Quantum Weibel instability

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Abstract

The Weibel instability is analyzed for quantum plasmas described by the Wigner-Maxwell model. For a suitable class of electromagnetic potentials, the Wigner-Maxwell system is linearized yielding a general dispersion relation for transverse electromagnetic waves. For a double Gaussian equilibrium with temperature anisotropy, the derived dispersion relation generalizes the classical Weibel instability equation. More detailed analytical results are obtained for the cases of extreme temperature anisotropy and for a three-dimensional water bag distribution. In all cases, quantum effects tends to weaken or suppress the instability. Applications are discussed for dense astrophysical objects like white dwarfs and neutron stars as well as for tunnel-ionized plasmas with controllable perpendicular plasma temperature.

1 Introduction

Quantum plasmas have attracted renewed attention in recent years due to the ongoing miniaturization of ultra small electronic devices and micro mechanical systems [1], to the relevance of quantum effects for dense laser-plasmas and micro plasmas [2] and for dense astrophysical objects [3]. Quantum phenomena are relevant for these systems for a variety of reasons, the most usual

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being the de Broglie wavelength of the charge carriers (electrons, positrons, holes and so on) becoming comparable to the characteristic dimensions of the system. Quantum ion-acoustic waves [4], a quantum magnetohydrodynamics model [5], shear Alfvén modes in ultra-cold quantum magneto plasmas [6], quantum corrections for the Zakharov system [7]-[9] and nonlinear solutions for quantum magneto plasmas [10] have been constructed. New quantum modes have also been identified for ultra-cold dusty plasmas [11]–[16], where quantum effects can be used for plasma diagnostics. The most recent developments take care of spin effects in non relativistic quantum plasmas [17] as well as the associated magnetohydrodynamics equations [18], with possible important applications for solid state plasmas as well as in the vicinity of pulsars and magnetars. In addition, there are the analysis of the nonlinear instability of polaritons [19], of the dynamics of dark solitons and vortices in quantum electron plasma [20], of nonlinear interactions between intense circularly polarized electromagnetic waves and electron plasma oscillations [21], of the behavior of quantum diodes [22], of nonlinear quantum dust-acoustic waves [23], of quantum ion-acoustic double layers [24], of linear and nonlinear ion-acoustic waves in unmagnetized electron-positron-ion quantum plasmas [25], the construction of classes of solutions for the quantum Zakharov-Kuznetsov equation [26], of electron-acoustic solitary waves in dense quantum electron-ion plasmas [27] and of linear and nonlinear dust ion acoustic waves in ultra cold quantum dusty plasmas [28]. Finally, there are new experimental studies [29] of weakly degenerate quantum plasmas in a gaseous regime (i.e. non solid state plasmas). A recent review on quantum plasma models and their range of validity can be found in [30].

In some systems [31], the ultimate influence of quantum mechanics in plasmas is the stabilization of some classically unstable mode, for sufficiently strong quantum effects. However, in the intermediate regime where quantum effects are not too intense but are nevertheless not negligible, there are situations where unexpected quantum instabilities can arise. Examples on this are the quantum two-stream and three-stream instabilities [32]–[35], showing unstable modes of pure quantum nature and no classical counterpart. In addition, unlike classical plasmas, there is no Penrose functional determining the linear stability properties of quantum plasmas [36]. These considerations points to the subtle rôle played in plasmas by quantum diffraction effects like tunneling and wave-packet spreading. Therefore, it is a relevant subject, to work out well known classical instabilities now in the context of quantum plasma models. In this perspective, the present work considers Weibel's instability [37]. Weibel instability arises from temperature anisotropy in the equilibrium distribution function and is one of the fundamental instabilities of plasma physics. In the more recent years, it has been the central concept in several instances, like in fast ignitor scenarios [38], for particle acceleration and magnetic field generation in astrophysical settings [39, 40, 41], for collective non-Abelian Weibel instabilities in melting color glass condensates [42], in covariant relativistic scenarios [43, 44], in electron-positron relativistic shocks [45], with kappa and generalized (r, q) distributions [46] and in laser heated plasmas [47].

The aim of this contribution is to get detailed information about the influence of quantum effects on Weibel's instability. For this purpose, it is used the (kinetic) Wigner-Maxwell model, which is the quantum counterpart of the Vlasov-Maxwell system. In this context, the Wigner function plays the same rôle as the classical distribution function. Due to its mathematical complexity, frequently the electromagnetic Wigner equation [5] has not been taken as the basic tool in electromagnetic quantum plasmas. Rather, most works relies on the quantum hydrodynamic model [5, 33]. Here, however, the kinetic description is used in order to provide a easier comparison with previous results on Weibel's instability, which were constructed in terms of the Vlasov-Maxwell system. Nevertheless, notice that there are instances [48] where fluid models were also applied to Weibel's instability. Since the treatment is restricted to small amplitude waves, the electromagnetic Wigner equation can still provide meaningful results without too complicated analytical difficulties.

Sometimes, the Weibel instability is treated in conjunction with counterstreaming beams and/or ambient magnetic fields. However, as pointed out in [49], in these cases it is more appropriated to talk about filamentation instability. Indeed, the Weibel instability is prompted only by a single anisotropic system. So, here it is followed the original approach by Weibel [37], focusing only on the consequences of temperature anisotropy, but now allowing also for quantum effects. In addition, the analysis is restricted to non-relativistic systems. Notice that recently quantum effects were addressed for the filamentation (but not Weibel) instability [50]. In this case, quantum effects have been shown to reduce both the unstable wave-vector domain and the maximum growth rate.

This paper is organized as follows. In Section II, we write the dispersion relation for small amplitude transverse electromagnetic waves in quantum plasmas, as derived from the Wigner-Maxwell system. Then it is assumed a double Gaussian equilibrium distribution function where temperature anisotropy is allowed. The resulting dispersion relation is analyzed for several limiting cases in Section III. Namely, there are considered the ultra-quantum case, the semiclassic case for small wave-lengths, and the semiclassic case for long wave-lengths In Section IV an equilibrium with extreme temperature anisotropy and a three-dimensional water bag model are considered, allowing for more detailed analytic results. In all cases, quantum effects are stabilizing. Section V discuss possible applications in the case of neutron stars and white dwarfs as well as for tunnel-ionized plasmas. As is shown in the following, the quantum effects are enhanced for larger density and larger temperature anisotropy. For neutron stars and white dwarfs, the high densities tend to enhance quantum effects. For tunnel-ionized plasmas, the densities are not so high, but the temperature anisotropy can be significantly enough. Section VI is reserved to the conclusions.

2 Basic equations

Consider a plasma composed of electrons (charge $-e$, mass m) and a neutralizing immobile ionic background. In terms of the Wigner distribution function $f = f(\mathbf{r}, \mathbf{v}, t)$, the electron particle density $n = n(\mathbf{r}, t)$ and the current density $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ are given by

$$n = \int d\mathbf{v} f \,, \quad \mathbf{J} = -e \int d\mathbf{v}\, f\, \mathbf{v}\,. \tag{1}$$

All integrals are from minus to plus infinity unless otherwise stated. To proceed, it is necessary to work in terms of the electromagnetic potentials $(\phi(\mathbf{r}, t), \mathbf{A}(\mathbf{r}, t))$, since the electromagnetic Wigner equation [5, 51] is written in terms of them and not the fields. In tangent space with coordinates $(\mathbf{r}, \mathbf{v})$, and time variable t, it reads

$$\begin{aligned}
0 \;&=\; \frac{\partial f}{\partial t} + (v_i - \frac{eA_i}{m})\left(\frac{\partial f}{\partial r_i} + \frac{e}{m}\frac{\partial A_j}{\partial r_i}\frac{\partial f}{\partial v_j}\right) + \frac{e}{m}\frac{\partial \mathbf{A}}{\partial t}\cdot\frac{\partial f}{\partial \mathbf{v}} + \qquad (2)\\
&+\; \frac{ie}{\hbar}\left(\frac{m}{2\pi\hbar}\right)^3 \int\!\!\int d\mathbf{s}\, d\mathbf{v}' e^{im(\mathbf{v}-\mathbf{v}')\cdot\mathbf{s}/\hbar}\left[\phi\left(\mathbf{r}+\frac{\mathbf{s}}{2}\right) - \phi\left(\mathbf{r}-\frac{\mathbf{s}}{2}\right)\right] f(\mathbf{r},\mathbf{v}')\\
&-\; \frac{ie^2}{2\hbar m}\left(\frac{m}{2\pi\hbar}\right)^3 \int\!\!\int d\mathbf{s}\, d\mathbf{v}' e^{im(\mathbf{v}-\mathbf{v}')\cdot\mathbf{s}/\hbar}\left[A^2\left(\mathbf{r}+\frac{\mathbf{s}}{2}\right) - A^2\left(\mathbf{r}-\frac{\mathbf{s}}{2}\right)\right] f(\mathbf{r},\mathbf{v}')\\
&+\; \frac{e}{2m}\left(\frac{m}{2\pi\hbar}\right)^3 \left[\frac{\partial}{\partial r_i} + \frac{e}{m}\frac{\partial A_j}{\partial r_i}\frac{\partial}{\partial v_j}\right] \times
\end{aligned}$$

$$\times \int\int d\mathbf{s}\, d\mathbf{v} e^{im(\mathbf{v}-\mathbf{v}')\cdot\mathbf{s}/\hbar} \left[A_i\left(\mathbf{r}+\frac{\mathbf{s}}{2}\right)+A_i\left(\mathbf{r}-\frac{\mathbf{s}}{2}\right)\right] f(\mathbf{r},\mathbf{v}')$$

$$-\frac{ie}{\hbar}\left(\frac{m}{2\pi\hbar}\right)^3\left(\mathbf{v}-\frac{e\mathbf{A}}{m}\right)\cdot\int\int d\mathbf{s}\, d\mathbf{v}' e^{im(\mathbf{v}-\mathbf{v}')\cdot\mathbf{s}/\hbar}\left[\mathbf{A}\left(\mathbf{r}+\frac{\mathbf{s}}{2}\right)-\mathbf{A}\left(\mathbf{r}-\frac{\mathbf{s}}{2}\right)\right] f(\mathbf{r},\mathbf{v}'),$$

omitting the time-dependence of the several quantities and using summation convention in some terms. For the derivation of the electromagnetic Wigner equation in this form, it was assumed the Coulomb gauge, $\nabla\cdot\mathbf{A} = 0$. Also, notice that the electromagnetic Wigner equation as in Eq. (3) of Ref. [5] was been written in phase space with canonical coordinates $(\mathbf{q},\mathbf{p})$ and a time variable $\tau \equiv t$. To derive (2), it is necessary to transform using $\mathbf{q} = \mathbf{r}$, $\mathbf{p} = m\mathbf{v} - e\mathbf{A}$ and $\tau = t$, applying the chain rule, which implies, in particular,

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial t} + \frac{e}{m}\frac{\partial\mathbf{A}}{\partial t}\cdot\frac{\partial}{\partial\mathbf{v}}\,, \tag{3}$$

$$\frac{\partial}{\partial q_i} = \frac{\partial}{\partial r_i} + \frac{e}{m}\frac{\partial A_j}{\partial r_i}\frac{\partial}{\partial v_j}\,. \tag{4}$$

In the formal classical limit, $\hbar \to 0$, (2) reduces to the Vlasov equation,

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{e}{m}\left(\frac{\partial\phi}{\partial r_i} + \frac{\partial A_i}{\partial t} + v_j\left[\frac{\partial A_i}{\partial r_j} - \frac{\partial A_j}{\partial r_i}\right]\right)\frac{\partial f}{\partial v_i} = 0\,. \tag{5}$$

As a last remark, here we also have corrected a mistyping in one of the signals at the third term in the right-hand side of Eq. (3) of Ref. [5]. This mistyping produce no harm for the conclusions of this reference.

Now take linear wave propagation along the $O\hat{z}$ axis, with wave-vector $\mathbf{k} = k\hat{z}$, identically zero scalar potential and

$$\mathbf{A} = \mathbf{A}_\perp \exp(i[kz - \omega t])\,, \tag{6}$$

where $\mathbf{A}_\perp$ is a first-order object satisfying $\mathbf{k}\cdot\mathbf{A}_\perp \equiv 0$. This form for the vector potential is consistent with a transverse magnetic field. Also, the Wigner function expands as $f = f_0 + f_1\exp(i[kz-\omega t])$, where $f_0 = f_0(\mathbf{v})$ is the equilibrium Wigner function, satisfying

$$\int d\mathbf{v} f_0 = n_0\,, \quad \int d\mathbf{v}\, f_0\, \mathbf{v} = 0\,, \tag{7}$$

where n_0 is the ambient ion density and f_1 is a first order perturbation. Accordingly, the current density is given by

$$\mathbf{J} = -e\int d\mathbf{v} f_1 \mathbf{v}\,, \tag{8}$$

omitting the $\exp(i[kz-\omega t])$ dependence.

In this context, Ampère's law yields

$$(\omega^2 - c^2k^2)\mathbf{A}_\perp = \frac{e}{\varepsilon_0}\int d\mathbf{v} f_1 \mathbf{v}\,. \tag{9}$$

The homogeneous Maxwell equations are identically satisfied when working with the electromagnetic potentials, while Poisson equation is satisfied since it can be shown that there are no charge density fluctuations ($\int d\mathbf{v} f_1 = 0$), as in the classical Weibel instability. Therefore, the only remaining equation is the linearized Wigner equation. Using (2), the result is

$$\begin{aligned}(\omega &- kv_z)\left(f_1 - \frac{e}{m}\mathbf{A}_\perp\cdot\frac{\partial f_0}{\partial \mathbf{v}}\right) \\ &= \frac{e\,\mathbf{v}\cdot\mathbf{A}_\perp}{\hbar}\left(f_0(v_x,v_y,v_z-\frac{\hbar k}{2m}) - f_0(v_x,v_y,v_z+\frac{\hbar k}{2m})\right)\,,\end{aligned} \tag{10}$$

for $\mathbf{v} = (v_x,v_y,v_z)$ and $\hbar = h/(2\pi)$ being the scaled Planck's constant. Combining (7), (9) and (10), assuming that the equilibria f_0 are even functions of all velocity components also satisfying the condition $f_0(v_x,v_y,v_z) = f_0(v_y,v_x,v_z)$ and doing an integration by parts, there follows the quantum dispersion relation for transverse waves ($\mathbf{k}\cdot\mathbf{E} = 0$),

$$\begin{aligned}\omega^2 - \omega_p^2 &- c^2k^2 + \frac{m\omega_p^2}{2n_0\hbar}\int d\mathbf{v}\left(\frac{v_x^2+v_y^2}{\omega - kv_z}\right)\times \\ &\times\left(f_0(v_x,v_y,v_z+\frac{\hbar k}{2m}) - f_0(v_x,v_y,v_z-\frac{\hbar k}{2m})\right) = 0\,,\end{aligned} \tag{11}$$

where $\omega_p = (n_0e^2/(m\varepsilon_0))^{1/2}$ is the plasma frequency.

Consider the specific case of the equilibrium Wigner function

$$f_0 = \frac{n_0}{T_\parallel^{1/2}T_\perp}\left(\frac{m}{2\pi}\right)^{3/2}\exp\left(-\frac{m}{2T_\perp}(v_x^2+v_y^2) - \frac{mv_z^2}{2T_\parallel}\right)\,, \tag{12}$$

showing temperature anisotropy between parallel and perpendicular directions. The temperatures $T_\perp$ and $T_\parallel$ are measured in terms of energy units (Boltzmann's constant is $\kappa_B \equiv 1$). Then the dispersion relation (11) develops into

$$\omega^2 - c^2k^2 - \omega_p^2\left(1 + \frac{T_\perp}{T_\parallel}W_Q\right) = 0\,, \tag{13}$$

where

$$W_Q = \frac{mv_\parallel}{2\hbar k}\left(Z(\frac{\omega}{kv_\parallel} + \frac{\hbar k}{2mv_\parallel}) - Z(\frac{\omega}{kv_\parallel} - \frac{\hbar k}{2mv_\parallel})\right) , \tag{14}$$

for $v_\parallel = (2T_\parallel/m)^{1/2}$ and where Z is the plasma dispersion function. The function W_Q can be appropriately named a "quantum Weibel function". Apart from a scale factor, it is a centered finite difference version of the derivative of the plasma dispersion function. In the formal classical limit when $\hbar \to 0$,

$$W_Q \to W\left(\frac{\omega}{kv_\parallel}\right) \equiv -1 - \frac{\omega}{kv_\parallel} Z\left(\frac{\omega}{kv_\parallel}\right) , \tag{15}$$

where the function $W(\omega/(kv_\parallel))$ is the same (Weibel) function as the one in Eq. (5) of Ref. [48]. Hence, in this formal classical limit, the dispersion relation (13) reduces to the classical one, in accordance with the correspondence principle. Notice that quantum effects appears only through the non-dimensional parameter

$$H = \frac{\hbar k}{mv_\parallel} , \tag{16}$$

basically depending only on the longitudinal quantities k and $v_\parallel$.

3 Limiting cases

We consider separately the ultra-quantum case, the small quantum parameter and wave-number case and the small quantum parameter and large wave-number case.

3.1 Ultra-quantum case ($H \gg 1$)

It is interesting to check the behavior of the dispersion relation in the case of very large or very small quantum effects. In this regard, an useful alternative (exact) expression for W_Q is

$$W_Q = \frac{1}{2\sqrt{\pi}} \int \frac{d\xi e^{-\xi^2}}{(\xi - \frac{\omega}{kv_\parallel})^2 - H^2/4} , \tag{17}$$

for $\xi = \omega/(kv_\parallel)$. Assuming large quantum effects, so that $H^2 \gg |\omega/(kv_\parallel)|^2$, it follows from (17) that $W_Q \simeq -2/H^2$, so that (13) imply

$$\omega^2 = c^2k^2 + \omega_p^2\left(1 - \frac{2m^2v_\parallel^2 T_\perp}{\hbar^2 k^2 T_\parallel}\right) . \tag{18}$$

There can be instability ($\omega^2 < 0$), provided there is also sufficient temperature anisotropy,

$$\frac{T_\perp}{T_\parallel} > \frac{H^2}{2}\left(1 + \frac{c^2k^2}{\omega_p^2}\right) . \tag{19}$$

However, since the right hand side of the last inequality is an increasing function of H, one conclude that quantum effects play an stabilizing rôle. Another way to derive this conclusion comes from rewriting (19) in terms of the instability condition

$$k^2 < k_c^2 \equiv \frac{\omega_p^2}{2c^2}\left(-1 + (1 + \frac{16mc^2T_\perp}{\hbar^2\omega_p^2})^{1/2}\right) . \tag{20}$$

For increasing quantum effects, the critical wave-number k_c becomes smaller, eventually dropping to zero. Notice also that (20) and $H^2 \gg 1$ are conflicting conditions. A detailed but cumbersome analysis shows that both conditions can be satisfied only for $\hbar\omega_p/(mc^2) \gg 1$, violating the non-relativistic assumption of the present model.

3.2 Semiclassic case at small wave-lengths ($H \ll 1$ and $|\xi| \ll 1$)

Now retaining only the first-order quantum correction, one get

$$W_Q = -1 - \xi Z(\xi) + \frac{H^2}{12}\Big(2 + 3\xi Z(\xi) - 2\xi^2 - 2\xi^3 Z(\xi)\Big) + O(H^4) . \tag{21}$$

For such small quantum effects and also taking small wave-lengths so that $|\xi| \ll 1$ and $Z(\xi) \simeq i\sqrt{\pi}$, there follows from (21) that

$$W_Q \simeq -1 - i\xi\sqrt{\pi} + \frac{H^2}{6} , \tag{22}$$

while the dispersion relation (13) produces

$$\omega = \frac{ikv_\parallel T_\parallel}{\sqrt{\pi}T_\perp}\left(\frac{T_\perp}{T_\parallel}(1 - \frac{H^2}{6}) - 1 - \frac{c^2k^2}{\omega_p^2}\right) . \tag{23}$$

In the derivation of (23), it was used $|\omega^2|/(c^2k^2) = |\xi^2|v_\parallel^2/c^2 \ll 1$, also consistent with the non-relativistic approximation. Assuming positive wave-numbers, it is apparent from (23) that purely growing waves (frequencies composed only by a positive imaginary part) can exist for sufficiently large temperature anisotropy, and that this necessary anisotropy becomes larger for increasing H. Specifically, for small quantum effects and wave-lengths, there will be a purely growing wave only if

$$\frac{T_\perp}{T_\parallel} > \frac{1}{1 - H^2/6} \simeq 1 + \frac{H^2}{6}\,, \tag{24}$$

showing the need of extra anisotropy for instability, due to quantum effects. In addition, in this combined regime of small quantum effects and wave-lengths, it can be proven that the unstable wave-numbers are restricted to

$$k^2 < k_c^2 \equiv \frac{\omega_p^2 (T_\perp/T_\parallel - 1)}{c^2\left(1 + \frac{T_\perp \hbar^2 \omega_p^2}{12 T_\parallel^2 m c^2}\right)}\,. \tag{25}$$

Once again, the stabilizing nature of (now small) quantum effects is apparent, since the allowable unstable wave-numbers occurs for a smaller range for increasing H^2.

3.3 Semiclassic case at large wave-lengths ($H \ll 1$ and $|\xi| \gg 1$)

From the expansion (21) and using $Z(\xi) \simeq -1/\xi - 1/(2\xi^3)$ when $|\xi| \gg 1$, for large wave-lengths, it follows the dispersion relation

$$\omega^2 - c^2 k^2 - \omega_p^2 \left(1 + \frac{k^2 T_\perp}{m\omega^2}(1 - \frac{H^2}{4})\right) = 0\,. \tag{26}$$

For $|\omega| \ll ck$, (26) yields the purely growing mode

$$\omega = ik \left(\frac{T_\perp (1 - H^2/4)}{m}\right)^{1/2} \left(\frac{\omega_p^2}{c^2 k^2 + \omega_p^2}\right)^{1/2}\,. \tag{27}$$

Notice that the growth rate becomes smaller for larger H. Also, in view of (27), the condition $|\xi| \gg 1$ can be attained only for $T_\perp \gg T_\parallel/(1 - H^2/4)$. Equations (26) and (27) are formally the same as Eqs. (8) and (9) of Ref. [48], making the replacement $T_\perp \to T_\perp(1 - H^2/4)$, indicating the need of extra temperature anisotropy, in view of quantum effects.

4 Toy models and more detailed analytical results

Anisotropic Gaussian distributions are amenable to analytical results only for limiting situations. This Section consider some toy models which behave more friendly in this respect. As a first example where full analytical results are available, consider the equilibrium Wigner function

$$f_0 = \frac{n_0 m}{2\pi T_\perp}\delta(v_z)\exp\left(-\frac{m}{2T_\perp}(v_x^2+v_y^2)\right)\,, \tag{28}$$

which can be viewed as a distribution with extreme temperature anisotropy ($T_\parallel \to 0$). Inserting into (11) and proceeding as before, it results

$$\omega^2 = \frac{1}{2}\left(\omega_p^2 + c^2k^2 + \frac{\hbar^2 k^4}{4m^2} \pm \left[(\omega_p^2 + c^2k^2 - \frac{\hbar^2 k^4}{4m^2})^2 + \frac{4k^2\omega_p^2 T_\perp}{m}\right]^{1/2}\right)\,. \tag{29}$$

One of the roots is unstable ($\omega^2 < 0$), provided

$$k^2 < k_c^2 \equiv \frac{\omega_p^2}{2c^2}\left(1 + \frac{16mc^2T_\perp}{\hbar^2\omega_p^2}\right)^{1/2} - \frac{\omega_p^2}{2c^2}\,, \tag{30}$$

showing, once again, stabilization due to increasing quantum effects.

The classical transverse Weibel instability has sometimes considered in terms of water bag distributions [38, 49, 52]. As a second example, also amenable to detailed calculations, take the following three-dimensional water bag [49] equilibrium,

$$f_0 = \frac{n_0}{8v_\perp^2 v_\parallel}\left(\theta(v_x+v_\perp) - \theta(v_x - v_\perp)\right)\left(\theta(v_y+v_\perp) - \theta(v_y - v_\perp)\right) \times \\ \times \left(\theta(v_z+v_\parallel) - \theta(v_z - v_\parallel)\right)\,, \tag{31}$$

where θ is the Heaviside function and in this context $v_\perp$ and $v_\parallel$ are related to dispersion of velocities in the perpendicular plane and along the $O\hat{z}$ axis, as before. Then the dispersion relation (11) yields

$$\omega^2 - \omega_p^2 - c^2k^2 = \frac{m\omega_p^2 v_\perp^2}{6\hbar k v_\parallel}\ln\left(\frac{\omega^2 - (kv_\parallel - \hbar k^2/(2m))^2}{\omega^2 - (kv_\parallel + \hbar k^2/(2m))^2}\right)\,, \tag{32}$$

not in polynomial form as in the formal classical limit [49] for the corresponding equilibrium. However, some analytical results are still available. Assuming purely growing instabilities with a growth rate γ, so that $\omega = i\gamma$, and also disregarding γ^2 at the left-hand side of (32) assuming $\gamma^2 \ll \omega_p^2$, there follows

$$\gamma^2 = \frac{\hbar k^3 v_\parallel}{m} \coth\left(\frac{3\hbar k v_\parallel (c^2 k^2 + \omega_p^2)}{m\omega_p^2 v_\perp^2}\right) - k^2 v_\parallel^2 - \frac{\hbar^2 k^4}{4m^2} \,. \qquad (33)$$

Notice that (33) is not valid in the formal classical limit $\hbar \equiv 0$ because then the associated growth rate would not be small. Rather, in this limit one has to Taylor expand the right-hand side of (32), and then the results from [49] are recovered.

To proceed, it is convenient to adopt the following rescaling,

$$\bar\gamma = \frac{\gamma}{\omega_p} \,, \quad \bar k = \frac{k v_\parallel}{\omega_p} \,, \quad \bar H = \frac{\hbar \omega_p}{m v_\parallel^2} \,, \quad \bar v_\perp = \frac{v_\perp}{c} \,, \quad \bar v_\parallel = \frac{v_\parallel}{c} \,, \qquad (34)$$

so that (33) becomes

$$\bar\gamma^2 = \bar H \bar k^3 \coth\left(\frac{3\bar H \bar k}{\bar v_\perp^2}(\bar k^2 + \bar v_\parallel^2)\right) - \bar k^2 - \frac{\bar H^2 \bar k^4}{4} \,. \qquad (35)$$

It is possible to estimate the maximum wave-number for instability, using $\coth(\xi) \simeq 1/\xi$ for $|\xi| \ll 1$. Assuming that this expansion is valid, one will conclude from (35) that $\bar\gamma^2 > 0$ provided

$$\bar k^2 < \bar k_m^2 \equiv \frac{1}{2\bar H^2}\left([(\bar H^2 \bar v_\parallel^2 - 4)^2 + \frac{16\bar H^2 \bar v_\perp^2}{3}]^{1/2} - \bar H^2 \bar v_\parallel^2 - 4\right) \,. \qquad (36)$$

From (36), it can be shown that there will exist some unstable mode ($\bar k_m^2 > 0$) if and only if there is sufficient temperature anisotropy, $\bar v_\perp > \sqrt{3}\bar v_\parallel$, independently of the strength of the quantum effects, and in accordance with the scaling found for classical plasma [49]. However, it can be deduced from (36) that the critical wave-number shrinks to zero as $\bar H \to \infty$.

5 Applications

Consider now the case of anisotropic Maxwellian equilibria. For semiclassic and small wave-lengths conditions (Section III.2), one can use (23) to show that the maximal value of the growth rate happens at $k = k_m = k_c/\sqrt{3}$, with k_c given by (25). The corresponding maximal growth rate is then

$$\gamma_m = \left(\frac{8}{27\pi}\right)^{1/2} \omega_p \left(\frac{T_\parallel}{mc^2}\right)^{1/2} \frac{T_\parallel}{T_\perp} \frac{(T_\perp/T_\parallel - 1)^{3/2}}{\left(1 + \frac{T_\perp \hbar^2 \omega_p^2}{12 T_\parallel^2 mc^2}\right)^{1/2}} \,. \tag{37}$$

The calculations leading to (37) were semiclassic in the sense that the parameter $H = \hbar k/mv_\parallel$ is taken as small. However, there can be significant deviations from the classical expression, coming from the $\hbar^2$ term in the denominator of (37), provided there is sufficient temperature anisotropy. These deviations are also enhanced by large densities. For instance, consider white dwarfs and neutron stars, with typical values $n_0 \sim 10^{32} m^{-3}$, $T_\perp = 10^7 K$, $T_\parallel = T_\perp/100$. The origin of the temperature anisotropy can be, for instance, the propagation of a shock wave. For these parameters, for hydrogen plasma, one finds that the quantum corrected maximal growth rate is about 11% smaller than the classical one. However, these calculations have to be taken with care, because they suppose that H^2 from (16) at $k = k_m$ and $|\xi| = \gamma_m/(k_m v_\parallel)$ are small quantities. For the chosen parameters one get $H^2 \sim 0.41$ and $|\xi| \sim 0.37$. For larger densities, H^2 would be even greater.

Another interesting system where quantum corrections for Weibel instability can be significant are tunnel-ionized plasmas with negligible longitudinal temperature and where the perpendicular temperature can be controlled by a varying laser polarization. It has been argued [53] that the Weibel instability could be a mechanism for further increase of $T_\parallel$ with time. For a typical value [53] of $T_\parallel \sim 1eV$, one find

$$\frac{T_\perp \hbar^2 \omega_p^2}{12 T_\parallel^2 mc^2} = 2.3 \times 10^{-34} n_0 T_\perp \,, \tag{38}$$

using SI units for n_0 and $T_\perp$. Although it is not easy to get large values for the quantity at the right-hand side of (38), the fast progress in next generation intense laser-solid density plasma interaction experiments can be such that quantum effects stops the Weibel instability, specially for ultradense systems. For the largest densities [29, 54] feasible now, $n_0 \sim 10^{29} m^{-3}$, and for $T_\perp \sim 100eV$, the right-hand side of (38) has already a significant value of the order of 25. Another way to get even larger quantum effects is the use of smaller values of $T_\parallel$. The results of this section, however, have to be taken with care, since they deal with dense plasmas which should be more properly treated by Fermi-Dirac statistics.

6 Conclusions

In general terms, quantum effects produces smaller growth rates and smaller ranges for unstable wave-numbers, in the case of equilibria with distribution functions anisotropic in temperature. Some applications in astrophysical scenarios and in tunnel-ionized plasma were discussed. Eventually, in the ultra-quantum case, the unstable region shrinks to zero. We can understand this result in an heuristic way as follows. Due to wave-particle spreading and tunneling, quantum effects tends to enhance the dispersion of particles in phase space. This corresponds to an effectively smaller temperature anisotropy, or thermalization, so that the original ratio $T_\perp/T_\parallel$ have to be greater to produce the same instability results as in classical plasma. Similar spreading in phase space also occurs in the case of quantum corrected Bernstein-Greene-Kruskal modes [55]. However, as mentioned in the Introduction, sometimes quantum effects can also gives unexpected enhancement of plasma instabilities. Therefore, it is interesting to pursue this trend, looking at the behavior of additional well-known classical plasma instabilities, in the context of quantum plasma models. Also, an important issue is the inclusion of relativistic and spin effects.

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