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Quantum and classical dynamics of Langmuir wave packets

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Abstract

The quantum Zakharov system in three-spatial dimensions and an associated Lagrangian description, as well as its basic conservation laws are derived. In the adiabatic and semiclassical case, the quantum Zakharov system reduces to a quantum modified vector nonlinear Schrödinger (NLS) equation for the envelope electric field. The Lagrangian structure for the resulting vector NLS equation is used to investigate the time-dependence of the Gaussian shaped localized solutions, via the Rayleigh-Ritz variational method. The formal classical limit is considered in detail. The quantum corrections are shown to prevent the collapse of localized Langmuir envelope fields, in both two and three-spatial dimensions. Moreover, the quantum terms can produce an oscillatory behavior of the width of the approximate Gaussian solutions. The variational method is shown to preserve the essential conservation laws of the quantum modified vector NLS equation.

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I. INTRODUCTION

The Zakharov system [1], describing the coupling between Langmuir and ion-acoustic waves, is one of the basic plasma models, see Ref. [2, 3] for reviews. Recently [4], a quantum modified Zakharov system was derived, by means of the quantum plasma hydrodynamic model [5]–[7]. In this context, enhancement of the quantum effects was then shown *e. g.* to suppress the four-wave decay instability. Subsequently [8], a kinetic treatment of the quantum Zakharov system has shown that the modulational instability growth rate can be increased in comparison to the classical case, for partially coherent Langmuir wave electric fields. Also [9], a variational formalism was obtained and used to study the radiation of localized structures described by the quantum Zakharov system. Bell shaped electric field envelopes of electron plasma oscillations in dense quantum plasmas obeying Fermi statistics were analyzed in Ref. [10]. More mathematically-oriented works on the quantum Zakharov equations concern its Lie symmetry group [11] and the derivation of exact solutions [12]–[14]. Finally, there is evidence of hyperchaos in the reduced temporal dynamics arising from the quantum Zakharov equations [15].

All these paper refer to quantum Zakharov equations in one-spatial-dimension only. In the present work, we extend the quantum Zakharov system to fully three-dimensional space, allowing also for the magnetic field perturbation. In the classical case, both heuristic arguments and numerical simulations indicate that the ponderomotive force can produce finite-time collapse of Langmuir wave packets in two- or three-dimensions [2], [16, 17]. This is in contrast to the one-dimensional case, whose solutions are smooth for all time. A dynamic rescaling method was used for the time-evolution of electrostatic self-similar and asymptotically self-similar solutions in two- and three-dimensions, respectively [18]. Allowing for transverse fields shows that singular solutions of the resulting vector Zakharov equations are weakly anisotropic, for a large class of initial conditions [19]. The electrostatic nonlinear collapse of Langmuir wave packets in the ionospheric and laboratory plasmas has been observed [20, 21]. Also, the collapse of Langmuir wave packets in beam plasma experiments verifies the basic concepts of strong Langmuir turbulence, as introduced by Zakharov [22]. The analysis of the coupled longitudinal and transverse modes in the classical strong Langmuir turbulence has been less studied [23]–[25], as well as the intrinsically magnetized case [26], which can lead to upper-hybrid wave collapse [27]. Finally, Zakharov-like equations

have been proposed for the electromagnetic wave collapse in a radiation background [28].

It is expected that the ponderomotive force causing the collapse of localized solutions in two- or three-space dimensions could be weakened by the inclusion of quantum effects, making the dynamics less violent. This conjecture is checked after establishing the quantum Zakharov system in higher-dimensional space and using its variational structure in association with a (Rayleigh-Ritz) trial function method.

The manuscript is organized in the following fashion. In Section 2, the quantum Zakharov system in three-spatial-dimensions is derived by means of the usual two-time scale method applied to the fully 3D quantum hydrodynamic model. In Section 3, the 3D quantum Zakharov system is shown to be described by a Lagrangian formalism. The basic conservation laws are then also derived. When the density fluctuations are so slow in time so that an adiabatic approximation is possible, and treating the quantum term of the low-frequency equation as a perturbation, a quantum modified vector nonlinear Schrödinger equation for the envelope electric field is obtained. In Section 4, the variational structure is used to analyze the temporal dynamics of localized (Gaussian) solutions of this quantum NLS equation, through the Rayleigh-Ritz method, in two-spatial-dimensions. Section 5 follows the same strategy, extended to fully 3D space. Special attention is paid to the comparison between the classical and quantum cases, with considerable qualitative and quantitative differences. Section 6 contains conclusions.

II. QUANTUM ZAKHAROV EQUATIONS IN $3+1$ DIMENSIONS

The starting point for the derivation of the electromagnetic quantum Zakharov equations is the quantum hydrodynamic model for an electron-ion plasma, Equations (20)-(28) of Ref. [7]. For the electron fluid pressure p_e, consider the equation of state for spin 1/2 particles at zero temperature,

$$p_e = \frac{3}{5}\,\frac{m_e v_{Fe}^2\, n_e^{5/3}}{n_0^{2/3}}\,, \tag{1}$$

where m_e is the electron mass, v_{Fe} is the Fermi electron thermal speed, n_e is the electron number density and n_0 is the equilibrium particle number density both for electron and ions. The pressure and quantum effects (due to their larger mass) are neglected for the ions. Also due to the larger ion mass, it is possible to introduce a two-time scale decomposition, $n_e = n_0 + \delta n_s + \delta n_f$, $n_i = n_0 + \delta n_s$, $\mathbf{u}_e = \delta\mathbf{u}_s + \delta\mathbf{u}_f$, $\mathbf{u}_i = \delta\mathbf{u}_s$, $\mathbf{E} = \delta\mathbf{E}_s + \delta\mathbf{E}_f$, $\mathbf{B} = \delta\mathbf{B}_f$,

where the subscripts s and f refer to slowly and rapidly changing quantities, respectively. Also, $\mathbf{u}_e$ is the electron fluid velocity, n_i the ion number density, $\mathbf{u}_i$ the ion fluid velocity, $\mathbf{E}$ the electric field, and $\mathbf{B}$ the magnetic field. Notice that it is assumed that there is no slow contribution to the magnetic field, a restriction which allows to get $\mathbf{B} = (m_e/e)\,\nabla\times\delta\mathbf{u}_f$ (see Equation (2.21) of Ref. [3]), where $-e$ is the electron charge. Including a slow contribution to the magnetic field could be an important improvement, but this is outside the scope of the present work.

Following the usual approximations [3, 4], the quantum corrected 3D Zakharov equations read

$$\begin{aligned} 2i\omega_{pe}\frac{\partial\tilde{\mathbf{E}}}{\partial t} &- c^2\,\nabla\times(\nabla\times\tilde{\mathbf{E}}) + v_{Fe}^2\nabla(\nabla\cdot\tilde{\mathbf{E}}) = \\ &= \frac{\delta n_s}{n_0}\,\omega_{pe}^2\,\tilde{\mathbf{E}} + \frac{\hbar^2}{4m_e^2}\nabla\left[\nabla^2(\nabla\cdot\tilde{\mathbf{E}})\right], \qquad (2) \\ \frac{\partial^2\delta n_s}{\partial t^2} &- c_s^2\,\nabla^2\delta n_s - \frac{\varepsilon_0}{4m_i}\nabla^2(|\tilde{\mathbf{E}}|^2) + \frac{\hbar^2}{4m_e m_i}\,\nabla^4\delta n_s = 0\,. \qquad (3) \end{aligned}$$

Here $\tilde{\mathbf{E}}$ is the slowly varying envelope electric field defined via

$$\mathbf{E}_f = \frac{1}{2}\,(\tilde{\mathbf{E}}\,e^{-i\omega_{pe}t} + \tilde{\mathbf{E}}^*\,e^{i\omega_{pe}t})\,, \qquad (4)$$

where ω_{pe} is the electron plasma frequency. Also, in Eqs. (2–3) c is the speed of light in vacuum, $\hbar$ the scaled Planck constant, ε_0 the vacuum permittivity and m_i the ion mass. In addition, $c_s^2 = \kappa_B T_{Fe}/m_i$, where $\kappa_B T_{Fe} = m_e v_{Fe}^2$. Therefore, c_s is a Fermi ion-acoustic speed, with the Fermi temperature replacing the thermal temperature for the electrons.

In comparison to the classical Zakharov system (see Eqs. (2.48a)–(2.48b) of Ref. [3]), there is the inclusion of the extra dispersive terms proportional to $\hbar^2$ in Eqs. (2)–(3). Other quantum difference is the presence of the Fermi speed instead of the thermal speed in the last term at the left hand side of Eq. (2). From the qualitative point of view, the terms proportional to $\hbar^2$ are responsible for extra dispersion which can avoid collapsing of Langmuir envelopes, at least in principle. This possibility is investigated in Sections 4 and 5. Finally, notice the non trivial form of the fourth order derivative term in Eq. (2). It is not simply proportional to $\nabla^4\tilde{\mathbf{E}}$ as could be wrongly guessed from the quantum Zakharov equations in $1+1$ dimensions, where there is a $\sim\partial^4\tilde{\mathbf{E}}/\partial x^4$ contribution [4].

It is useful to consider the rescaling

$$\bar{\mathbf{r}} = \frac{2\sqrt{\mu}\,\omega_{pe}\,\mathbf{r}}{v_{Fe}}\,, \quad \bar{t} = 2\,\mu\,\omega_{pe} t\,, \tag{5}$$
$$n = \frac{\delta n_s}{4\mu n_0}\,, \quad \mathcal{E} = \frac{e\,\tilde{\mathbf{E}}}{4\,\sqrt{\mu}\,m_e \omega_{pe} v_{Fe}}\,,$$

where $\mu = m_e/m_i$. Then, dropping the bars in $\mathbf{r}, t$, we obtain

$$i\frac{\partial \mathcal{E}}{\partial t} - \frac{c^2}{v_{Fe}^2}\nabla \times (\nabla \times \mathcal{E}) + \nabla(\nabla \cdot \mathcal{E}) = = n\,\mathcal{E} + \Gamma\,\nabla\left[\nabla^2(\nabla \cdot \mathcal{E})\right]\,, \tag{6}$$

$$\frac{\partial^2 n}{\partial t^2} - \nabla^2 n - \nabla^2(|\mathcal{E}|^2) + \Gamma\,\nabla^4 n = 0\,, \tag{7}$$

where

$$\Gamma = \frac{m_e}{m_i}\left(\frac{\hbar\,\omega_{pe}}{\kappa_B T_{Fe}}\right)^2 \tag{8}$$

is a non-dimensional parameter associated with the quantum effects. Usually, it is an extremely small quantity, but it is nevertheless interesting to retain the $\sim \Gamma$ terms, specially for the collapse scenarios. The reason is not only due to a general theoretical motivation, but also because from some simple estimates one concludes that these terms become of the same order as some of other terms in Eqs. (2)–(3) provided that the characteristic length l for the spatial derivatives becomes as small as the mean inter-particle distance, $l \sim n_0^{-1/3}$. Of course, the Zakharov equations are not able to describe the late stages of the collapse, since they do not include dissipation, which is unavoidable for short scales. But even Landau damping would be irrelevant for a zero-temperature Fermi plasma, where the main influence comes from the Pauli pressure. In the left-hand side of Eq. (6), the $\nabla(\nabla \cdot \mathcal{E})$ term is retained because the $\sim c^2/v_{Fe}^2$ transverse term disappears in the electrostatic approximation.

In the adiabatic limit, neglecting $\partial^2 n/\partial t^2$ in Eq. (7) and under appropriated boundary conditions, it follows that

$$n = -|\mathcal{E}|^2 + \Gamma\,\nabla^2 n\,, \tag{9}$$

When $\Gamma \neq 0$, it is not easy to directly express n as a function of $|\mathcal{E}|$ as in the classical case. Therefore, the adiabatic limit is not enough to derive a vector nonlinear Schrödinger equation, due to the coupling in Eq. (9).

III. LAGRANGIAN STRUCTURE AND CONSERVATION LAWS

The quantum Zakharov equations (6)–(7) can be described by the Lagrangian density

$$\begin{aligned}\mathcal{L} \;=\; & \frac{i}{2}\Big(\mathcal{E}^*\cdot\frac{\partial\mathcal{E}}{\partial t}-\mathcal{E}\cdot\frac{\partial\mathcal{E}^*}{\partial t}\Big)-\frac{c^2}{v_{Fe}^2}|\nabla\times\mathcal{E}|^2-|\nabla\cdot\mathcal{E}|^2-\Gamma\,|\nabla(\nabla\cdot\mathcal{E})|^2\\ & +\; n\Big(\frac{\partial\alpha}{\partial t}-|\mathcal{E}|^2\Big)-\frac{1}{2}\Big(n^2+\Gamma|\nabla n|^2+|\nabla\alpha|^2\Big)\,,\end{aligned} \tag{10}$$

where n, the auxiliary function α and the components of $\mathcal{E},\mathcal{E}^*$ are regarded as independent fields. Remark: for the particular form (10) and for a generic field ψ, one computes the functional derivative as

$$\frac{\delta\mathcal{L}}{\delta\psi}=\frac{\partial\mathcal{L}}{\partial\psi}-\frac{\partial}{\partial r_i}\frac{\partial\mathcal{L}}{\partial\psi/\partial r_i}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\psi/\partial t}+\frac{\partial^2}{\partial r_i\,\partial r_j}\frac{\partial\mathcal{L}}{\partial^2\psi/\partial r_i\partial r_j}\,, \tag{11}$$

using the summation convention and where r_i are cartesian components.

Taking the functional derivatives with respect to n and α, we have

$$\frac{\partial\alpha}{\partial t}=n+|\mathcal{E}|^2-\Gamma\nabla^2 n, \tag{12}$$

and

$$\frac{\partial n}{\partial t}=\nabla^2\alpha\,, \tag{13}$$

respectively. Eliminating α from Eqs. (12) and (13) we obtain the low frequency equation. In addition, the functional derivatives with respect to $\mathcal{E}^*$ and $\mathcal{E}$ produce the high-frequency equation and its complex conjugate. The present formalism is inspired by the Lagrangian formulation of the classical Zakharov equations [29].

The quantum Zakharov equations admit as exact conserved quantities the "number of plasmons" of the Langmuir field,

$$N=\int|\mathcal{E}|^2\,d\mathbf{r}\,, \tag{14}$$

the linear momentum (with components P_i, $i=x,y,z$),

$$P_i=\int\Big[\frac{i}{2}\Big(\mathcal{E}_j\,\frac{\partial\mathcal{E}_j^*}{\partial r_i}-\mathcal{E}_j^*\,\frac{\partial\mathcal{E}_j}{\partial r_i}\Big)-n\,\frac{\partial\alpha}{\partial r_i}\Big]\,d\mathbf{r} \tag{15}$$

and the Hamiltonian,

$$\begin{aligned}\mathcal{H} \;=\; & \int\Big[n|\mathcal{E}|^2+\frac{c^2}{v_{Fe}^2}\,|\nabla\times\mathcal{E}|^2+|\nabla\cdot\mathcal{E}|^2+\Gamma\,|\nabla(\nabla\cdot\mathcal{E})|^2\\ & +\;\frac{1}{2}\Big(n^2+\Gamma|\nabla n|^2+|\nabla\alpha|^2\Big)\Big]\,d\mathbf{r}\,.\end{aligned} \tag{16}$$

Furthermore, there is also a preserved angular momenta functional, but it is not relevant in the present work. These four conserved quantities can be associated, through Noether's theorem, to the invariance of the action under gauge transformation, time translation, space translation and rotations, respectively. The conservation laws can be used *e. g.* to test the accuracy of numerical procedures. Also, observe that equations (7) and (9) for the adiabatic limit are described by the same Lagrangian density (10). In this approximation, it suffices to set $\alpha \equiv 0$.

In addition to the adiabatic limit, Eq. (9) can be further approximated to

$$n = -|\mathcal{E}|^2 - \Gamma \nabla^2(|\mathcal{E}|^2)\,, \tag{17}$$

assuming that the quantum term is a perturbation. In this way and using Eq. (6), a quantum modified vector nonlinear Schrödinger equation is derived

$$\begin{aligned} i\frac{\partial \mathcal{E}}{\partial t} \; &+ \; \nabla(\nabla\cdot\mathcal{E}) - \frac{c^2}{v_{Fe}^2}\nabla\times(\nabla\times\mathcal{E}) + |\mathcal{E}|^2\mathcal{E} = \\ &= \; \Gamma\nabla\left[\nabla^2(\nabla\cdot\mathcal{E})\right] - \Gamma\,\mathcal{E}\nabla^2(|\mathcal{E}|^2)\,. \end{aligned} \tag{18}$$

The appropriate Lagrangian density $\mathcal{L}_{ad,sc}$ for the semiclassical equation (18) is given by

$$\begin{aligned} \mathcal{L}_{ad,sc} \; &= \; \frac{i}{2}\left(\mathcal{E}^*\cdot\frac{\partial\mathcal{E}}{\partial t} - \mathcal{E}\cdot\frac{\partial\mathcal{E}^*}{\partial t}\right) - \frac{c^2}{v_{Fe}^2}|\nabla\times\mathcal{E}|^2 - |\nabla\cdot\mathcal{E}|^2 \\ &- \; \Gamma\,|\nabla(\nabla\cdot\mathcal{E})|^2 + \frac{1}{2}\,|\mathcal{E}|^4 - \frac{\Gamma}{2}\left|\nabla[\,|\mathcal{E}|^2]\right|^2, \end{aligned} \tag{19}$$

where the independent fields are taken as $\mathcal{E}$ and $\mathcal{E}^*$ components.

The expression N for the number of plasmons in Eq. (14) remains valid as a constant of motion in the joint adiabatic and semiclassical limit, as well as the momentum $\mathbf{P}$ in Eq. (15) with $\alpha \equiv 0$. Finally, the Hamiltonian

$$\begin{aligned} \mathcal{H}_{ad,sc} = \int\Big[\,\frac{c^2}{v_{Fe}^2}\,|\nabla\;\times\;\mathcal{E}|^2 + |\nabla\cdot\mathcal{E}|^2 + \Gamma\,|\nabla(\nabla\cdot\mathcal{E})|^2 \\ - \;\frac{1}{2}\,|\mathcal{E}|^4 + \frac{\Gamma}{2}\left|\nabla[\,|\mathcal{E}|^2\,]\right|^2\,\Big]\,d\mathbf{r} \end{aligned} \tag{20}$$

is also a conserved quantity.

In the following, the influence of the quantum terms in the right-hand side of Eq. (18) are investigated, assuming adiabatic conditions for collapsing quantum Langmuir envelopes. Other scenarios for collapse, like the supersonic one [18, 19], could also be relevant and shall be investigated in the future.

IV. VARIATIONAL SOLUTION IN TWO DIMENSIONS

Consider the adiabatic semiclassical system defined by Eq. (18). We refer to localized solution for this vector NLS equation as (quantum) "Langmuir wave packets", or envelopes. As discussed in detail in [29] in the purely classical case, Langmuir wave packets will become singular in a finite time, provided the energy is not bounded from below. Of course, explicit analytic Langmuir envelopes are difficult to derive. A fruitful approach is to make use of the Lagrangian structure for deriving approximate solutions. This approach has been pursued in [30] for the classical and in [9] for the quantum Zakharov system. Both studies considered the internal vibrations of Langmuir envelopes in one-spatial-dimension. Presently, we shall apply the time-dependent Rayleigh-Ritz method for the higher-dimensional cases. A priori, it is expected that the quantum corrections would inhibit the collapse of localized solutions, in view of wave-packet spreading. To check this conjecture, and to have more definite information on the influence of the quantum terms, first we consider the following *Ansatz*,

$$\mathcal{E} = \left(\frac{N}{\pi}\right)^{1/2} \frac{1}{\sigma} \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \exp\left(i(\Theta + k\rho^2)\right) \; (\cos\phi, \sin\phi, 0)\,, \tag{21}$$

which is appropriate for two-spatial-dimensions. Here σ, k, Θ and ϕ are real functions of time, and $\rho = \sqrt{x^2+y^2}$. The normalization condition (14) is automatically satisfied (in 2D the spatial integrations reduce to integrations on the plane). Other localized forms, involving *e. g.* a *sech* type dependence, could have been also proposed. Here a Gaussian form was suggested mainly for the sake of simplicity [31]. Notice that the envelope electric field (21) is not necessarily electrostatic: it can carry a transverse ($\nabla \times \mathcal{E} \neq 0$) component.

The free functions in Eq. (21) should be determined by extremization of the action functional associated with the Lagrangian density (19). A straightforward calculation gives

$$\begin{aligned} L_2 \equiv \int \mathcal{L}_{ad,sc}\, dx\, dy \; &= \; -N \left[\dot{\Theta} + \sigma^2 \dot{k} + \frac{2c^2}{v_{Fe}^2} k^2\sigma^2 + \frac{1}{2}\left(\frac{c^2}{v_{Fe}^2} - \frac{N}{2\pi}\right)\frac{1}{\sigma^2}\right. \\ &+ \; \left. 8\Gamma k^2 + 16\Gamma k^4\sigma^4 + \left(1 + \frac{N}{2\pi}\right)\frac{\Gamma}{\sigma^4}\right], \end{aligned} \tag{22}$$

where only the main quantum contributions are retained. Now L_2 is the Lagrangian for a mechanical system, after the spatial form of the envelope electric field was defined in advance via Eq. (21). Of special interest is the behavior of the dispersion σ. For a collapsing solution one could expect that σ goes to zero in a finite time. The phase Θ and the chirp function

k should be regarded as auxiliary fields. Notice that L_2 is not dependent on the angle ϕ, which remains arbitrary as far as the variational method is concerned.

Applying the functional derivative of L_2 with respect to Θ, we obtain

$$\frac{\delta L_2}{\delta \Theta} = 0 \quad \rightarrow \quad \dot{N} = 0\,, \tag{23}$$

so that the variational solution preserves the number of plasmons, as expected. The remaining Euler-Lagrange equations are

$$\frac{\delta L_2}{\delta k} = 0 \quad \rightarrow \quad \sigma\dot{\sigma} \;=\; \frac{2c^2}{v_{Fe}^2}\,\sigma^2 k + 8\Gamma k + 32\Gamma\sigma^4 k^3\,, \tag{24}$$

$$\begin{aligned}\frac{\delta L_2}{\delta \sigma} = 0 \quad \rightarrow \quad \sigma\dot{k} \;=\; & -\frac{2c^2}{v_{Fe}^2}\,k^2\sigma + \frac{1}{2}\left(\frac{c^2}{v_{Fe}^2} - \frac{N}{2\pi}\right)\frac{1}{\sigma^3} - 32\Gamma k^4\sigma^3 \\ & + \left(1 + \frac{N}{2\pi}\right)\frac{2\Gamma}{\sigma^5}\,.\end{aligned} \tag{25}$$

The exact solution of the nonlinear system (24–25) is difficult to obtain, but at least the dynamics was reduced to ordinary differential equations.

It is instructive to analyze the purely classical ($\Gamma \equiv 0$) case first. This is specially true, since to our knowledge the Rayleigh-Ritz method was not applied to the vector NLS equation (18), even for classical systems. The reason can be due to the calculational complexity induced by the transverse term. When $\Gamma = 0$, Eq. (24) gives $k = v_{Fe}^2\dot{\sigma}/2c^2\sigma$. Inserting this in Eq. (25) we have

$$\ddot{\sigma} = -\frac{\partial V_{2c}}{\partial \sigma}\,, \tag{26}$$

where the pseudo-potential V_{2c} is

$$V_{2c} = \frac{c^2}{2v_{Fe}^2}\left(\frac{c^2}{v_{Fe}^2} - \frac{N}{2\pi}\right)\frac{1}{\sigma^2}\,. \tag{27}$$

From Eq. (27) it is evident that the repulsive character of the pseudo-potential will be converted into an attractive one, whenever the number of plasmons exceeds a threshold,

$$N > \frac{2\pi c^2}{v_{Fe}^2}\,, \tag{28}$$

a condition for Langmuir wave packet collapse in the classical two-dimensional case. The interpretation of the result is as follows. When the number of plasmons satisfy Eq. (28), the refractive $\sim |\mathcal{E}|^4$ term dominates over the dispersive terms in the Lagrangian density (19), producing a singularity in a finite time. Finally, notice the ballistic motion when $N = 2\pi c^2/v_{Fe}^2$, which can also lead to singularity.

Further insight follows after evaluating the energy integral (20) with the *Ansatz* (21), which gives, after eliminating k,

$$\mathcal{H}_{ad,sc,2c} = \frac{N v_{Fe}^2}{c^2} \left[\frac{\dot{\sigma}^2}{2} + V_{2c} \right] \quad (\Gamma \equiv 0) \,. \tag{29}$$

Of course, this energy first integral could be obtained directly from Eq. (26). However, the plausibility of the variational solution is reinforced, since Eq. (29) shows that it preserves the exact constant of motion $\mathcal{H}_{ad,sc}$. In addition, in the attractive (collapsing) case the energy (29) is not bounded from bellow.

In the quantum ($\Gamma \neq 0$) case, Eq. (24) becomes a cubic equation in k, whose exact solution is too cumbersome to be of practical use. It is better to proceed by successive approximations, taking into account that the quantum and electromagnetic terms are small. In this way, one arrives at

$$\ddot{\sigma} = -\frac{\partial V_2}{\partial \sigma} \,, \tag{30}$$

where the pseudo-potential V_2 is

$$V_2 = \frac{c^2}{2 v_{Fe}^2} \left(\frac{c^2}{v_{Fe}^2} - \frac{N}{2\pi} \right) \frac{1}{\sigma^2} + \frac{\Gamma c^2}{v_{Fe}^2} \left(1 + \frac{N}{2\pi} \right) \frac{1}{\sigma^4} \,. \tag{31}$$

Now, even if the threshold (28) is exceeded, the repulsive $\sim \sigma^{-4}$ quantum term in V_2 will prevent singularities. This adds quantum diffraction as another physical mechanism, besides dissipation and Landau damping, so that collapsing Langmuir wave packets are avoided in vector NLS equation. Also, similar to Eq. (29), it can be shown that the approximate dynamics preserves the energy integral, even in the quantum case. Indeed, calculating from Eq. (20) and the variational solution gives $\mathcal{H}_{ad,sc}$ as

$$\mathcal{H}_{ad,sc,2} = \frac{N v_{Fe}^2}{c^2} \left[\frac{\dot{\sigma}^2}{2} + V_2 \right] \quad (\Gamma \geq 0) \,. \tag{32}$$

From Eq. (30), obviously $\dot{\mathcal{H}}_{ad,sc,2} = 0$.

It should be noticed that oscillations of purely quantum nature are obtained when the number of plasmons exceeds the threshold (28). Indeed, in this case the pseudo-potential V_2 in Eq. (31) assumes a potential well form as shown in Figure 1, which clearly admits oscillations around a minimum $\sigma = \sigma_m$. Here,

$$\sigma_m = 2 \left[\frac{\Gamma(1 + N/2\pi)}{N/2\pi - c^2/v_{Fe}^2} \right]^{1/2} \,. \tag{33}$$

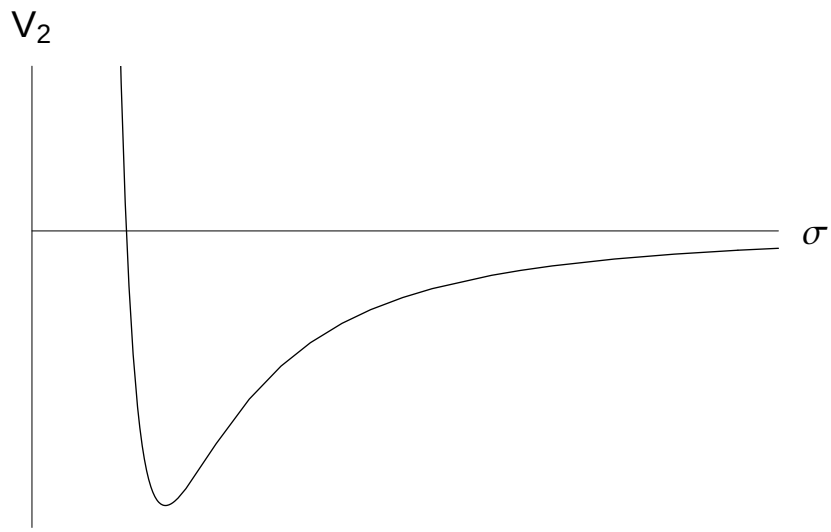

FIG. 1: The qualitative form of the pseudo-potential in Eq. (31) for $N > 2\pi c^2/v_{Fe}^2$.

Also, the minimum value of V_2 is

$$V_2(\sigma_m) = -\frac{c^2}{16\Gamma\, v_{Fe}^2}\,\frac{(N/2\pi - c^2/v_{Fe}^2)^2}{1+N/2\pi} > -\frac{1}{16\Gamma}\left(\frac{N}{2\pi} - \frac{c^2}{v_{Fe}^2}\right)^2 , \tag{34}$$

the last inequality follows since Eq. (28) is assumed. Therefore, a deepest potential well is obtained when N is increasing. Also, for too large quantum effects the trapping of the localized electric field in this potential well would be difficult, since $V_2(\sigma_m) \to 0_-$ as Γ increases. This is due to the dispersive nature of the quantum corrections.

The frequency ω of the small amplitude oscillations is derived linearizing Eq. (30) around the equilibrium point (33). Restoring physical coordinates via Eq. (5) this frequency is calculated as

$$\begin{aligned} \omega &= \frac{c}{\sqrt{2}\, v_{Fe}}\left(\frac{\kappa_B T_{Fe}}{\hbar\,\omega_{pe}}\right)^2 \frac{(N/2\pi - c^2/v_{Fe}^2)^{3/2}}{1+N/2\pi}\,\omega_{pe} \\ &< \frac{v_{Fe}}{\sqrt{2}\, c}\left(\frac{\kappa_B T_{Fe}}{\hbar\,\omega_{pe}}\right)^2 \left(\frac{N}{2\pi} - \frac{c^2}{v_{Fe}^2}\right)^{3/2} \omega_{pe}\,. \end{aligned} \tag{35}$$

To conclude, the variational solution suggests that the extra dispersion arising from the quantum terms would inhibit the collapse of Langmuir wave packets in two-spatial-dimensions. Moreover, for sufficient electric field energy (which is proportional to N), instead of collapse there will be oscillations of the width of the localized solution, due to the competition between the classical refraction and the quantum diffraction. The frequency of

linear oscillations is then given by Eq. (35). The emergence of a pulsating Langmuir envelope is a qualitatively new phenomena, which could be tested quantitatively in experiments.

V. VARIATIONAL SOLUTION IN THREE-DIMENSIONS

It is worth to study the dynamics of localized solutions for the vector NLS equation (18) in fully three-dimensional space. For this purpose, we consider the Gaussian form

$$\mathcal{E} = \left(\frac{N}{(\sqrt{\pi}\,\sigma)^3}\right)^{1/2} \exp\left[-\frac{r^2}{2\sigma^2} + i(\Theta + k\,r^2)\right] (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)\,, \tag{36}$$

where $\sigma, k, \Theta, \theta$ and ϕ are real functions of time and $r = \sqrt{x^2+y^2+z^2}$, applying the Rayleigh-Ritz method just like in the last Section. The normalization condition (14) is automatically satisfied with Eq. (36), which, occasionally, can also support a transverse ($\nabla \times \mathcal{E} \neq 0$) part.

Proceeding as before, the Lagrangian

$$\begin{aligned} L_3 \equiv \int \mathcal{L}_{ad,sc}\, d\mathbf{r} \;=\; & -N\Big[\dot{\Theta} + \frac{3}{2}\,\sigma^2\dot{k} + \frac{4\,c^2}{v_{Fe}^2}\,k^2\sigma^2 + \frac{c^2}{v_{Fe}^2\,\sigma^2} - \frac{N}{4\sqrt{2}\,\pi^{3/2}\,\sigma^3} \\ & + \; 10\,\Gamma k^2 + 20\,\Gamma k^4\sigma^4 + \frac{5\,\Gamma}{4\,\sigma^4} + \frac{3\,\Gamma N}{4\sqrt{2}\,\pi^{3/2}\,\sigma^5}\Big] \end{aligned} \tag{37}$$

is derived. In comparison to the reduced 2D-Lagrangian in Eq. (22), there are different numerical factors as well as qualitative changes due to higher-order nonlinearities. Also, the angular variables θ and ϕ don't appear in L_3.

The main remaining task is to analyze the dynamics of the width σ as a function of time. This is achieved from the Euler-Lagrange equations for the action functional associated to L_3. As before, $\delta L_3/\delta\Theta = 0$ gives $\dot{N} = 0$, a consistency test satisfied by the variational solution. The other functional derivatives yield

$$\frac{\delta L_3}{\delta k} = 0 \rightarrow \sigma\dot{\sigma} \;=\; \frac{4k}{3}\left[\frac{2\,c^2}{v_{Fe}^2}\,\sigma^2 + 5\Gamma\,(1+4k^2\sigma^4)\right]\,, \tag{38}$$

$$\begin{aligned} \frac{\delta L_3}{\delta \sigma} = 0 \rightarrow \sigma\dot{k} \;=\; & \frac{1}{3}\Big[-\frac{8\,c^2}{v_{Fe}^2}\,k^2\sigma + \frac{2\,c^2}{v_{Fe}^2\sigma^3} - \frac{3\,N}{4\,\sqrt{2}\,\pi^{3/2}\,\sigma^4} \\ & - \; 80\,\Gamma k^4\sigma^3 + \frac{5\,\Gamma}{\sigma^5} + \frac{15\,\Gamma N}{4\,\sqrt{2}\,\pi^{3/2}\,\sigma^6}\Big]\,. \end{aligned} \tag{39}$$

In the formal classical limit ($\Gamma \equiv 0$), and using Eq. (38) to eliminate k, we obtain

$$\ddot{\sigma} = -\frac{\partial V_{3c}}{\partial \sigma}\,, \tag{40}$$

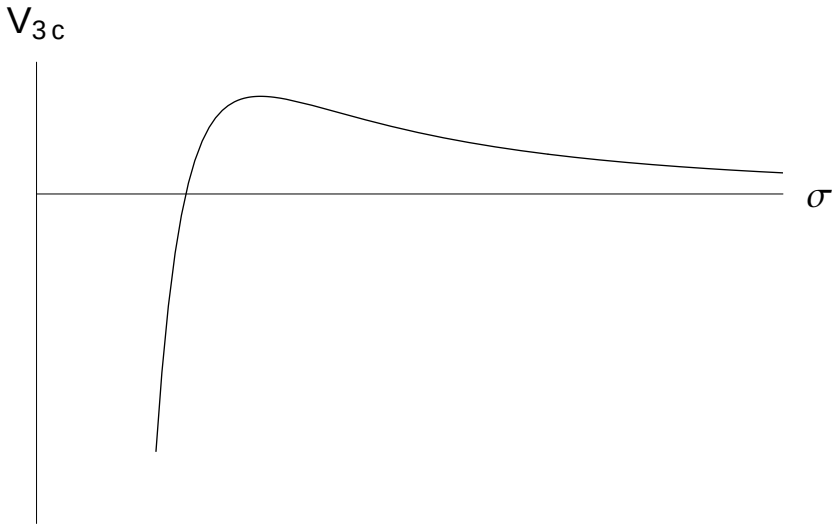

FIG. 2: The qualitative form of the pseudo-potential V_{3c} in Eq. (41).

where now the pseudo-potential V_{3c} is

$$V_{3c} = \frac{c^2}{v_{Fe}^2} \left(\frac{8\,c^2}{9\,v_{Fe}^2\,\sigma^2} - \frac{2\,N}{9\,\sqrt{2}\,\pi^{3/2}\,\sigma^3} \right) . \tag{41}$$

The form (41) shows a generic singular behavior, since the attractive $\sim \sigma^{-3}$ term will dominate for sufficiently small σ, irrespective of the value of N. Hence, in fully three-dimensional space there is more "room" for a collapsing dynamics. Figure 2 shows the qualitative form of V_{3c}, attaining a maximum at $\sigma = \sigma_M$, where

$$\sigma_M = \frac{3\,v_F^2\,N}{8\,\sqrt{2}\,\pi^{3/2}\,c^2} . \tag{42}$$

By Eq. (39) and using successive approximations in the parameter Γ to eliminate k via Eq. (38), we obtain

$$\ddot{\sigma} = -\frac{\partial V_3}{\partial \sigma} , \tag{43}$$

where

$$V_3 = \frac{8\,c^2}{3\,v_{Fe}^2} \left[\frac{c^2}{3\,v_{Fe}^2\,\sigma^2} - \frac{N}{12\,\sqrt{2}\,\pi^{3/2}\,\sigma^3} + \frac{5\,\Gamma}{12\,\sigma^4} + \frac{\Gamma\,N}{4\,\sqrt{2}\,\pi^{3/2}\,\sigma^5} \right] . \tag{44}$$

The quantum terms are repulsive and prevent collapse, since they dominate for sufficiently small σ. Moreover, when $\Gamma \neq 0$ an oscillatory behavior is possible, provided a certain condition, to be explained in the following, is meet.

To examine the possibility of oscillations, consider $V_3'(\sigma) = 0$, the equation for the critical points of V_3. Under the rescaling $s = \sigma/\sigma_M$, where σ_M (defined in Eq. (42)) is the maximum

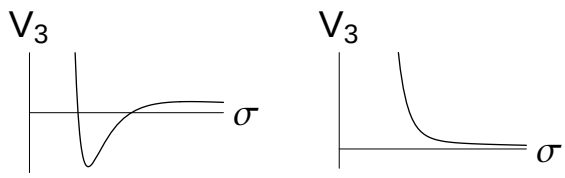

FIG. 3: The qualitative form of the pseudo-potential V_3 in Eq. (44) for $g < 1$ (on the left) and $g > 1$ (on the right).

of the purely classical pseudo-potential, the equation for the critical points read

$$V_3' = 0 \quad \rightarrow \quad s^3 - s^2 + \frac{4\,g}{27} = 0\,, \tag{45}$$

where

$$g = \frac{480\,\pi^3\,\Gamma\,c^4}{N^2\,v_{Fe}^4} \tag{46}$$

is a new dimensionless parameter. In deriving Eq. (45), it was omitted a term negligible except if $s \sim c^2/v_{Fe}^2$, which is unlikely.

The quantity g plays a decisive rôle on the shape of V_3. Indeed, calculating the discriminant shows that the solutions to the cubic in Eq. (45) are as follows: (a) $g < 1 \rightarrow$ three distinct real roots (one negative and two positive); (b) $g = 1 \rightarrow$ one negative root, one (positive) double root; (c) $g > 1 \rightarrow$ one (negative) real root, two complex conjugate roots. Therefore, $g < 1$ is the condition for the existence of a potential well, which can support oscillations. This is shown in Figure 3. The analytic formulae for the solutions of the cubic in Eq. (45) are cumbersome and will be omitted.

Restoring physical coordinates, the necessary condition for oscillations is rewritten as

$$g < 1 \quad \rightarrow \quad \frac{\varepsilon_0}{2} \int |\tilde{\mathbf{E}}|^2\, d\mathbf{r} > \frac{\sqrt{30\pi}}{\gamma}\, m_e\, v_{Fe}\, c\,, \tag{47}$$

where $\gamma = e^2/4\,\pi\varepsilon_0\,\hbar\,c \simeq 1/137$ is the fine structure constant. From Eq. (47) it is seen that for sufficient electrostatic energy the width σ of the localized envelope field can show oscillations, supported by the competition between classical refraction and quantum diffraction. Also, due to the Fermi pressure, for large particle densities the inequality (47) becomes more difficult to be met, since $v_{Fe} \sim n_0^{1/3}$. For example, when $n_0 \sim 10^{36}\,m^{-3}$ (white dwarf), the right-hand-side of Eq. (47) is 0.6 GeV. For $n_0 \sim 10^{33}\,m^{-3}$ (the next generation intense laser-solid density plasma experiments), it is 57.5 MeV.

Finally, notice that $\mathcal{H}_{ad,sc}$ from Eq. (20), evaluated with the variational solution (36), is proportional to $\dot{\sigma}^2/2 + V_3$, which is a constant of motion for Eq. (43). Therefore, the approximate solution preserves one of the basic first integrals of the vector NLS equation (18), as it should be.

VI. CONCLUSION

In this paper, the quantum Zakharov system in fully three-dimensional space has been derived. An associated Lagrangian structure was found, as well as the pertinent conservation laws. From the Lagrangian formalism, many possibilities are opened. Here, the variational description was used to analyze the behavior of localized envelope electric fields of Gaussian shape, in both two- and three-space dimensions. It was shown that the quantum corrections induce qualitative and quantitative changes, inhibiting singularities and allowing for oscillations of the width of the Langmuir envelope field. This new dynamics can be tested in experiments. In particular, the rôle of the parameter g and the inequality in Eq. (47) should be investigated. However, the variational method was applied only for the adiabatic and semiclassical case, which allows to derive the quantum modified vector NLS equation (18). Other, more general, scenarios for the solutions of the fully three-dimensional quantum Zakharov system are also worth to study, with numerical and real experiments.

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