

# Soliton solutions of 3D Gross-Pitaevskii equation by a potential control method

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We present a class of three-dimensional solitary waves solutions of the Gross-Pitaevskii (GP) equation, which governs the dynamics of Bose-Einstein condensates (BECs). By imposing an external controlling potential, a desired time-dependent shape of the localized BEC excitation is obtained. The stability of some obtained localized solutions is checked by solving the time-dependent GP equation numerically with analytic solutions as initial conditions. The analytic solutions can be used to design external potentials to control the localized BECs in experiment.

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## I. INTRODUCTION

Solitons are nonlinear localized wave packets sustained by the balance between wave dispersion and medium nonlinearity. Solitons propagate over large distances without changing their shape [1, 2, 3]. Solitary waves or solitons have been observed in several areas of physics including fluids, plasmas, optics, biology, and condensed matters (e.g. Bose-Einstein condensates). Many types of solitons have been studied, starting with classical examples found in integrable models, such as the Korteweg-de Vries, sine-Gordon, Toda-lattice, and nonlinear Schrödinger equations, and their non-integrable extensions. Solitons are robust against collisions due to the integrability of the underlying equations.

Recent observations of matter-wave solitons [4, 5, 6, 7, 8, 9, 10, 11] have been among the most groundbreaking achievements in the burgeoning fields of Bose-Einstein condensation (BEC) of dilute atomic gases. In the latter, bosonic atoms below a certain temperature suddenly develop in the lowest quantum mechanical state. The balance between the spatial dispersion of matter waves and repulsive or attractive atomic interactions in Bose-Einstein condensates (BECs) ensures the existence of dark [4, 5, 6, 7] or bright [8, 9, 10, 11] solitons, respectively. A dark (bright) matter wave soliton is a localized BEC having a minimum (maximum) condensate density at the center. However, if the atomic condensate is embedded into a periodic potential created by standing light waves, i.e. optical lattice [12], there exists possibility of reversing the matter wave group dispersion sign (e.g. from positive group dispersion to negative group dispersion), and of the possible observation of the bright matter-wave soliton in the BECs with repulsive inter-atomic interaction. The concept of the dispersion control by periodic potentials is also well known in solid state physics [13] and a very active topic of research in nonlinear optics [14]. The dynamics and stability of the matter wave solitons in the BECs is governed by the nonlinear Schrödinger equation [1], known as the Gross-Pitaevskii (GP) equation [15, 16] in the context of the BECs. For BECs with positive (repulsive) interparticle interactions, dark solitons with locally depleted density have been studied theoretically [17] and have been observed in many experiments [4, 5, 6, 7, 8, 9, 10, 11]. Numerical and theoretical studies revealed the dynamics and stability the BECs in

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a magneto-trap both for repulsive and attractive interactions in a limited parameter range in spherically symmetric trap [18]. The stabilization and controlling of the BECs in asymmetric traps have been investigated by considering the time-dependent solutions of the GP equation [20]. Stable condensates with a limit number of atoms of  ${}^7\text{Li}$  with an attractive interaction have been observed in a magnetically trapped gas [19]. The formation of matter-wave solitons [4, 5] and trains of solitons [8, 9] have been observed in BECs of  ${}^7\text{Li}$  atoms that are confined in a quasi-one-dimensional trap and magnetically tuned from repulsive to attractive interactions. The solitons are predicted to either collapse or explode, depending on the parameters of the BECs and on the confining or repulsive potential [21]. The collective collapse [22, 23] and explosion [23] of BECs with attractive interactions have been observed in experiments. Several theoretical and experimental studies of coherent matter waves are contained in Ref. [24]. Wang *et al.* [25] have presented the analytical dark and bright solitons of the one-dimensional GP equation with a confining potential. The generation of matter wave dark and bright solitons in a prescribed external potential for confining the BECs have been investigated with a periodically varying nonlinear coefficient [26, 27]. A tight transverse trap with a gradually varying local frequency along the longitudinal direction induces an effective potential for one-dimensional solutions in a self-attracted BEC [28]. The propagation of a dark soliton in a quasi-1D BEC in the presence of a random potential has been studied by Bilas and Pavloff [29]. Furthermore, there is a recent theoretical study of exact one-dimensional solitary wave solutions in a radially confining potential [30]. The dynamics of the one-dimensional bright matter wave soliton in a lattice potential has been studied by Poletti *et al.* [31]. Two-dimensional dark solitons to the nonlinear Schrödinger equation are numerically created by two processes to show their robustness (e.g. stable against the head-on collision). Stellmer *et al.* [32] present experimental data exhibiting the head-on collision of dark solitons generated in an elongated BEC. Experiments do not report discernible interaction among solitons, demonstrating the fundamental theoretical concepts of solitons as quasiparticles.

In this paper, we theoretically study possibility of controlling the time-dependent dynamics of the BECs in three-dimensions with a carefully spatially shaped, time-dependent controlling potential. This idea is theoretically supported by our recent mathematical investigations [33] and on the improvement of the recently-proposed 'controlling potential method' [34]. From the experimental point of view, this idea could be realized by techniques involving lithographically designed circuit patterns that provide electromagnetic guides and microtraps for ultracold systems of atoms in BEC experiments [35], and by optically induced "exotic" potentials [36]. The stability of the obtained analytic solutions are investigated numerically by direct integration of the time-dependent GP equation.

## II. THEORY

The dynamics of BECs in a non-uniform potential is governed by the three-dimensional Gross-Pitaevskii equation [15, 16]

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + gN|\Psi|^2\Psi + V_{\text{ext}}(\mathbf{r}, t)\Psi, \quad (1)$$

where  $m$  is the atomic mass, and  $V_{\text{ext}}$  is the external potential,  $g = 4\pi\hbar^2 a/m$  where  $a$  is the short-range scattering length, which can be either positive or negative, giving rise to either repulsive or attractive interactions, and  $N$  is the number of atoms in the BECs. Typical parameters values where solitons have been observed in BECs of  ${}^7\text{Li}$  atoms are  $N = 10^4\text{--}10^5$  at a temperature of 1–10  $\mu\text{K}$  and a magnetic field  $\sim 400\text{--}600$  G, leading to a small scattering length of  $a \approx -0.2$  nm [4, 9]. In Eq. (1),  $\Psi$  is normalized such that  $\int |\Psi|^2 d^3r = 1$ . Equation (1) can be cast into the dimensionless form

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \nabla^2 \psi + \tilde{g}|\psi|^2\psi + U_{\text{ext}}(\mathbf{r}, t)\psi, \quad (2)$$

where the time  $t$  is normalized by  $t_0 = (4\pi)^2 ma^2 N^2 / \hbar$ , space  $\mathbf{r}$  by  $r_0 = 4\pi|a|N$ , the external potential  $U_{\text{ext}}$  by  $\hbar^2 / (4\pi)^2 ma^2 N^2$ , and  $\psi = r_0^{3/2} \Psi$ , so that  $\int |\psi|^2 d^3r = 1$  in the normalized spatial variables. With this normalization, we have  $\tilde{g} = +1$  for  $a > 0$  and  $\tilde{g} = -1$  for  $a < 0$ . For typical experimental values  $N = 6000$ ,  $|a| = 0.2$  nm and  $m = 1.17 \times 10^{-26}$  kg ( ${}^7\text{Li}$ ) [9], we would have  $r_0 = 16$   $\mu\text{m}$  and  $t_0 = 0.028$  s. Our goal is to design the external potential to obtain a desired time-dependent shape of the solution.

We concentrate on a sub-class of solutions, in which the GP equation can be separated into one linear, two-dimensional equation, and one nonlinear, one-dimensional equation. In doing so, we first make a decomposition of the external potential according to [33]

$$U_{\text{ext}}(\mathbf{r}, t) = U_{\perp}(\mathbf{r}_{\perp}, t) + U_z(\mathbf{r}_{\perp}, z, t), \quad (3)$$

where  $\mathbf{r}_\perp$  is the position vector perpendicular to the  $z$  direction, and make the ansatz that the solution can be separated as

$$\psi(\mathbf{r}, t) = \psi_\perp(\mathbf{r}_\perp, t)\psi_z(z, t), \quad (4)$$

where the normalization conditions  $\int |\psi_\perp|^2 d^2 r_\perp = 1$  and  $\int |\psi_z|^2 dz = 1$  are imposed.

Inserting Eqs. (3) and (4) into Eq. (2), and reordering the terms, we have

$$\begin{aligned} & \psi_z \left[ i \frac{\partial \psi_\perp}{\partial t} + \frac{1}{2} \nabla_\perp^2 \psi_\perp - U_\perp(\mathbf{r}_\perp, t) \psi_\perp \right] \\ &= -\psi_\perp \left[ i \frac{\partial \psi_z}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi_z}{\partial z^2} - \tilde{g} |\psi_\perp|^2 |\psi_z|^2 \psi_z - U_z(\mathbf{r}_\perp, z, t) \psi_z \right], \end{aligned} \quad (5)$$

where we have denoted  $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ . By requiring that  $\psi_\perp$  satisfies the linear Schrödinger equation

$$i \frac{\partial \psi_\perp}{\partial t} + \frac{1}{2} \nabla_\perp^2 \psi_\perp - U_\perp(\mathbf{r}_\perp, t) \psi_\perp = 0, \quad (6)$$

we obtain the one-dimensional GP equation

$$i \frac{\partial \psi_z}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi_z}{\partial z^2} - \tilde{g} |\psi_\perp|^2 |\psi_z|^2 \psi_z - U_z(\mathbf{r}_\perp, z, t) \psi_z = 0. \quad (7)$$

For the linear Eq. (6) for  $\psi_\perp(\mathbf{r}_\perp, t)$ , we assume a parabolic potential well in the form

$$U_\perp = \frac{1}{2} K_x(t) x^2 + \frac{1}{2} K_y(t) y^2, \quad (8)$$

which yields solutions of Eq. (6) in terms of functions of the form  $\psi_{\perp mn}(x, y, t) = \psi_{xm}(x, t) \psi_{yn}(y, t)$  where  $\psi_{\alpha\beta}$  (where  $\alpha = x, y$  and  $\beta = 0, 1, \dots$ ) are Hermite-Gauss functions in the form

$$\psi_{\alpha\beta}(\alpha, t) = \frac{\exp[-\alpha^2/4\sigma_\alpha^2(t)]}{[2\pi\sigma_\alpha^2(t)2^{2\beta}(\beta!)^2]^{1/4}} H_\beta \left[ \frac{\alpha}{\sqrt{2}\sigma_\alpha(t)} \right] \exp \left[ i \frac{\gamma_\alpha(t)\alpha^2}{2} + i\phi_{\alpha\beta}(t) \right], \quad (9)$$

(normalized so that  $\int_{-\infty}^{\infty} |\psi_{\alpha\beta}(\alpha, t)|^2 d\alpha = 1$ ) where  $H_\beta(\xi)$  are Hermite polynomials of order  $\beta$ . The first few Hermite polynomials are listed in Table I in the Appendix A. Here the phases are given in terms of  $\sigma_\alpha$  as

$$\gamma_\alpha(t) = \frac{1}{\sigma_\alpha} \frac{d\sigma_\alpha}{dt}, \quad (10)$$

and  $\phi_{\alpha\beta}(t) = (2\beta + 1)\phi_{\alpha 0}(t)$ , where

$$\frac{d\phi_{\alpha 0}(t)}{dt} = -\frac{1}{4\sigma_\alpha^2(t)}. \quad (11)$$

We have that  $\sigma_\alpha$  is related to  $K_\alpha$  by the Pinney equation [39]

$$\frac{d^2\sigma_\alpha}{dt^2} + K_\alpha(t)\sigma_\alpha - \frac{1}{4\sigma_\alpha^3} = 0. \quad (12)$$

Solving for  $K_\alpha(t)$  in (12) and inserting the result into (8) we have the potential well

$$U_\perp = \frac{1}{2} \left( -\frac{1}{\sigma_x} \frac{d^2\sigma_x}{dt^2} + \frac{1}{4\sigma_x^4} \right) x^2 + \frac{1}{2} \left( -\frac{1}{\sigma_y} \frac{d^2\sigma_y}{dt^2} + \frac{1}{4\sigma_y^4} \right) y^2. \quad (13)$$

in terms of  $\sigma_x(t)$  and  $\sigma_y(t)$ . Hence, we have the possibility to arbitrarily choosing the time-dependent widths  $\sigma_x(t)$  and  $\sigma_y(t)$  (and the indices  $m$  and  $n$ ) of our solution, and obtain the potential  $U_{perp}$  necessary to sustain that solution. Now, the solution of Eq. (7) for  $\psi_z$  must be such that the total  $\psi(\mathbf{r}, t)$  in (4) solves the original GP equation (2). A special solution is

$$\psi_z(z, t) = \left[ -\frac{\tilde{g}\delta_m\delta_n}{4\sigma_x(t)\sigma_y(t)} \right]^{1/2} \operatorname{sech} \left[ -\frac{\tilde{g}\delta_m\delta_n}{2\sigma_x(t)\sigma_y(t)} z \right] \exp \left[ \frac{i}{2} g(t) z^2 + i\theta_0(t) \right], \quad (14)$$

where phase functions are given by

$$g = \frac{1}{\sigma_x(t)\sigma_y(t)} \frac{d[\sigma_x(t)\sigma_y(t)]}{dt}. \quad (15)$$

and

$$\frac{d\theta_0}{dt} = \frac{1}{8} \left[ \frac{\tilde{g}\delta_m\delta_n}{\sigma_x(t)\sigma_y(t)} \right]^2. \quad (16)$$

The external potential  $U_z$  to sustain this solution is given by

$$U_z(\mathbf{r}_\perp, z, t) = -\tilde{g} \left[ |\psi_\perp|^2 - \frac{\delta_m\delta_n}{\sigma_x(t)\sigma_y(t)} \right] |\psi_z|^2 + \frac{1}{2}K(t)z^2, \quad (17)$$

where

$$K(t) = -\frac{1}{\sigma_x(t)\sigma_y(t)} \frac{d^2}{dt^2} [\sigma_x(t)\sigma_y(t)]. \quad (18)$$

The details of the derivation of  $\psi_z$  and  $U_z$  are given in the Appendix A. The perpendicular solutions  $\psi_\perp$  in (9) and parallel solution  $\psi_z$  in (14) can now be used to construct the total, three-dimensional solution  $\psi$  in (4) of the GP equation (2), and where the external controlling potential  $U_{ext}$  in (3) is the sum of  $U_\perp$  in (13) and  $U_z$  in (17).

### III. ANALYTICAL AND NUMERICAL EXAMPLES

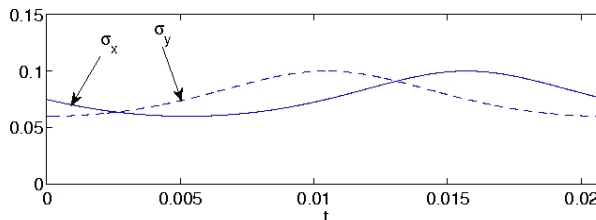


FIG. 1: The soliton widths  $\sigma_x$  (solid line) and  $\sigma_y$  (dashed line) as functions of time, during one oscillation period, for the equilibrium widths  $\sigma_{0x} = \sigma_{0y} = 0.075$ , the oscillation amplitudes  $a_{0x} = a_{0y} = 0.25$ , and the frequency  $\omega = 300$ .

We will here exemplify our analytic results with exact time-dependent solutions of the GP equation. As mentioned above, we are free to choose arbitrary time-dependencies of the widths  $\sigma_x(t)$  and  $\sigma_y(t)$ . In Eq. (9) we also need the Hermite polynomials  $H_m$  and in Eqs. (14), (16) and (17) we need the constants  $\delta_m$ , which are given in Table I in Appendix A, for different excited states  $m$ . As an example, we choose  $\sigma_x(t)$  and  $\sigma_y(t)$  to be periodic in time, of the form

$$\sigma_x(t) = \frac{\sigma_{0x}}{1 + a_{0x} \sin(\omega t)}, \quad (19)$$

and

$$\sigma_y(t) = \frac{\sigma_{0y}}{1 + a_{0y} \cos(\omega t)}, \quad (20)$$

where we investigate cases of relatively large amplitude oscillations with  $a_{0x} = a_{0y} = 0.25$ , and we set  $\sigma_{0x} = \sigma_{0y} = 0.075$  and  $\omega = 300$ . In dimensional units with  $r_0 = 16 \mu\text{m}$  and  $t_0 = 0.028 \text{s}$ , this corresponds to a typical soliton width of  $\sim 0.075 \times 16 \mu\text{m} \approx 1.5 \mu\text{m}$  and the frequency  $300/(2\pi \times 0.028) \text{Hz} = 1.7 \text{kHz}$ . The choice of the functions (19) and (20) describes periodically in time pulsating solutions widths, illustrated in Fig. 1. Small values of  $\sigma_x$  and  $\sigma_y$  correspond to spatially localized solitary waves, while large values correspond to wider solitary waves. The amplitude of the solitary wave also varies so as to keep the total number of condensates constant,  $\int |\psi|^2 d^3r = 1$ . In Figs. 2-4, we have plotted the solutions together with the corresponding controlling potentials in the  $xy$ -plane at  $z = 0$ , for

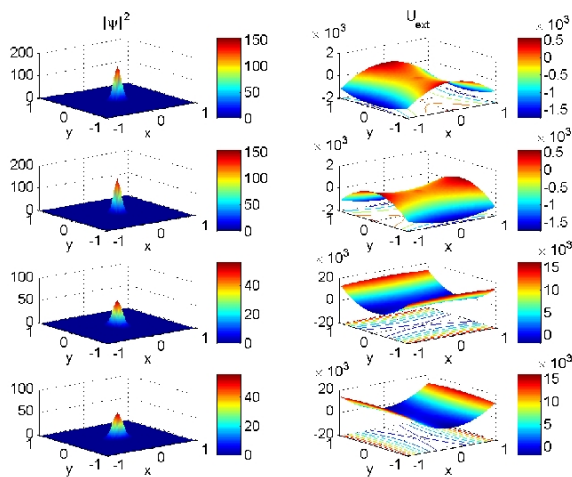


FIG. 2: The condensate density  $|\psi|^2$  (left) and the external potential  $U_{ext}$  (right) in the  $xy$  plane at  $z = 0$ , at times  $t = 0$ ,  $t = 0.25 \times 2\pi/\omega$ ,  $t = 0.5 \times 2\pi/\omega$ , and  $t = 0.75 \times 2\pi/\omega$  (top to bottom rows), for the ground state  $(m, n) = (0, 0)$ .

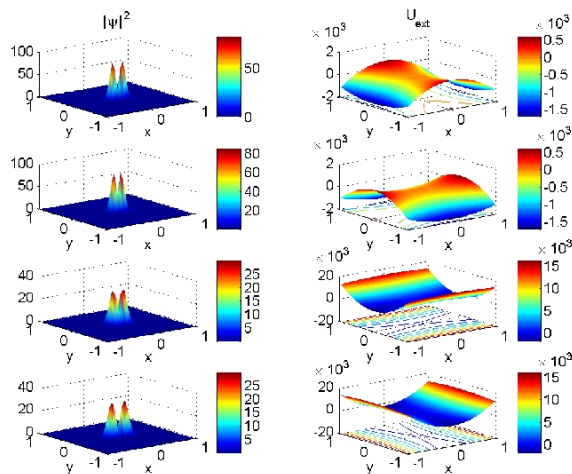


FIG. 3: The condensate density  $|\psi|^2$  (left) and the external potential  $U_{ext}$  (right) in the  $xy$  plane at  $z = 0$ , at times  $t = 0$ ,  $t = 0.25 \times 2\pi/\omega$ ,  $t = 0.5 \times 2\pi/\omega$ , and  $t = 0.75 \times 2\pi/\omega$  (top to bottom rows), for the excited state  $(m, n) = (1, 0)$ .

the ground state  $(m, n) = (0, 0)$  and the excited states  $(m, n) = (1, 0)$  and  $(m, n) = (1, 1)$ , where  $(m, n)$  refers to the orders of the Hermite polynomials in the perpendicular solution  $\psi_{\perp mn}(x, y, t)$ . The widths are varying such that at time  $t = 0$ , the solitons are localized and large amplitude while at later times their amplitudes decrease and their widths increase, first in the  $y$  direction and then in the  $x$  direction. We note that the external potential is sometimes large amplitude and confining, and sometimes small amplitude and non-confining.

For the analytic results to be observable in experiments, it is necessary that they are stable. To assess the stability of the solutions, we have therefore solved the time-dependent GP equation (2) in three dimensions with the analytic solution as initial condition at time  $t = 0$ . We used a box length  $L_x = L_y = 2$  in the  $x$  and  $y$  dimensions and  $L_z = 3$  in the  $z$  direction, with periodic boundary conditions. The spatial derivatives were approximated with a pseudospectral method, and the time stepping was performed with the standard 4th-order Runge-Kutta method. The space was resolved with 64 grid points in the  $x$  and  $y$  directions and with 200 grid points in the  $z$  direction, and the timestep was taken to be  $\Delta t = \pi/300\,000$ .

In order to seed any instability in the system, we added random perturbations in phase to the initial condition of the order 0.01 rad. We then simulated the system for 5 oscillation periods and measured the maximum density  $|\psi|_{num,max}^2$  obtained in the numerical solution as well as the maximum relative error in density fluctuations,  $\varepsilon = |(|\psi|^2 - |\psi|_{num}^2)|_{max}/|\psi|_{max}^2$ , as a function of time, and plotted the results in Fig. 5, for the ground state  $(m, n) = (0, 0)$  and the excited states  $(m, n) = (1, 0)$  and  $(m, n) = (1, 1)$ . We see that the numerical solution follows almost exactly

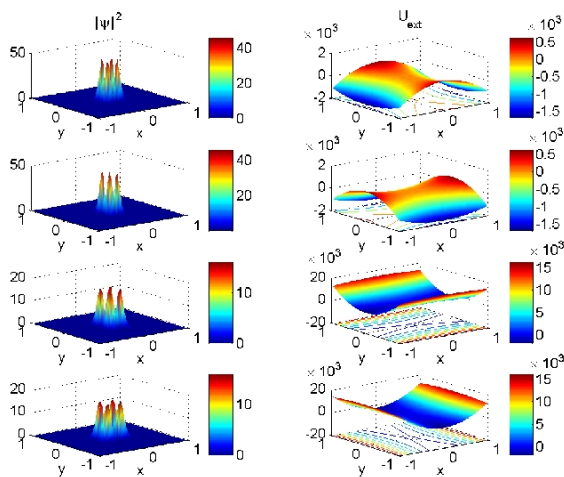


FIG. 4: The condensate density  $|\psi|^2$  (left) and the external potential  $U_{ext}$  (right) in the  $xy$  plane at  $z = 0$ , at times  $t = 0$ ,  $t = 0.25 \times 2\pi/\omega$ ,  $t = 0.5 \times 2\pi/\omega$ , and  $t = 0.75 \times 2\pi/\omega$  (top to bottom rows), for the excited state  $(m, n) = (1, 1)$ .

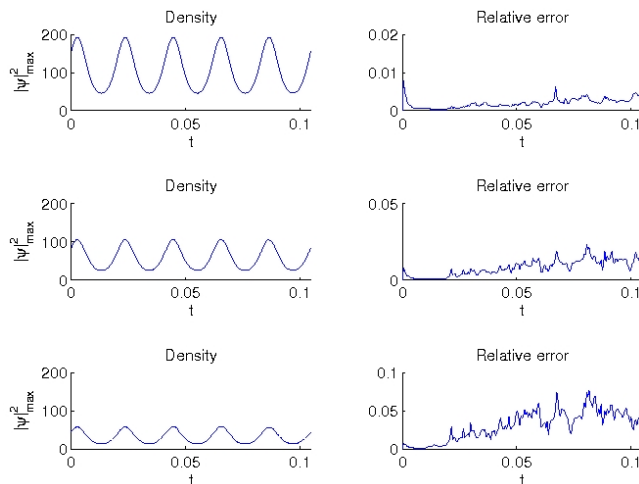


FIG. 5: Numerical simulation results of the GP equation for the amplitude  $a_{0x} = a_{0y} = 0.25$  and  $(m, n) = (0, 0)$ ,  $(1, 0)$  and  $(1, 1)$  (top to bottom panels), showing the maximum density (left column) and the relative deviation of the numerical solution from the exact analytic solution (right column).

the analytic solution, without increasing substantially throughout the simulation. The ground state  $(m, n) = (0, 0)$  shows a relative error less than 1 % throughout the simulation while the excited state  $(m, n) = (1, 1)$  shows a somewhat larger error but still less than 10 %. Hence the controlled BEC seems to be stable enough to be observed in experiment.

#### IV. SUMMARY AND CONCLUSIONS

In summary, we have presented a class of three-dimensional solitary waves solutions of the Gross-Pitaevskii (GP) equation, which governs the dynamics of Bose-Einstein condensates. This has been done on the basis of our recent mathematical investigations [33] and improving the formulation of the recently-proposed controlling potential method [34]. By imposing an external controlling potential, a desired time-dependent shape of the localized BECs is obtained. The stability of the exact solutions were checked with direct simulations of the time-dependent, three-dimensional GP equation. Our simulations show that the localized condensates are stable with respect to perturbed initial conditions. We propose that our findings could be tested experimentally by techniques involving lithographically designed circuit patterns that provide electromagnetic guides and microtraps for ultracold systems of atoms in BEC experiments [35], and by optically induced “exotic” potentials [36]. Furthermore, numerical simulations [40] reveal that soliton emission

from a BEC can be controlled by a shallow optical dipole trap. Here, the emission of matter wave bursts is triggered by spatial variation of the scattering length along the trapping axis. The motion of the 1D dark soliton can also be controlled by means of periodic potentials in optical lattices [41]. Finally, it should be noted that repulsive BECs confined by an optical lattice and a parabolic magnetic trap can appear in the form of vortices [42] as well. Computer simulations [43] reveal the condensation of a finite temperature Bose gas in the form of a single vortex. Interactions of solitary waves and vortex rings in a cylindrically controlled BECs exhibit robustness during head-on collisions [44].

### APPENDIX A: DETAILS IN THE DERIVATION OF THE PARALLEL SOLUTION $\psi_z$ AND THE POTENTIAL $V_z$

We here present a special solution of Eq. (7) for  $\psi_z$ , such that the total  $\psi(\mathbf{r}, t) = \psi_\perp(\mathbf{r}_\perp, t)\psi_z(z, t)$  solves the original GP equation (2). Multiplying Eq. (7) by  $|\psi_\perp|^2$  and integrating over  $\mathbf{r}_\perp$  space, we obtain

$$i\frac{\partial\psi_z}{\partial t} + \frac{1}{2}\frac{\partial^2\psi_z}{\partial z^2} - q_0(t)|\psi_z|^2\psi_z - V(z, t)\psi_z = 0, \quad (\text{A1})$$

where  $V(z, t) = \int U_z(\mathbf{r}_\perp, z, t)|\psi_\perp|^2 d^2r_\perp$  and

$$q_0(t) = \tilde{g} \int |\psi_\perp|^4 dx dy = \frac{\tilde{g}\delta_m\delta_n}{\sigma_x(t)\sigma_y(t)}, \quad (\text{A2})$$

where the numerical factor

$$\delta_m = \frac{1}{\sqrt{2\pi}2^{2m}(m!)^2} \int_{-\infty}^{\infty} \exp(-2\xi^2)|H_m(\xi)|^4 d\xi \quad (\text{A3})$$

is evaluated for given values of  $m$ . The first few Hermite polynomials and values of  $\delta_m$  are listed in Table I.

$m$	$H_m(\xi)$	$\delta_m$
0	1	$1/(2\sqrt{\pi})$
1	$2\xi$	$3/(8\sqrt{\pi})$
2	$4\xi^2 - 2$	$41/(128\sqrt{\pi})$

TABLE I: The Hermite polynomial  $H_m(\xi)$  and the value of  $\delta_m$  for  $m = 0, 1$ , and  $2$ .

Equations (7) and (A1) should give the same solution for  $\psi_z$ , and this imposes restrictions on the external potential. Subtracting Eq. (7) from Eq. (A1) we obtain the compatibility relation

$$U_z(\mathbf{r}_\perp, z, t) = -\tilde{g}|\psi_\perp|^2|\psi_z|^2 + q_0(t)|\psi_z|^2 + V. \quad (\text{A4})$$

We will consider the special case where the external potential for the one-dimensional GP equation (A1) has the form  $V(z, t) = (1/2)K(t)z^2$  so that

$$i\frac{\partial\psi_z}{\partial t} + \frac{1}{2}\frac{\partial^2\psi_z}{\partial z^2} - q_0(t)|\psi_z|^2\psi_z - \frac{1}{2}K(t)z^2\psi_z = 0. \quad (\text{A5})$$

Using the Madelung fluid ansatz [37]  $\psi_z = \sqrt{\rho(z, t)}\exp[i\theta(z, t)]$ , we obtain after some manipulations [cf. Eq. (11) of Ref. [38]]

$$-\rho\frac{\partial v}{\partial t} + v\frac{\partial\rho}{\partial t} + 2\left[c_0(t) - \int^z \frac{\partial v}{\partial t} dz\right]\frac{\partial\rho}{\partial z} - \left(\rho\frac{\partial U}{\partial z} + 2U\frac{\partial\rho}{\partial z}\right) + \frac{1}{4}\frac{\partial^3\rho}{\partial z^3} = 0, \quad (\text{A6})$$

and

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial z}(\rho v) = 0, \quad (\text{A7})$$

where

$$v = \frac{\partial\theta}{\partial z}. \quad (\text{A8})$$

In Eq. (A6),  $c_0(t)$  is an arbitrary function of  $t$  to be determined below and we have denoted  $U = q_0(t)|\psi_z|^2 + V(z, t) = q_0(t)\rho + V(z, t)$ . In order to find solitary wave solutions of Eqs. (A6) and (A7), we now assume that

$$v(z, t) = g(t)z, \quad (\text{A9})$$

where  $g(t)$  (not to be confused with the coupling parameter of the GP equation) is a function to be determined, and we have from Eq. (A8) that

$$\theta(z, t) = \frac{1}{2}g(t)z^2 + \theta_0(t), \quad (\text{A10})$$

where  $\theta_0(t)$  is an arbitrary function of  $t$ . Even if a time-dependent phase is included in the transverse solutions  $\psi_\perp$  below, it is necessary to keep  $\theta_0$ , since the parallel solutions are required to satisfy a separate equation. Using Eq. (A9), together with the relations  $\partial v/\partial t = g'(t)z$  and  $\int(\partial v/\partial t)dz = (1/2)g'(t)z^2$  (where the primes denote derivatives) in Eqs. (A6) and (A7), we have

$$\begin{aligned} & -g'(t)z\rho + g(t)z\frac{\partial\rho}{\partial t} + 2\left[c_0(t) - \frac{1}{2}g'(t)z^2\right]\frac{\partial\rho}{\partial z} \\ & - 3q_0\rho\frac{\partial\rho}{\partial z} - \left(Kz\rho + Kz^2\frac{\partial\rho}{\partial z}\right) + \frac{1}{4}\frac{\partial^3\rho}{\partial z^3} = 0 \end{aligned} \quad (\text{A11})$$

and

$$\frac{\partial\rho}{\partial t} + g(t)z\frac{\partial\rho}{\partial z} + g(t)\rho = 0, \quad (\text{A12})$$

respectively. Multiplying Eq. (A12) by  $g(t)z$  and subtracting the result from Eq. (A11), and reordering the terms, we have

$$-z[g'(t) + g^2(t) + K(t)]\left(z\frac{\partial\rho}{\partial z} + \rho\right) + 2c_0(t)\frac{\partial\rho}{\partial z} - 3q_0\rho\frac{\partial\rho}{\partial z} + \frac{1}{4}\frac{\partial^3\rho}{\partial z^3} = 0. \quad (\text{A13})$$

If  $g(t)$  obeys the Riccati equation

$$g'(t) + g^2(t) + K(t) = 0, \quad (\text{A14})$$

then Eq. (A13) simplifies to

$$2c_0(t)\frac{\partial\rho}{\partial z} - 3q_0\rho\frac{\partial\rho}{\partial z} + \frac{1}{4}\frac{\partial^3\rho}{\partial z^3} = 0. \quad (\text{A15})$$

We now introduce the change of variables  $\xi = G(t)z + R(t)$  and  $\tau = t$ , or

$$z = [\xi - R(\tau)]/G(\tau) \quad (\text{A16})$$

and

$$t = \tau, \quad (\text{A17})$$

which can be introduced into Eqs. (A13) and (A12) to obtain

$$2c_0G\frac{\partial\rho}{\partial\xi} - 3q_0G\rho\frac{\partial\rho}{\partial\xi} + \frac{1}{4}G^3\frac{\partial^3\rho}{\partial\xi^3} = 0, \quad (\text{A18})$$

and

$$(\xi - R)\left[\frac{G'(\tau)}{G} + g(\tau)\right]\frac{\partial\rho}{\partial\xi} + R'(\tau)\frac{\partial\rho}{\partial\xi} + g\rho + \frac{\partial\rho}{\partial\tau} = 0, \quad (\text{A19})$$

respectively. Choosing

$$G'(\tau)/G(\tau) + g(\tau) = 0, \quad (\text{A20})$$



and  $R = \text{constant}$  ( $R = 0$  without loss of generality), Eq. (A19) becomes

$$\frac{\partial \rho}{\partial \tau} + g\rho = 0. \quad (\text{A21})$$

We now look for a solution in the form  $\rho(\xi, \tau) = A(\tau)F(\xi)$ . With this ansatz, Eqs. (A18) and (A21) can be written as

$$2c_0(\tau)F'(\xi) - 3q_0(\tau)A(\tau)F(\tau)F'(\xi) + \frac{1}{4}G^2(\tau)F'''(\xi) = 0, \quad (\text{A22})$$

and

$$A'(\tau) + g(\tau)A(\tau) = 0, \quad (\text{A23})$$

respectively. For consistency, Eq. (A22) for  $F(\tau)$  should be reduced to an equation with the coefficients independent of  $\tau$ , i.e.  $c_0(\tau)$  and  $q_0(\tau)A(\tau)$  must be proportional to  $G^2(\tau)$ . Equations (A20) and (A23) also imply that  $A$  is proportional to  $G$ . Integrating Eq. (A20) as

$$G(\tau) = G_0 \exp\left(-\int_0^\tau g(s) ds\right), \quad (\text{A24})$$

where we without loss of generality can choose  $G_0 = 1$ , we then have

$$A(\tau) = A_0 G(\tau), \quad (\text{A25})$$

$$c_0(\tau) = C_0 G^2(\tau), \quad (\text{A26})$$

and

$$q_0(\tau) = Q_0 G(\tau), \quad (\text{A27})$$

so that Eq. (A22) takes the form (after eliminating the common exponential factor)

$$2C_0 F'(\xi) - 3A_0 Q_0 F'(\xi)F(\xi) + \frac{1}{4}F'''(\xi) = 0. \quad (\text{A28})$$

Equation (A28), which is the time-independent Korteweg-de Vries equation, admits solitary wave solutions for  $C_0 < 0$  and  $A_0 Q_0 < 0$  in the form

$$F(\xi) = \frac{2C_0}{A_0 Q_0} \text{sech}^2\left(\frac{\xi}{\Delta}\right), \quad (\text{A29})$$

where  $\Delta = 1/\sqrt{2|C_0|}$ . It follows that

$$\rho(z, t) = \frac{1}{2\Delta} G(t) \text{sech}^2\left[\frac{G(t)z}{\Delta}\right]. \quad (\text{A30})$$

with  $A_0 \neq 0$ , and where we have chosen  $Q_0 = -2/\Delta$  so that  $q_0(t) = -(2/\Delta)G$  to ensure that  $\int_{-\infty}^{\infty} |\psi_z|^2 dz = \int_{-\infty}^{\infty} \rho dz = 1$ . From (A2) we then have

$$-\frac{2}{\Delta}G = \frac{\tilde{g}\delta_m\delta_n}{\sigma_x(t)\sigma_y(t)}. \quad (\text{A31})$$

Since  $G = 1$  at  $t = 0$ , we find the width

$$\Delta = -\frac{2\sigma_x(0)\sigma_y(0)}{\tilde{g}\delta_m\delta_n}, \quad (\text{A32})$$

so that

$$G = \frac{\sigma_x(0)\sigma_y(0)}{\sigma_x(t)\sigma_y(t)}, \quad (\text{A33})$$

and

$$g = \frac{1}{\sigma_x(t)\sigma_y(t)} \frac{d[\sigma_x(t)\sigma_y(t)]}{dt}. \quad (\text{A34})$$

Finally we have

$$\psi_z(z, t) = \left[ -\frac{\tilde{g}\delta_m\delta_n}{4\sigma_x(t)\sigma_y(t)} \right]^{1/2} \text{sech} \left[ -\frac{\tilde{g}\delta_m\delta_n}{2\sigma_x(t)\sigma_y(t)} z \right] \exp \left[ \frac{i}{2}g(t)z^2 + i\theta_0(t) \right], \quad (\text{A35})$$

where  $g$  is given by (A34). To obtain an expression for  $\theta_0(t)$  in terms of  $\sigma_x$  and  $\sigma_y$ , we use Eq. (A35) to calculate  $U(z, t)$ ,  $\rho(z, t)$  and  $v(z, t)$ , substitute the result into Eq. (A6), taking into account that  $g = -\dot{G}/G$ . This gives  $d\theta_0/dt = -c_0$ , which together with (A26), (A32) and (A33) yields the result

$$\frac{d\theta_0}{dt} = \frac{1}{8} \left[ \frac{\tilde{g}\delta_m\delta_n}{\sigma_x(t)\sigma_y(t)} \right]^2. \quad (\text{A36})$$

Using (A2) into (A4), we obtain the external potential

$$U_z(\mathbf{r}_\perp, z, t) = -\tilde{g} \left[ |\psi_\perp|^2 - \frac{\delta_m\delta_n}{\sigma_x(t)\sigma_y(t)} \right] |\psi_z|^2 + \frac{1}{2}K(t)z^2, \quad (\text{A37})$$

where  $K$  is found from the Riccati equation (A14) and Eq. (A34) as

$$K(t) = -\frac{1}{\sigma_x(t)\sigma_y(t)} \frac{d^2}{dt^2} [\sigma_x(t)\sigma_y(t)]. \quad (\text{A38})$$

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- [1] C. Sulem and P. L. Sulem, *The Nonlinear Schrödinger Equation* (Springer, Berlin, 1999).
  - [2] G. B. Whitham, *Linear and Nonlinear Waves* (John Wiley & Sons, New York, 1999)
  - [3] T. Dauxois and M. Peyrard, *Physics of Solitons* (Cambridge University Press, Cambridge, 2006).
  - [4] S. Burger *et al.*, Phys. Rev. Lett. **83**, 5198 (1999).
  - [5] J. Denschlag *et al.*, Science **287**, 97 (2000).
  - [6] B. P. Anderson *et al.*, Phys. Rev. Lett. **86**, 2926 (2001).
  - [7] A. Weller *et al.*, Phys. Rev. Lett. **101**, 130401 (2008).
  - [8] K. E. Strecker *et al.*, Nature (London) **417**, 150 (2002); New J. Phys. **5**, 73 (2003).
  - [9] L. Khaykovich *et al.*, Science **296** 1290 (2002).
  - [10] B. Eiermann *et al.*, Phys. Rev. Lett. **92**, 230401 (2004).
  - [11] G. Herring *et al.*, Phys. Lett. A **345**, 144 (2005).
  - [12] B. P. Anderson and M. A. Kasevich, Science **282**, 1686 (1998); F. S. Cataliotti *et al.*, Science **419**, 51 (2002); M. Greiner *et al.*, Nature (London) **419**, 51 (2002); B. Eiermann *et al.*, Phys. Rev. Lett. **91**, 060402 (2003).
  - [13] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders College, New York, 1976).
  - [14] *Nonlinear Photonic Crystals*, edited by R. E. Slusher and B. J. Eggleton (Springer, Berlin, 2003).
  - [15] E. P. Gross, Nuovo Cimento **20**, 454 (1961); L. P. Pitaevsky, Sov. Phys. JETP **13**, 451 (1961).
  - [16] F. Dalfovo *et al.*, Rev. Mod. Phys. **71** **463** (1999).
  - [17] A. D. Jackson, G. M. Kavoulakis, and C. J. Pethick, Phys. Rev. A **58**, 2417 (1998).
  - [18] P. A. Ruprecht, M. J. Holland, K. Burnett, and M. Edwards, Phys. Rev. A **51**, 4704 (1995).
  - [19] C. C. Bradley, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. **78**, 985 (1997).
  - [20] V. M. Pérez-García, H. Michinel, and H. Herrero, Phys. Rev. A **57**, 3837 (1998); A. E. Muryshev, H. B. van Linden van den Heuvell, and G. V. Shlyapnikov, Phys. Rev. A **60**, R2665 (1999).
  - [21] L. D. Carr and Y. Castin, Phys. Rev. A **66**, 063602 (2002).

- [22] C. A. Sackett, J. M. Gerton, M. Welling, and R. G. Hulet, *Phys. Rev. Lett.* **82**, 876 (1999); J. M. Gerton, D. Strekalov, I. Prodan, and R. Hulet, *Nature (London)* **408**, 692 (2000).
- [23] E. A. Donley, N. R. Claussen, S. L. Cornish, J. L. Roberts, E. A. Cornell, and C. E. Wiemann, *Nature (London)* **412**, 295 (2001).
- [24] See the Special issue on coherent matter waves in *J. Phys. B* **33**, Number 19 (2000).
- [25] S. J. Wang *et al.*, *Phys. Rev. A* **68**, 015601 (2003).
- [26] P. G. Kevrekidis *et al.*, *Phys. Rev. Lett.* **90**, 230401 (2003); D. E. Pelinovsky *et al.*, *ibid.* **91**, 240201 (2003).
- [27] H. Saito and M. Ueda, *Phys. Rev. Lett.* **90**, 040403 (2003).
- [28] S. De Nicola, B. A. Malomed, and R. Fedele, *Phys. Lett. A* **360**, 164 (2006).
- [29] N. Bilas and N. Pavloff, *Phys. Rev. Lett.* **95**, 130403 (2005).
- [30] R. Atre, P. K. Panigrahi, and G. S. Agarwal, *Phys. Rev. E* **73**, 056611 (2006).
- [31] D. Poletti *et al.*, *Phys. Rev. Lett.* **101**, 150403 (2008).
- [32] S. Stellmer *et al.*, *Phys. Rev. Lett.* **101**, 120406 (2008).
- [33] R. Fedele, D. Jovanovic, S. De Nicola, B. Eliasson and P.K. Shukla, arXiv:0907.1240v1 [nlin.SI], submitted to *J. Phys. A: Math. and Theor.* (2009)
- [34] R.Fedele, P.K. Shukla, S. De Nicola, M.A. Man'ko, V.I. Man'ko, and F.S. Cataliotti, *JEPT Lett* **80**, 535 (2004); *Physica Scripta*, **T116**, 10, (2005); S. De Nicola, R.Fedele, D. Jovanovic, B. Malomed, M.A. Man'ko, V.I. Man'ko and P.K. Shukla, *Eur. Phys. J. B* **54**, 113 (2006)
- [35] J. Fortagh, A. Grossman, C. Zimmermann, and T. W. Hänsch, *Phys. Rev. Lett.* **81**, 5310 (1998); J. Reichel, *Appl. Phys. B: Lasers Opt.* **74**, 469 (2002); R. Folman, P. Krüger, J. Schmiedmayer, J. Denschlag, and C. Henkel, *Adv. At. Mol. Opt. Phys.* **48**, 263 (2002).
- [36] R. Grimm, M. Weidenmüller, and Y. Ovchinnikov, *Adv. At. Mol. Opt. Phys.* **42**, 95 (2000).
- [37] R. Fedele, *Physica Scripta* **65**, 502 (2002).
- [38] R. Fedele and H. Schamel, *Eur. Phys. J. B* **27**, 313 (2002).
- [39] E. Pinney, *Proc. Am. Math. Soc.* **1**, 681 (1950).
- [40] M. I. Rodas-Verde *et al.*, *Phys. Rev. Lett.* **95**, 153903 (2005).
- [41] G. Theocharis *et al.*, *Phys. Rev. E* **71**, 017602 (2005).
- [42] P. G. Kevrekidis *et al.*, *J. Phys. B.: Mol. Opt. Phys.* **36**, 3467 (2003); *Mod. Phys. Lett. B* **18**, 1481 (2004).
- [43] C. N. Weiler *et al.*, *Nature (London)* **455**, 948 (2008).
- [44] S. Komineas and J. Brand, *Phys. Rev. Lett.* **95**, 110401 (2005).