The outlier eigenpair of oriented random graphs

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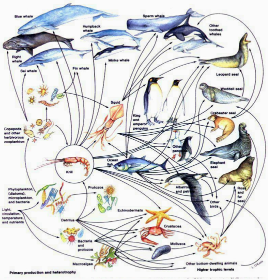
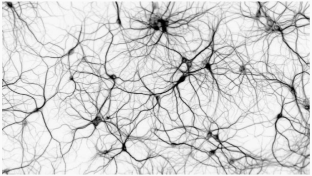
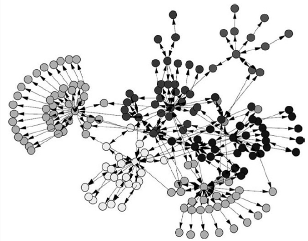
Directed complex networks







Random graph models ⇒ **Adjacency random matrix** ⇒ **Sparse and asymmetric**

Spectrum ⇒ community detection, diffusion, stability, etc

Ensemble of random graphs

$\mathbf{A} \Rightarrow N \times N$ asymmetric random matrix or graph

$\{\lambda_\alpha\}_{\alpha=1,\ldots N} \Rightarrow$ complex eigenvalues

$\{|r_\alpha\rangle, \langle l_\alpha|\}_{\alpha=1,\ldots,N} \Rightarrow$ left/right eigenvectors

- **Main assumptions**:

  $\Rightarrow$ Independent entries

  $\Rightarrow$ Local tree-like structure

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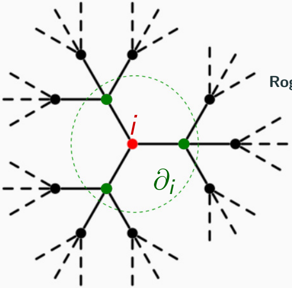


cavity method

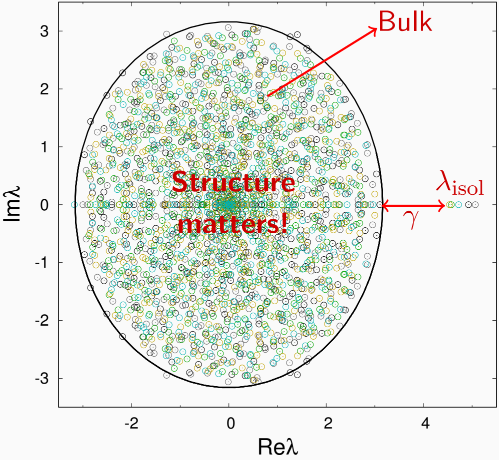
Rogers *et al*, PRE, 2008

Rogers and Castillo, PRE, 2009

FLM *et al*, PRE, 2010

Asymmetric sparse random matrices

Directed Poisson graphs ($N = 500$)



$$\rho(\lambda) = \frac{1}{N} \sum_{\alpha=1}^{N} \delta(\lambda - \lambda_\alpha)$$

Bulk: continuous $\rho(\lambda) > 0$

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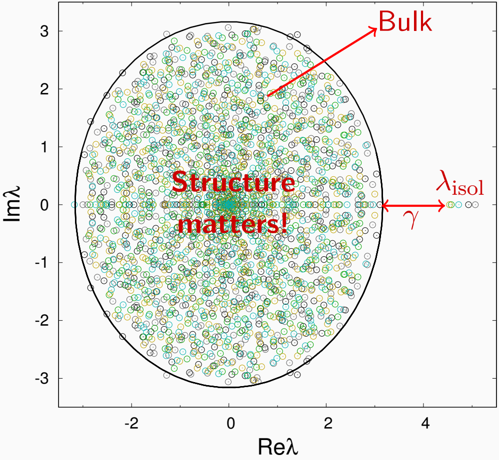

$\rho(\lambda) = \frac{1}{N} \sum_{\alpha=1}^{N} \delta(\lambda - \lambda_\alpha)$

Bulk: continuous $\rho(\lambda) > 0$

Outside the bulk: $\rho(\lambda) = 0$

$\Rightarrow$ **Boundary of continuum**

$\Rightarrow$ **Average outlier λ_{isol}**

$\Rightarrow$ **Spectral gap γ**

The resolvent

Spectral observables $\Rightarrow \mathbf{G}(\lambda) = (\mathbf{A} - \mathbf{I}\lambda)^{-1}, \quad \lambda \in \mathbb{C}$

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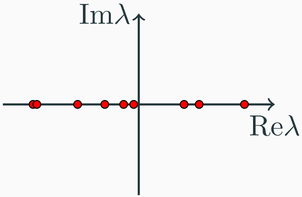

Hermitian matrices

$\mathbf{G}$ is analytic for $\mathrm{Im}\lambda \neq 0$

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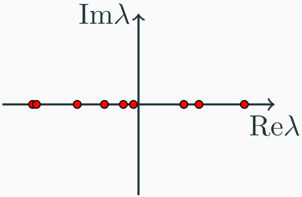

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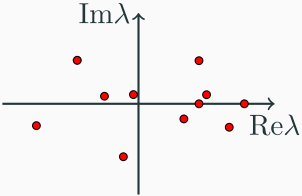

non-Hermitian matrices

$\mathbf{G}$ is singular for $\mathrm{Im}\lambda \neq 0$

Hermitization method

- $2N \times 2N$ block **normal** matrix:

$$\mathbf{B}(\lambda, \eta) = \begin{pmatrix} \eta \mathbf{I} & -i(\mathbf{A} - \lambda \mathbf{I}) \\ -i(\mathbf{A}^\dagger - \lambda^* \mathbf{I}) & \eta \mathbf{I} \end{pmatrix}, \quad \textbf{Regularizer: } \eta > 0$$

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- Left and right eigenvectors of λ_α:

$$\begin{pmatrix} |l_\alpha\rangle \\ |r_\alpha\rangle \end{pmatrix} = \lim_{\eta\to 0^+} \eta \mathbf{B}^{-1}(\lambda,\eta)\boldsymbol{U}|_{\lambda=\lambda_\alpha}$$

$$\boldsymbol{U} = (1\ 1\ \ldots\ 1)^T$$

Local recursive equations - outside the bulk

- $\eta \to 0^+$ is taken analytically

- **Left and right eigenvectors**:

$$r_i = -G_{ii} \sum_{k\in\partial_i} A_{ik} r_k^{(i)} \quad l_i = -G_{ii}^* \sum_{k\in\partial_i} l_k^{(i)} A_{ki}$$

$$r_i^{(\ell)} = -G_{ii}^{(\ell)} \sum_{k\in\partial_i\setminus\ell} A_{ik} r_k^{(i)} \quad l_i^{(\ell)} = -\left(G_{ii}^{(\ell)}\right)^* \sum_{k\in\partial_i\setminus\ell} l_k^{(i)} A_{ki}$$

- **Resolvent diagonal elements**:

$$G_{ii} = \left(-\lambda - \sum_{k\in\partial_i} A_{ik} G_{kk}^{(i)} A_{ki}\right)^{-1} \; G_{ii}^{(\ell)} = \left(-\lambda - \sum_{k\in\partial_i\setminus\ell} A_{ik} G_{kk}^{(i)} A_{ki}\right)^{-1}$$

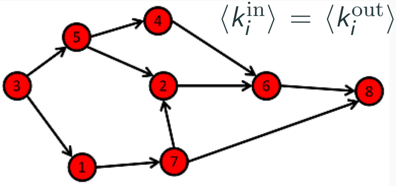
Eigenvector moments of oriented random graphs

No bidirected edges

$$A_{ij}A_{ji} = 0 \; \forall \, (i,j)$$

Nonzero $A_{ij} \to$ distribution $p_A(A)$

$\langle k_i^{\mathrm{in}} \rangle = \langle k_i^{\mathrm{out}} \rangle = c$



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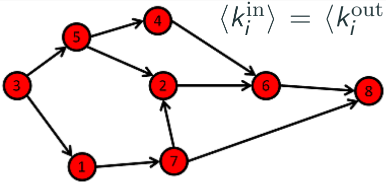

$$\langle k_i^{\mathrm{in}} \rangle = \langle k_i^{\mathrm{out}} \rangle = c$$

- Self-consistent equations for the first two moments:

$$\langle r \rangle = \frac{\langle A \rangle c}{\lambda} \langle r \rangle$$

$$\langle |r|^2 \rangle = \frac{\langle A^2 \rangle c}{|\lambda|^2} \langle |r|^2 \rangle + \frac{\langle A \rangle^2 \langle k^{\mathrm{out}}(k^{\mathrm{out}}-1) \rangle}{|\lambda|^2} |\langle r \rangle|^2$$

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$\langle f(X) \rangle \rightarrow$ average over X $\qquad \lambda \in \mathbb{C} \rightarrow$ arbitrary parameter

Solving the equations

- $\langle r \rangle \neq 0$:

  $\Rightarrow$ **Average outlier** $\rightarrow$ $\boxed{\lambda = c\langle A \rangle}$

  $\Rightarrow$ **Moments** $\rightarrow$ $\boxed{\frac{\langle r \rangle^2}{\langle r^2 \rangle} = \alpha^2 \left[1 - \frac{\langle A^2 \rangle}{c\langle A \rangle^2}\right]}$ $\alpha = \frac{c}{\sqrt{\langle k^{\mathrm{out}}(k^{\mathrm{out}}-1)\rangle}}$

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I. Neri and FLM, PRL, 2016

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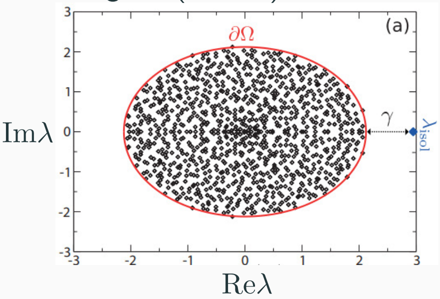
UNIVERSAL FUNCTIONS OF $y = \frac{\langle A^2 \rangle}{c\langle A \rangle^2}$**!**

I. Neri and FLM, PRL, 2016

Universal transition - $N = 1000$

Regular ($c = 3$), Gaussian



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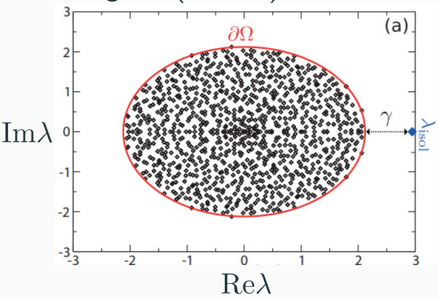

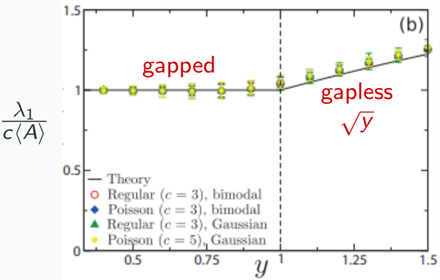

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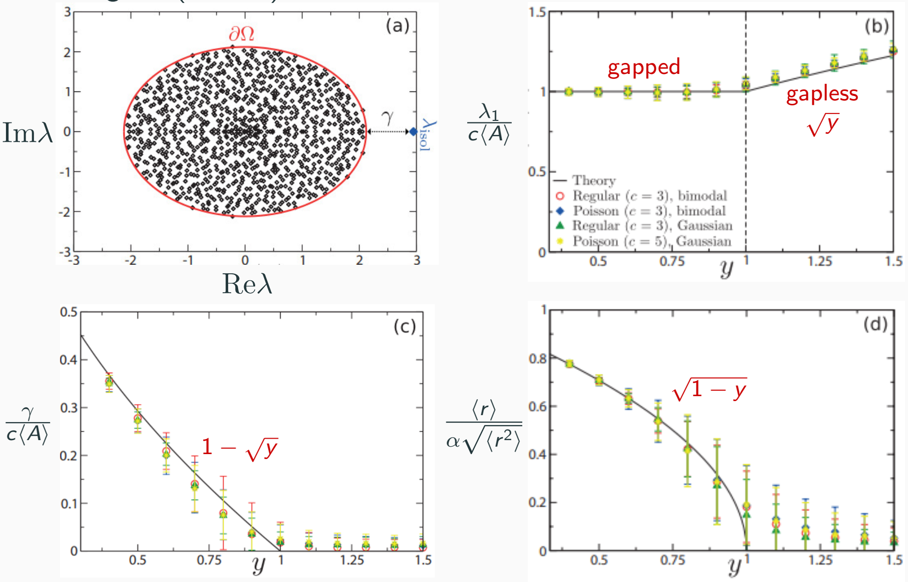

Eigenvector distributions

- Distribution $p_R(r)$ of the right eigenvector for $N \to \infty$:

$$p_R(r) = \sum_{k=0}^{\infty} p_k^{\mathrm{out}} \int \left[\prod_{j=1}^{k} dA_j dr_j p_A(A_j) p_R(r_j)\right] \delta\left(r - \frac{1}{\lambda} \sum_{j=1}^{k} A_j r_j\right)$$

$\Rightarrow$ Choose $\lambda \to$ boundary or outlier

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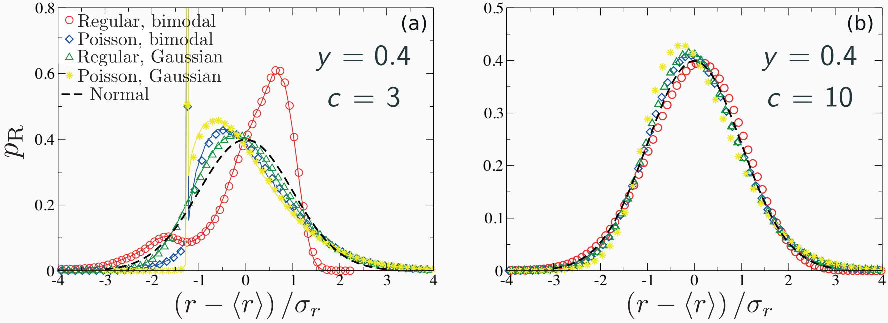


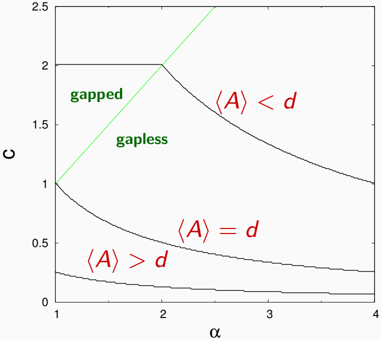
Stability of complex systems

$$\frac{dx_i(t)}{dt} = F_i[\mathbf{x}(t)] \qquad i = 1, \ldots, N$$

$x_i(t) \rightarrow$ population density, neural activity, etc

Linearization around equilibrium $\mathbf{x}(t) = \mathbf{x}_{\mathrm{eq}}$: $A_{ij} \equiv \left.\frac{\partial F_i}{\partial x_j}\right|_{x_{\mathrm{eq}}}$

$A_{ii} = -d \Rightarrow$ **self-regulation**



$$\alpha = \langle A^2 \rangle / \langle A \rangle^2$$

Stability:

Gapless $\Rightarrow c < \frac{d^2}{\alpha \langle A \rangle^2}$

Gapped $\Rightarrow c < \frac{d}{\langle A \rangle}$

Final remarks

- **Universality in sparse random matrices $\rightarrow$ gap/gapless**
- Evidence in empirical data?
- What else...

$\Rightarrow$ Breaking of rotational symmetry of $p_R(r)$

$\Rightarrow$ Universality for higher moments

$\Rightarrow$ Non-universal behaviour of $p_R(r)$ for $c \gg 1$!