pi i}{\omega}\sum_a e_a \int d\mathbf{v}\, v_\parallel\, \delta N^{a0}_{k_\parallel,\omega},$$
$$\Lambda_\pm(k_\parallel,\omega)\,\delta E^\pm_{k_\parallel,\omega} = -\frac{2\pi i}{\omega}\sum_a e_a \int d\mathbf{v}\, v_\perp\, e^{\mp i\varphi}\, \delta N^{a0}_{k_\parallel,\omega}, \quad (13)$$

where $\epsilon_\parallel(k_\parallel,\omega)$ and $\Lambda_\pm(k_\parallel,\omega)$ are longitudinal and transverse linear response functions defined by

$$\epsilon_\parallel(k_\parallel,\omega) = 1 + \sum_a \frac{4\pi e_a^2}{m_a k_\parallel}\int d\mathbf{v}\, \frac{\partial f_a/\partial v_\parallel}{\omega - k_\parallel v_\parallel + i0},$$
$$\Lambda_\pm(k_\parallel,\omega) = 1 - \frac{c^2k_\parallel^2}{\omega^2} + \sum_a \frac{4\pi e_a^2}{m_a\omega^2}\int d\mathbf{v} \times \frac{v_\perp/2}{\omega - k_\parallel v_\parallel \pm \Omega_a + i0} \times \left((\omega - k_\parallel v_\parallel)\frac{\partial f_a}{\partial v_\perp} + k_\parallel v_\perp\frac{\partial f_a}{\partial v_\parallel}\right). \quad (14)$$

Without the right-hand sides in Eq. (13), the wave equations simply define the dispersion relations for parallel-propagating longitudinal and transverse waves (with the plus and minus signs signifying the two circular polarizations). The presence of the right-hand sides leads to the corrections due to spontaneous thermal effects or equivalently, the effects due to discrete particles. However, as we shall see, the assumption of ignoring the $k_\perp$ dependence on the source fluctuation $\delta N^{a0}_{k_\parallel,\omega}$ is incomplete. For the moment, however, let us further review the procedure as outlined in Ref. 26.

A. Wave kinetic equation

Upon multiplying $\delta E^\parallel_{-k_\parallel,-\omega}$ to the first equation of (13), $\delta E^\mp_{-k_\parallel,-\omega}$ to the second equation, taking the ensemble averages, and making use of the relations

$$\langle \delta E^\parallel_{k_\parallel,\omega}\, \delta E^\parallel_{-k_\parallel,-\omega}\rangle = \langle \delta E^2_\parallel\rangle_{k_\parallel,\omega},$$
$$\langle \delta E^+_{k_\parallel,\omega}\, \delta E^-_{-k_\parallel,-\omega}\rangle = \langle \delta E^2_+\rangle_{k_\parallel,\omega},$$
$$\langle \delta E^-_{k_\parallel,\omega}\, \delta E^+_{-k_\parallel,-\omega}\rangle = \langle \delta E^2_-\rangle_{k_\parallel,\omega},$$
$$\langle \delta E^2_\parallel\rangle_{-k_\parallel,-\omega} = \langle \delta E^2_\parallel\rangle_{k_\parallel,\omega},$$
$$\langle \delta E^2_+\rangle_{-k_\parallel,-\omega} = \langle \delta E^2_-\rangle_{k_\parallel,\omega},$$
$$\langle \delta E^2_-\rangle_{-k_\parallel,-\omega} = \langle \delta E^2_+\rangle_{k_\parallel,\omega}. \quad (15)$$

Equation (13) can be expressed as

$$\epsilon_\parallel(k_\parallel,\omega)\langle \delta E^2_\parallel\rangle_{k_\parallel,\omega} = -\frac{4\pi i}{\omega}\sum_a e_a\int d\mathbf{v}\, v_\parallel \langle \delta N^{a0}_{k_\parallel,\omega}\delta E^\parallel_{-k_\parallel,-\omega}\rangle,$$
$$\Lambda_\pm(k_\parallel,\omega)\langle \delta E^2_\pm\rangle_{k_\parallel,\omega} = -\frac{4\pi i}{\omega}\sum_a e_a\int d\mathbf{v}\, \frac{v_\perp}{2} \times e^{\mp i\varphi}\langle \delta N^{a0}_{k_\parallel,\omega}\delta E^\mp_{-k_\parallel,-\omega}\rangle. \quad (16)$$

To obtain the source fluctuations $\langle \delta N^{a0}_{k_\parallel,\omega}\delta E^\parallel_{-k_\parallel,-\omega}\rangle$ and $\langle \delta N^{a0}_{k_\parallel,\omega}\delta E^\mp_{-k_\parallel,-\omega}\rangle$, we go back to Eq. (13), change $(k_\parallel,\omega)$ to $(-k_\parallel,-\omega)$, multiply $\delta N^{a0}_{k_\parallel,\omega}$, and take ensemble averages

$$\langle \delta N^{a0}_{k_\parallel,\omega}\, \delta E^\parallel_{-k_\parallel,-\omega}\rangle = \frac{4\pi i}{\omega\epsilon_\parallel(-k_\parallel,-\omega)}\sum_b e_b\int d\mathbf{v}'\, v'_\parallel \times \langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\, \delta N^{b0}_{-k_\parallel,-\omega}(\mathbf{v}')\rangle,$$
$$\langle \delta N^{a0}_{k_\parallel,\omega}\, \delta E^\pm_{-k_\parallel,-\omega}\rangle = \frac{2\pi i}{\omega\Lambda_\pm(-k_\parallel,-\omega)}\sum_b e_b\int d\mathbf{v}'\, v'_\perp \times e^{\pm i\varphi'}\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\, \delta N^{b0}_{-k_\parallel,-\omega}(\mathbf{v}')\rangle. \quad (17)$$

In Ref. 26 we made use of Eq. (7) by simply taking the limit $k_\perp \to 0$. If we allow such a procedure, then of course, we obtain the simple expressions found in Ref. 26, that is,

$$\langle \delta N^{a0}_{k_\parallel,\omega}\, \delta E^\parallel_{-k_\parallel,-\omega}\rangle = \frac{ie_a}{2\pi^2 k_\parallel \epsilon^*_\parallel(k_\parallel,\omega)}\,\delta(\omega - k_\parallel v_\parallel) f_a,$$
$$\langle \delta N^{a0}_{k_\parallel,\omega}\, \delta E^\pm_{-k_\parallel,-\omega}\rangle = \frac{ie_a}{4\pi^2\omega\Lambda^*_\pm(k_\parallel,\omega)}\,\mathrm{Re}\, v_\perp\, e^{\mp i\varphi} \times \delta(\omega - k_\parallel v_\parallel \mp \Omega_a) f_a, \quad (18)$$

where we have made use of $\epsilon_\parallel(-k_\parallel,-\omega) = \epsilon^*_\parallel(k_\parallel,\omega)$ and $\Lambda_\pm(-k_\parallel,-\omega) = \Lambda^*_\mp(k_\parallel,\omega)$. However, this is where the correction should be made by taking into account the fact that the source fluctuation (7) should not have any preferred direction in the wave vector space for 3D situation, and that in order to formulate a kinetic theory for turbulence propagating in quasi-parallel direction, the proper way is to consider the full expression for source fluctuation (7) and integrate over $k_\perp$.

Within the context of Eq. (17), the relevant quantities to be re-examined are therefore

$$\int d\mathbf{v}' v_\parallel' \langle \delta N_a^0(\mathbf{v})\, \delta N_b^0(\mathbf{v}')\rangle_{k_\parallel,\omega} = (2\pi)^{-2}\delta_{ab} v_\parallel \int_0^\infty dk_\perp k_\perp \sum_{n=-\infty}^{\infty} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right) \times \delta\left(\omega - k_\parallel v_\parallel - n\Omega_a\right) f_a(\mathbf{v}),$$
$$\int d\mathbf{v}' v_\perp' e^{\mp i\varphi'} \langle \delta N_a^0(\mathbf{v})\, \delta N_b^0(\mathbf{v}')\rangle_{k_\parallel,\omega} = (2\pi)^{-2}\delta_{ab} v_\perp e^{\mp i\varphi} \int_0^\infty dk_\perp k_\perp \sum_{n=-\infty}^{\infty} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right) \times \delta\left[\omega - k_\parallel v_\parallel - (n\pm 1)\Omega_a\right] f_a(\mathbf{v}). \tag{19}$$

Upon inserting Eq. (19) to the right-hand side of Eq. (17), we now have a proper expression that should replace Eq. (18)

$$\langle \delta N^{a0}_{k_\parallel,\omega}\, \delta E^{\parallel}_{-k_\parallel,-\omega}\rangle = \frac{i e_a v_\parallel}{\pi\omega\epsilon_\parallel(-k_\parallel,-\omega)} \int_0^\infty dk_\perp k_\perp \times \sum_{n=-\infty}^{\infty} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right) \delta\left(\omega - k_\parallel v_\parallel - n\Omega_a\right) f_a,$$
$$\langle \delta N^{a0}_{k_\parallel,\omega}\, \delta E^{\pm}_{-k_\parallel,-\omega}\rangle = \frac{i e_a v_\perp}{2\pi\omega\Lambda_\pm(-k_\parallel,-\omega)} \operatorname{Re} \int_0^\infty dk_\perp k_\perp \times \sum_{n=-\infty}^{\infty} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right) e^{\mp i\varphi}\, \delta\left[\omega - k_\parallel v_\parallel - (n\pm 1)\Omega_a\right] f_a. \tag{20}$$

In Eq. (20), if we arbitrarily take the limit of $k_\perp \to 0$, then we recover Eq. (18).[26] We reiterate that if the system under consideration is truly 1D such that the particle thermal motion is confined in 1D space, as in 1D computer simulation space, then Eq. (18) is the proper way to deal with the spectral representation of the source fluctuation. However, for a real 3D world, even if we are interested in waves excited along the direction of ambient magnetic field vector, one nevertheless must treat the source fluctuation as having both $k_\perp$ and $k_\parallel$, and subsequently integrate over $k_\perp$.

Note that the $k_\perp$ integral is a divergent quantity. To avoid the divergence, we introduce the upper cutoff

$$\int_0^\infty dk_\perp k_\perp J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right) \to \int_0^{k_m} dk_\perp k_\perp J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right), \tag{21}$$

where k_m is to be determined. Appendix A discusses the evaluation of the $k_\perp$ integral. In the present section, we take a more heuristic approach. We argue that the characteristic value of v close to thermal speed is the most important range over which the argument of the Bessel function $k_\perp v_\perp/\Omega_a$ is to be considered. In this case, we may assume that $k_\perp v_\perp/\Omega_a$ is small so that the Bessel function factor $J_n^2(k_\perp v_\perp/\Omega_a)$ can be replaced by $\delta_{n,0}$. Under such a prescription, Eq. (20) greatly simplifies

$$\langle \delta N^{a0}_{k_\parallel,\omega}\, \delta E^{\parallel}_{-k_\parallel,-\omega}\rangle = \frac{i e_a k_m^2}{2\pi k_\parallel \epsilon_\parallel^*(k_\parallel,\omega)} \delta\left(\omega - k_\parallel v_\parallel\right) f_a,$$
$$\langle \delta N^{a0}_{k_\parallel,\omega}\, \delta E^{\pm}_{-k_\parallel,-\omega}\rangle = \frac{i e_a k_m^2}{4\pi\omega\Lambda_\pm^*(k_\parallel,\omega)} \operatorname{Re} v_\perp e^{\mp i\varphi} \times \delta\left(\omega - k_\parallel v_\parallel \mp \Omega_a\right) f_a. \tag{22}$$

Upon comparing the two expressions (18) and (22), one will notice that Eq. (22) is basically the same as Eq. (18), except for the overall multiplicative factor πk_m^2. We now substitute Eq. (22) to the right-hand side of Eq. (16) to arrive at

$$\epsilon_\parallel(k_\parallel,\omega)\langle \delta E_\parallel^2\rangle_{k_\parallel,\omega} = \sum_a \frac{2e_a^2 k_m^2}{k_\parallel^2 \epsilon_\parallel^*(k_\parallel,\omega)} \int d\mathbf{v}\, \delta\left(\omega - k_\parallel v_\parallel\right) f_a,$$
$$\Lambda_\pm(k_\parallel,\omega)\langle \delta E_\pm^2\rangle_{k_\parallel,\omega} = \sum_a \frac{e_a^2 k_m^2}{\omega^2 \Lambda_\pm^*(k_\parallel,\omega)} \int d\mathbf{v}\, v_\perp^2\, \delta\left(\omega - k_\parallel v_\parallel \pm \Omega_a\right) f_a. \tag{23}$$

Equation (23) can be used to discuss the correct dimensionality for 3D case, as is done in Appendix B.

To define cutoff k_m, we point out that k_m represents the maximum perpendicular wave number associated with the system at hand. As such, it can be determined, for instance, by requiring that the Bessel function argument defined with respect to the thermal speed is less than unity

$$\frac{k_\perp^2 v_a^2}{\Omega_a^2} = \frac{2k_\perp^2 m_a c^2 T_a}{e_a^2 B_0^2} \ll 1.$$

Then, we may define k_m^2 by

$$k_m^2 = \frac{\Omega_a^2}{v_a^2}, \tag{24}$$

where $v_a^2 = 2T_a/m_a$ is the thermal speed.

With the correct result, namely, Eq. (23) for 3D situation, the formal wave kinetic equation can be obtained. We introduce the slow-time dependence in the wave frequency

$$\omega \to \omega + i\frac{\partial}{\partial t}, \tag{25}$$

in the leading term, namely, the linear response functions

$$\epsilon_\parallel(k_\parallel,\omega)\langle \delta E_\parallel^2\rangle_{k_\parallel,\omega} \to \epsilon_\parallel(k_\parallel,\omega)\langle \delta E_\parallel^2\rangle_{k_\parallel,\omega} + \frac{i}{2}\frac{\partial \epsilon_\parallel(k_\parallel,\omega)}{\partial\omega}\frac{\partial}{\partial t}\langle \delta E_\parallel^2\rangle_{k_\parallel,\omega},$$
$$\omega^2\Lambda_\pm(k_\parallel,\omega)\langle \delta E_\pm^2\rangle_{k_\parallel,\omega} \to \omega^2\Lambda_\pm(k_\parallel,\omega)\langle \delta E_\pm^2\rangle_{k_\parallel,\omega} + \frac{i}{2}\frac{\partial \omega^2\Lambda_\pm(k_\parallel,\omega)}{\partial\omega}\frac{\partial}{\partial t}\langle \delta E_\pm^2\rangle_{k_\parallel,\omega}. \tag{26}$$

This leads to

$$i\frac{\partial \epsilon_\parallel(k_\parallel,\omega)}{\partial\omega}\frac{\partial}{\partial t}\langle \delta E_\parallel^2\rangle_{k_\parallel,\omega} + 2\epsilon_\parallel(k_\parallel,\omega)\langle \delta E_\parallel^2\rangle_{k_\parallel,\omega} = \sum_a \frac{4e_a^2 k_m^2}{k_\parallel^2\epsilon_\parallel^*(k_\parallel,\omega)} \int d\mathbf{v}\, \delta\left(\omega - k_\parallel v_\parallel\right) f_a, \tag{27}$$

$$i\frac{\partial \omega^2\Lambda_\pm(k_\parallel,\omega)}{\partial\omega}\frac{\partial}{\partial t}\langle \delta E_\pm^2\rangle_{k_\parallel,\omega} + 2\omega^2\Lambda_\pm(k_\parallel,\omega)\langle \delta E_\pm^2\rangle_{k_\parallel,\omega} = \sum_a \frac{2e_a^2 k_m^2}{\Lambda_\pm^*(k_\parallel,\omega)} \int d\mathbf{v} v_\perp^2\, \delta\left(\omega - k_\parallel v_\parallel \pm \Omega_a\right) f_a. \tag{28}$$

The above two equations constitute the spectral balance equation for longitudinal and transverse modes propagating in directions predominantly parallel/anti-parallel to the ambient magnetic field. They form the basis of the linear dispersion theory and quasilinear wave kinetic theory that includes the spontaneous emission of the eigenmodes. If we take the real parts of Eqs. (27) and (28) and ignore small real contributions from the right-hand sides, then we have the instantaneous dispersion relations for the two respective modes

$$\begin{aligned} \mathrm{Re}\,\epsilon_\parallel(k_\parallel,\omega) &= 0, \\ \mathrm{Re}\,\Lambda_\pm(k_\parallel,\omega) &= 0. \end{aligned} \tag{29}$$

Upon taking the imaginary parts, we have the formal wave kinetic equations

$$\begin{aligned} &\frac{\partial\,\mathrm{Re}\,\epsilon_\parallel(k_\parallel,\omega)}{\partial\omega}\frac{\partial\langle\delta E_\parallel^2\rangle_{k_\parallel,\omega}}{\partial t} + 2\,\mathrm{Im}\,\epsilon_\parallel(k_\parallel,\omega)\langle\delta E_\parallel^2\rangle_{k_\parallel,\omega} \\ &\quad = \sum_a 4e_a^2k_m^2\,\mathrm{Im}\,\frac{1}{k_\parallel^2\epsilon_\parallel^*(k_\parallel,\omega)}\int d\mathbf{v}\,\delta(\omega-k_\parallel v_\parallel)f_a, \end{aligned} \tag{30}$$

$$\begin{aligned} &\frac{\partial\,\mathrm{Re}\,\omega^2\Lambda_\pm(k_\parallel,\omega)}{\partial\omega}\frac{\partial\langle\delta E_\pm^2\rangle_{k_\parallel,\omega}}{\partial t} + 2\,\mathrm{Im}\,\omega^2\epsilon_\pm(k_\parallel,\omega)\langle\delta E_\pm^2\rangle_{k_\parallel,\omega} \\ &\quad = \sum_a 2e_a^2k_m^2\,\mathrm{Im}\,\frac{1}{\Lambda_\pm^*(k_\parallel,\omega)}\int d\mathbf{v}\,v_\perp^2\,\delta(\omega-k_\parallel v_\parallel\pm\Omega_a)f_a. \end{aligned} \tag{31}$$

The second terms on the left-hand sides of Eqs. (30) and (31) above correspond to the induced emission processes, while the right-hand sides, which arise due to discrete-particle effects, describe spontaneous emissions. Note that the above equations contain simplified forms of the spontaneous emission terms. If we resort to the more complete analysis as found in Appendix A, then one can show that the resulting formal wave kinetic equations generalize to

$$\begin{aligned} &\frac{\partial\mathrm{Re}\epsilon_\parallel(k_\parallel,\omega)}{\partial\omega}\frac{\partial\langle\delta E_\parallel^2\rangle_{k_\parallel,\omega}}{\partial t} + 2\mathrm{Im}\epsilon_\parallel(k_\parallel,\omega)\langle\delta E_\parallel^2\rangle_{k_\parallel,\omega} \\ &\quad = \sum_a \frac{4e_a^2k_m^2}{k_\parallel^2}\,\mathrm{Im}\left(\frac{1}{\epsilon_\parallel^*(k_\parallel,\omega)}\right)\int d\mathbf{v} \\ &\qquad \times\left[J_0^2\left(\frac{k_mv_\perp}{\Omega_a}\right)+J_1^2\left(\frac{k_mv_\perp}{\Omega_a}\right)\right]\delta(\omega-k_\parallel v_\parallel)f_a, \end{aligned} \tag{32}$$

$$\begin{aligned} &\frac{\partial\mathrm{Re}\,\omega^2\Lambda_\pm(k_\parallel,\omega)}{\partial\omega}\frac{\partial\langle\delta E_\pm^2\rangle_{k_\parallel,\omega}}{\partial t} + 2\mathrm{Im}\,\omega^2\Lambda_\pm(k_\parallel,\omega)\langle\delta E_\pm^2\rangle_{k_\parallel,\omega} \\ &\quad = \sum_a 2e_a^2k_m^2\,\mathrm{Im}\left(\frac{1}{\Lambda_\pm(k_\parallel,\omega)}\right)^*\int d\mathbf{v} \\ &\qquad \times\left[J_0^2\left(\frac{k_mv_\perp}{\Omega_a}\right)+J_1^2\left(\frac{k_mv_\perp}{\Omega_a}\right)\right]v_\perp^2\,\delta(\omega-k_\parallel v_\parallel\pm\Omega_a)f_a. \end{aligned} \tag{33}$$

This concludes the derivation of dimensionally correct formal wave kinetic equations. We next consider the particle kinetic equation.

B. Particle kinetic equation

The particle kinetic equation is given by Eq. (1). Upon making use of the general property

$$\begin{aligned} &\frac{\partial}{\partial v_i}\left[\left(1-\frac{\mathbf{k}\cdot\mathbf{v}}{\omega}\right)\delta_{ij}+\frac{k_iv_j}{\omega}\right]G_j \\ &\quad = \left[\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{\partial}{\partial v_\perp}+\frac{k_\parallel v_\perp}{\omega}\frac{\partial}{\partial v_\parallel}\right](G_x\cos\varphi+G_y\sin\varphi) \\ &\qquad +\frac{\partial G_z}{\partial v_\parallel}-\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{1}{v_\perp}\left(\sin\varphi\frac{\partial G_x}{\partial\varphi}-\cos\varphi\frac{\partial G_y}{\partial\varphi}\right), \end{aligned} \tag{34}$$

we have for quasi-1D situation

$$\begin{aligned} \frac{\partial f_a}{\partial t} &= -\frac{e_a}{m_a}\int d\mathbf{k}\int d\omega\left\{\sum_{+,-}\left[\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{\partial}{\partial v_\perp}+\frac{k_\parallel v_\perp}{\omega}\frac{\partial}{\partial v_\parallel}\right]\right. \\ &\quad \times\langle\delta E_{-k_\parallel,-\omega}^\mp\,\delta N_{k_\parallel,\omega}^a\rangle e^{\mp i\varphi}+\sum_{+,-}\frac{\mp i}{v_\perp}\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)e^{\mp i\varphi} \\ &\quad \left.\times\frac{\partial}{\partial\varphi}\langle\delta E_{-k_\parallel,-\omega}^\mp\,\delta N_{k_\parallel,\omega}^a\rangle+\frac{\partial}{\partial v_\parallel}\langle\delta E_{-k_\parallel,-\omega}^z\,\delta N_{k_\parallel,\omega}^a\rangle\right\}. \end{aligned} \tag{35}$$

From Eq. (10), we have the expression for $\delta N_{k_\parallel,\omega}^a$. We also make use of Eq. (22) for the source fluctuations, $\langle\delta E_{-k_\parallel,-\omega}^\parallel\,\delta N_{k_\parallel,\omega}^{a0}\rangle$ and $\langle\delta E_{-k_\parallel,-\omega}^\pm\,\delta N_{k_\parallel,\omega}^{a0}\rangle$. Taking these into account, we have

$$\begin{aligned} \langle\delta E_{-k_\parallel,-\omega}^\parallel\,\delta N_{k_\parallel,\omega}^a\rangle &= \frac{ie_ak_m^2}{2\pi k_\parallel\epsilon_\parallel^*(k_\parallel,\omega)}\,\delta(\omega-k_\parallel v_\parallel)f_a \\ &\quad -\frac{ie_a}{m_a}\frac{\langle\delta E_\parallel^2\rangle_{k_\parallel,\omega}}{\omega-k_\parallel v_\parallel}\frac{\partial f_a}{\partial v_\parallel}, \\ \langle\delta E_{-k_\parallel,-\omega}^\mp\,\delta N_{k_\parallel,\omega}^a\rangle &= \frac{ie_ak_m^2\,v_\perp e^{\pm i\varphi}}{4\pi\,\omega\Lambda_\pm^*(k_\parallel,\omega)}\,\delta(\omega-k_\parallel v_\parallel\pm\Omega_a)f_a \\ &\quad -\frac{ie_a}{m_a}\frac{e^{\pm i\varphi}\langle\delta E_\pm^2\rangle_{k_\parallel,\omega}}{\omega-kv_\parallel\pm\Omega_a+i0} \\ &\quad \times\left[\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{\partial f_a}{\partial v_\perp}+\frac{k_\parallel v_\perp}{\omega}\frac{\partial f_a}{\partial v_\parallel}\right]. \end{aligned} \tag{36}$$

Inserting Eq. (36) to Eq. (35), we obtain the particle kinetic equation

$$\begin{aligned} \frac{\partial f_a}{\partial t} &= \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[v_\perp\left(A_\perp f_a+D_{\perp\perp}\frac{\partial f_a}{\partial v_\perp}+D_{\perp\parallel}\frac{\partial f_a}{\partial v_\parallel}\right)\right] \\ &\quad +\frac{\partial}{\partial v_\parallel}\left(A_\parallel f_a+D_{\parallel\perp}\frac{\partial f_a}{\partial v_\perp}+D_{\parallel\parallel}\frac{\partial f_a}{\partial v_\parallel}\right), \end{aligned} \tag{37}$$

where $\mathbf{A}$ is the vector drag coefficient associated with the discrete-particle effects

$$\begin{aligned} \begin{pmatrix} A_\perp \\ A_\parallel \end{pmatrix} &= \frac{e_a^2k_m^2}{4\pi m_a}\int dk_\parallel\int d\omega\begin{pmatrix}\omega-k_\parallel v_\parallel \\ k_\parallel v_\perp\end{pmatrix} \\ &\quad \times\sum_{+,-}\mathrm{Im}\left(\frac{1}{\omega^2\Lambda_\pm(k_\parallel,\omega)}\right)^*v_\perp\,\delta(\omega-k_\parallel v_\parallel\pm\Omega_a) \\ &\quad +\frac{e_a^2k_m^2}{2\pi m_a}\int dk_\parallel\int d\omega\begin{pmatrix}0\\1\end{pmatrix}\mathrm{Im}\left(\frac{1}{k_\parallel\epsilon_\parallel(k_\parallel,\omega)}\right)^* \\ &\quad \times\delta(\omega-k_\parallel v_\parallel), \end{aligned} \tag{38}$$

and D is the diffusion tensor defined by

$$\begin{pmatrix} D_{\perp\perp} \\ D_{\perp\parallel}=D_{\parallel\perp} \\ D_{\parallel\parallel} \end{pmatrix} = \frac{\pi e_a^2}{m_a^2}\int dk_\parallel \int d\omega \sum_{+,-} \begin{pmatrix} (\omega-k_\parallel v_\parallel)^2 \\ (\omega-k_\parallel v_\parallel)(k_\parallel v_\perp) \\ (k_\parallel v_\perp)^2 \end{pmatrix} \frac{\langle \delta E_\mp^2\rangle_{k_\parallel,\omega}}{\omega^2}\,\delta(\omega-k_\parallel v_\parallel \pm \Omega_a)$$
$$+\frac{\pi e_a^2}{m_a^2}\int dk_\parallel \int d\omega \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \langle \delta E_\parallel^2\rangle_{k_\parallel,\omega}\,\delta(\omega-k_\parallel v_\parallel). \tag{39}$$

The above particle kinetic equation that contains the proper drag coefficient is the counterpart to the wave kinetic equations (30) and (31). As with Eqs. (30) and (31), however, the drag coefficient that has the dimensional correction factor k_m^2 is an approximation, which can be replaced by a more complete expression, as shown in Eqs. (32) and (33). If we rely on the more rigorous treatment as elucidated in Appendix A, the more complete particle kinetic equation emerges

$$\frac{\partial f_a}{\partial t} = \frac{e_a^2 k_m^2}{4\pi m_a}\int dk_\parallel \int d\omega \sum_{+,-}\frac{1}{v_\perp}\left[\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{\partial}{\partial v_\perp}+\frac{k_\parallel v_\perp}{\omega}\frac{\partial}{\partial v_\parallel}\right]\mathrm{Im}\left(\frac{1}{\omega\Lambda_\pm(k_\parallel,\omega)}\right)^*\left[J_0^2\left(\frac{k_m v_\perp}{\Omega_a}\right)+J_1^2\left(\frac{k_m v_\perp}{\Omega_a}\right)\right]v_\perp^2\,\delta(\omega-k_\parallel v_\parallel\pm\Omega_a) f_a$$
$$+\frac{\pi e_a^2}{m_a^2}\int dk_\parallel \int d\omega \sum_{+,-}\frac{1}{v_\perp}\left[\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{\partial}{\partial v_\perp}+\frac{k_\parallel v_\perp}{\omega}\frac{\partial}{\partial v_\parallel}\right] v_\perp \langle\delta E_\pm^2\rangle_{k_\parallel,\omega}\,\delta(\omega-k_\parallel v_\parallel\pm\Omega_a)\left[\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{\partial f_a}{\partial v_\perp}+\frac{k_\parallel v_\perp}{\omega}\frac{\partial f_a}{\partial v_\parallel}\right]$$
$$+\frac{e_a^2 k_m^2}{2\pi m_a}\frac{\partial}{\partial v_\parallel}\int dk_\parallel \int d\omega\,\mathrm{Im}\left(\frac{1}{k_\parallel \epsilon_\parallel(k_\parallel,\omega)}\right)^*\left[J_0^2\left(\frac{k_m v_\perp}{\Omega_a}\right)+J_1^2\left(\frac{k_m v_\perp}{\Omega_a}\right)\right]\delta(\omega-k_\parallel v_\parallel) f_a$$
$$+\frac{\pi e_a^2}{m_a^2}\frac{\partial}{\partial v_\parallel}\int dk_\parallel \int d\omega \langle\delta E_\parallel^2\rangle_{k_\parallel,\omega}\,\delta(\omega-k_\parallel v_\parallel)\frac{\partial f_a}{\partial v_\parallel}. \tag{40}$$

Note that Eq. (40) can also be alternatively expressed in the form of Fokker-Planck equation, as in Eq. (37). All one has to do is to redefine the drag coefficient **A** in Eq. (38) by replacing k_m^2 by $k_m^2[J_0^2(k_m v_\perp/\Omega_a)+J_2^2(k_m v_\perp/\Omega_a)]$.

III. SUMMARY

The equations for particles and waves presented in this paper rectify the shortcomings in the original formalism developed by Yoon and Fang,[26] whose purpose was to formulate a second-order nonlinear kinetic theory that describes the turbulence propagating in directions parallel/anti-parallel to the ambient magnetic field. However, in the original theory, the spontaneous emission (and spontaneous scattering) term(s) in the wave kinetic equation and the drag term in the particle kinetic equation, which arise from the discreteness of the plasma particles, were not treated properly. While their formalism may be applicable to an artificial one-dimensional world (such as 1D computer simulation environment), for three dimensions, one must not assume that the spontaneously emitted thermal fluctuations are directed only along the ambient magnetic field. Instead, all three directions of fluctuation emissions must be properly taken into account. This means that when one is interested in quasi-1D problems, one must integrate the spontaneous fluctuations over the perpendicular space.

The present paper carried out such an analysis, and the resulting wave and particle kinetic equations [Eqs. (30) and (31), or alternatively Eqs. (32) and (33), for the waves, and Eqs. (37)–(39), or more generally Eq. (40), for the particles] contain dimensionally correct coefficients for 3D case. In the present paper, we paid attention to the linear wave-particle interaction terms only, as a first step. We are planning on a subsequent paper that will address the problem of dimensionally correct nonlinear wave-particle interaction terms. While the essential theoretical formalism developed in the present paper is a formal one, specific application to actual applications will be made in a separate discussion. Note that in the original paper by Yoon and Fang,[26] the problem of low-frequency turbulence propagating along the ambient magnetic field and wave-particle interaction involving protons was considered. The formalism developed in the present paper can equally be employed for high-frequency waves propagating along the ambient magnetic field as well. For instance, whistler turbulence undergoing wave-particle interaction with the electrons can be addressed under the present formalism. An application of the present formalism was recently carried out. In a recent paper by Kim *et al.*,[29] an asymptotic theory of solar wind electron velocity distribution function was constructed based upon the present formalism. The referenced work was also motivated by a recent observation by Lacombe *et al.*[28] who identified whistler fluctuations in the solar wind with clearly quasi-parallel direction of propagation. For such a situation, the present formalism is applicable. Consequently, Ref. 29 formulated the model of the solar wind electrons that are in dynamical steady-state with the quasi-parallel whistler fluctuations.

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APPENDIX A: EVALUATION OF $k_\perp$ INTEGRAL

Let us rewrite Eq. (19) as follows:

$$\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\,\delta E^{\parallel}_{-k_\parallel,-\omega}\rangle = \frac{ie_a v_\parallel}{\pi^2\omega\,\epsilon^*_\parallel(k_\parallel,\omega)}\int_0^{k_m} dk_\perp k_\perp \sum_{n=-\infty}^{\infty} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\mathrm{Re}\,\frac{i}{\omega - k_\parallel v_\parallel - n\Omega_a + i0} f_a,$$
$$\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\,\delta E^{\pm}_{-k_\parallel,-\omega}\rangle = \frac{ie_a v_\perp}{2\pi^2\omega\,\Lambda^*_\pm(k_\parallel,\omega)}\, e^{\mp i\varphi}\,\mathrm{Re}\int_0^{k_m} dk_\perp k_\perp \sum_{n=-\infty}^{\infty} J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\frac{i}{\omega - k_\parallel v_\parallel - (n\mp 1)\,\Omega_a + i0} f_a. \quad (A1)$$

Upon making use of the integral identity[38]

$$\int dx\, x J_n^2(ax) = \frac{x^2}{2}\left[J_n^2(ax) - J_{n-1}(ax)J_{n+1}(ax)\right], \quad (A2)$$

we write

$$\int_0^{k_m} dk_\perp k_\perp J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right) = \frac{k_m^2}{2}\left[J_n^2(b_m) - J_{n-1}(b_m)J_{n+1}(b_m)\right],$$
$$b_m = \frac{k_m v_\perp}{\Omega_a}. \quad (A3)$$

We then make use of the Newberger Bessel function sum rule[39]

$$\sum_{n=-\infty}^{\infty}\frac{J_n(z)J_{n-m}(z)}{a-n} = \frac{(-1)^m\pi}{\sin\pi a} J_{m-a}(z)J_a(z), \quad (A4)$$

to write

$$\sum_{n=-\infty}^{\infty}\frac{J_n^2(b_m) - J_{n-1}(b_m)J_{n+1}(b_m)}{\omega - k_\parallel v_\parallel - n\Omega_a + i0} = \frac{\pi\left[J_{-a}(b_m)J_a(b_m) + J_{1-a}(b_m)J_{a+1}(b_m)\right]}{\Omega_a \sin\pi a},$$
$$\sum_{n=-\infty}^{\infty}\frac{J_n^2(b_m) - J_{n-1}(b_m)J_{n+1}(b_m)}{\omega - k_\parallel v_\parallel - (n\mp 1)\,\Omega_a + i0} = \frac{\pi\left[J_{-c}(b_m)J_c(b_m) + J_{1-c}(b_m)J_{c+1}(b_m)\right]}{\Omega_a \sin\pi c}, \quad (A5)$$

where

$$a = \frac{\omega - k_\parallel v_\parallel + i0}{\Omega_a},$$
$$c = \frac{\omega - k_\parallel v_\parallel \pm \Omega_a + i0}{\Omega_a}. \quad (A6)$$

Next, we make use of the integral involving products of the Bessel function[40]

$$J_\mu(z)J_\nu(z) = \frac{2}{\pi}\int_0^{\pi/2} d\theta\, J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\theta, \quad (A7)$$

where $[\mathrm{Re}(\mu+\nu) > -1]$, to write

$$\sum_{n=-\infty}^{\infty}\frac{J_n^2(b_m) - J_{n-1}(b_m)J_{n+1}(b_m)}{\omega - k_\parallel v_\parallel - n\Omega_a + i0} = \frac{2}{\Omega_a b_m}\int_0^{\pi/2} d\theta\, J_1(2b_m\cos\theta)\,\frac{\cos(2a\theta)}{\sin(\pi a)\cos\theta},$$
$$\sum_{n=-\infty}^{\infty}\frac{J_n^2(b_m) - J_{n-1}(b_m)J_{n+1}(b_m)}{\omega - k_\parallel v_\parallel - (n\mp 1)\,\Omega_a + i0} = \frac{2}{\Omega_a b_m}\int_0^{\pi/2} d\theta\, J_1(2b_m\cos\theta)\,\frac{\cos(2c\theta)}{\sin(\pi c)\cos\theta}. \quad (A8)$$

Inserting Eq. (A8) to Eq. (A1), we have

$$\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\,\delta E^{\parallel}_{-k_\parallel,-\omega}\rangle = \frac{ie_a v_\parallel}{\pi^2\omega\,\epsilon^*_\parallel(k_\parallel,\omega)}\,\mathrm{Re}\,i\,\frac{k_m^2}{\Omega_a b_m}\int_0^{\pi/2} d\theta\,J_1(2b_m\cos\theta)\,\frac{\cos(2a\theta)}{\sin(\pi a)\cos\theta}\,f_a,$$
$$\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\,\delta E^{\pm}_{-k_\parallel,-\omega}\rangle = \frac{ie_a v_\perp}{2\pi^2\omega\,\Lambda^*_\pm(k_\parallel,\omega)}\,e^{\pm i\varphi}\,\mathrm{Re}\,i\,\frac{k_m^2}{\Omega_a b_m}\int_0^{\pi/2} d\theta\,J_1(2b_m\cos\theta)\,\frac{\cos(2c\theta)}{\sin(\pi c)\cos\theta}\,f_a. \tag{A9}$$

Note that a and c have infinitesimally small positive imaginary parts and that in the context of Eq. (A9) we are only interested in Re i of everything to the right. As a consequence, we consider the imaginary part of the following quantity:

$$\lim_{a_i\to+0}\mathrm{Im}\left(\frac{\cos(2a\theta)}{\sin(\pi a)}\right) = -\lim_{a_i\to+0}\frac{1}{\cosh^2(\pi a_i)-\cos^2(\pi a_r)}\,[\cos(2a_r\theta)\cos(\pi a_r)\cosh(2a_i\theta)\sinh(\pi a_i)$$
$$+\sin(2a_r\theta)\sin(\pi a_r)\sinh(2a_i\theta)\cosh(\pi a_i)]$$
$$\approx -\lim_{a_i\to+0}\frac{a_i}{1+\pi^2a_i^2-\cos^2(\pi a_r)}\left[\pi\cos(2a_r\theta)\cos(\pi a_r)+2\theta\sin(2a_r\theta)\sin(\pi a_r)\right]. \tag{A10}$$

If the denominator is non-zero, then taking the limit of $a_i \to +0$ makes the above quantity to vanish. If on the other hand, $a_r \to 0$, then the denominator also approaches zero, in which case, the limit of $a_i \to +0$ is undetermined. To calculate such a limit, the above quantity is expanded around $a_r = 0$ by making use of trigonometric series expansion, $\sin x \sim x$ and $\cos x \sim 1 - x^2/2$

$$\lim_{a_i\to+0}\lim_{a_r\to 0}\mathrm{Im}\left(\frac{\cos(2a\theta)}{\sin(\pi a)}\right) = -\frac{1}{\pi}\lim_{a_i\to+0}\lim_{a_r\to 0}\left[1+\left(2\theta^2-\frac{\pi^2}{2}\right)a_r^2\right]\frac{a_i}{a_r^2+a_i^2}. \tag{A11}$$

We make use of the definition for the Dirac delta function

$$\delta(x-x_0) = \frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{(x-x_0)^2+\epsilon^2}, \tag{A12}$$

to first take care of the limit $a_i \to 0$. This leads to

$$\lim_{a_i\to+0}\lim_{a_r\to 0}\mathrm{Im}\left(\frac{\cos(2a\theta)}{\sin(\pi a)}\right) = -\lim_{a_r\to 0}\left[1+\left(2\theta^2-\frac{\pi^2}{2}\right)a_r^2\right]\delta(a_r) = -\delta(a_r). \tag{A13}$$

With this result, we now express Eq. (A9) as

$$\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\,\delta E^{\parallel}_{-k_\parallel,-\omega}\rangle = \frac{ie_a v_\parallel}{\pi^2\omega\,\epsilon^*_\parallel(k_\parallel,\omega)}\,\mathrm{Re}\,i\,\frac{k_m^2}{\Omega_a b_m}\int_0^{\pi/2} d\theta\,\frac{J_1(2b_m\cos\theta)}{\cos\theta}\,(-i)\,\delta(a_r)f_a,$$
$$\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\,\delta E^{\mp}_{-k_\parallel,-\omega}\rangle = \frac{ie_a v_\perp}{2\pi^2\omega\,\Lambda^*_\pm(k_\parallel,\omega)}\,e^{\pm i\varphi}\,\mathrm{Re}\,i\,\frac{k_m^2}{\Omega_a b_m}\int_0^{\pi/2} d\theta\,\frac{J_1(2b_m\cos\theta)}{\cos\theta}\,(-i)\,\delta(c_r)f_a, \tag{A14}$$

which can be re-expressed as

$$\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\,\delta E^{\parallel}_{-k_\parallel,-\omega}\rangle = \frac{ie_a k_m^2 v_\parallel}{\pi^2\omega\,\epsilon^*_\parallel(k_\parallel,\omega)}\int_0^{\pi/2} d\theta\,[J_0(2b_m\cos\theta)+J_2(2b_m\cos\theta)]\,\delta(\omega-k_\parallel v_\parallel)f_a,$$
$$\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\,\delta E^{\mp}_{-k_\parallel,-\omega}\rangle = \frac{ie_a k_m^2 v_\perp}{\pi^2\omega\,\Lambda^*_\pm(k_\parallel,\omega)}\,e^{\pm i\varphi}\int_0^{\pi/2} d\theta\,[J_0(2b_m\cos\theta)+J_2(2b_m\cos\theta)]\,\delta(\omega-k_\parallel v_\parallel\pm\Omega_a)f_a. \tag{A15}$$

Making use of the Bessel function integral identity[41]

$$\int_0^{\pi/2} d\theta\,J_{2n}(2b_m\sin\theta) = \int_0^{\pi/2} d\theta\,J_{2n}(2b_m\cos\theta) = \frac{\pi}{2}\,J_n^2(b_m), \tag{A16}$$

we finally obtain

$$\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\,\delta E^{\parallel}_{-k_\parallel,-\omega}\rangle = \frac{ie_a k_m^2}{2\pi k_\parallel\,\epsilon^*_\parallel(k_\parallel,\omega)}\left[J_0^2\left(\frac{k_m v_\perp}{\Omega_a}\right)+J_1^2\left(\frac{k_m v_\perp}{\Omega_a}\right)\right]\delta(\omega-k_\parallel v_\parallel)f_a,$$
$$\langle \delta N^{a0}_{k_\parallel,\omega}(\mathbf{v})\,\delta E^{\mp}_{-k_\parallel,-\omega}\rangle = \frac{ie_a k_m^2 v_\perp}{2\pi\omega\,\Lambda^*_\pm(k_\parallel,\omega)}\,e^{\pm i\varphi}\left[J_0^2\left(\frac{k_m v_\perp}{\Omega_a}\right)+J_1^2\left(\frac{k_m v_\perp}{\Omega_a}\right)\right]\delta(\omega-k_\parallel v_\parallel\pm\Omega_a)f_a. \tag{A17}$$

In the limit of

$$\frac{k_m v_\perp}{\Omega_a} \ll 1, \tag{A18}$$

we recover Eq. (22).

APPENDIX B: DIMENSIONALITY OF THE PRESENT FORMALISM

Equation (23) can be used to discuss the dimensionality in 3D space. The left-hand side has the dimensionality of the square of the spectral electric field amplitudes. Note that the electric field has the dimension

$$E \sim \frac{m^{1/2}}{l^{1/2} t}, \tag{B1}$$

where m is the mass, l is the distance, and t represents time. The one-dimensional Fourier component of the electric field spectral energy density has a dimensionality

$$E^2_{k_\parallel,\omega} \sim \frac{m}{t}. \tag{B2}$$

Consequently, the left-hand side of Eq. (23) has the dimensionality of

$$\text{lhs of Eq. (23)} = \frac{m}{t}. \tag{B3}$$

On the other hand, the right-hand side of Eq. (23) has the dimensionality of

$$\frac{e^2 n}{k v_T} \sim \frac{m\omega_{pe}^2}{k_\parallel v_T}, \tag{B4}$$

where v_T is the thermal speed. Since $k_\parallel v_T$ has the dimension of frequency and thus t^{-1}, it readily follows that the right-hand side also has the same dimensionality of the left-hand side. Note that this consistency in the dimensional analysis was achieved by introducing k_m^2, which shows that without the integration over perpendicular wave numbers one cannot achieve the proper dimensionality.

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