Nonlinear ion-acoustic solitons in a magnetized quantum plasma with arbitrary degeneracy of electrons

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Abstract

Nonlinear ion-acoustic waves are analyzed in a non-relativistic magnetized quantum plasma with arbitrary degeneracy of electrons. Quantum statistics is taken into account by means of the equation of state for ideal fermions at arbitrary temperature. Quantum diffraction is described by a modified Bohm potential consistent with finite temperature quantum kinetic theory in the long wavelength limit. The dispersion relation of the obliquely propagating electrostatic waves in magnetized quantum plasma with arbitrary degeneracy of electrons is obtained. Using the reductive perturbation method, the corresponding Zakharov-Kuznetsov equation is derived, describing obliquely propagating two-dimensional ion-acoustic solitons in a magnetized quantum plasma with degenerate electrons having arbitrary electron temperature. It is found that in the dilute plasma case only electrostatic potential hump structures are possible, while in dense quantum plasma in principle both hump and dip soliton structures are obtainable, depending on the electron plasma density and its temperature. The results are validated by comparison with the quantum hydrodynamic model including electron inertia and magnetization effects. Suitable physical parameters for observations are identified.

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I. INTRODUCTION

The ion-acoustic wave, which is the fundamental low frequency mode of plasma physics, is a prime focus of many current studies of localized electrostatic disturbances in laboratory, space and astrophysical plasmas. The study of ion-acoustic waves has also gained its importance in quantum plasmas to understand electrostatic wave propagation in microscopic scales. During the last decade, there has been a renewed interest to study collective wave phenomenon in quantum plasma, motivated by applications in semiconductors [1], high intensity laser-plasma experiments [2–4] and high density astrophysical plasmas such as in the interior of massive planets and white dwarfs, neutron stars or magnetars [5–7]. The quantum or degeneracy effects appears in plasmas when the de Broglie wavelength associated with the charged carriers becomes of the order of the inter-particle distances. The quantum effects in plasmas are more frequently due to electrons, which are lighter than ions, and it includes both Pauli's exclusion principle (for half spin particles) and Heisenberg's uncertainty principle, due to wave like nature of the particles.

Quantum ion-acoustic waves in unmagnetized dense plasma have been investigated using quantum hydrodynamic models [8]. In quantum hydrodynamics, the momentum equation for degenerate electrons contains a pressure term compatible with a Fermi-Dirac distribution function, while the Bohm potential term is included to account for quantum diffraction [9–11]. Later on the quantum hydrodynamics model for plasmas was extended to include magnetic fields, with the associated quantum magnetohydrodynamics theory developed and discussed in connection to astrophysical dense plasmas [12]. Quantum Trivelpiece-Gould modes in a dense magnetized quantum plasma were derived [13]. The exchange effects on low frequency excitations in plasma have been discussed [14], using a modified Vlasov equation incorporating the exchange interaction [15, 16].

The Zakharov-Kuznetsov (ZK) equation was derived in 1974, to study nonlinear propagation of ion-acoustic waves in magnetized plasmas [17]-[19]. The ZK equation is a multi-dimensional extension of the well-known Korteweg-de Vries equation for studying solitons (or single pulse structures). In the degenerate magnetized plasma case, the cold Fermi electron gas assumption has been applied in the derivation of the appropriate ZK equation [20–22], restricted to the fully degenerate case of negligible thermodynamic temperature in comparison to the Fermi temperature.

Linear ion-acoustic and electron Langmuir waves in a plasma with arbitrary degeneracy of electrons were studied using quantum kinetic theory [23]. The nonlinear theory of the isothermal ion-acoustic waves in degenerate unmagnetized electron plasmas was investigated [24]. The ranges of the phase velocities of the periodic ion-acoustic waves and the soliton speed were determined in degenerate plasma, but ignoring quantum diffraction effects. Also nonlinear Langmuir waves in a dense plasma with arbitrary degeneracy of electrons in the absence as well as in the presence of quantum diffraction effects in the model have been studied [25]. Eliasson and Shukla [26] derived certain nonlinear quantum electron fluid equation by taking into account the moments of the Wigner equation and using the Fermi-Dirac distribution function for electrons with arbitrary temperature. The relativistic description of localized wavepackets in electrostatic plasma [27] as well as the associated ZK equation for dense relativistic plasma [28] were obtained, in the limit of negligible thermodynamic temperature. Recently, the hydrodynamic equations for ion-acoustic excitations in electrostatic quantum plasma with arbitrary degeneracy were put forward [29]. The purpose of the present communication is to achieve a notable generalization of this work, obtaining the corresponding fluid theory for quantum magnetized ion-acoustic waves (MIAWs), both in the linear and nonlinear realms. The extension has a definite interest since magnetized degenerate plasmas are ubiquitous in astrophysics as well as in laboratory [30]. Specifically, it is of fundamental interest to access the nonlinear aspects of quantum MIAWs, which is a more accessible trend using hydrodynamic methods.

The manuscript is organized in the following way. In Section II, the set of dynamic equations or studying ion-acoustic waves in magnetized quantum plasmas with arbitrary degeneracy of electrons is presented. In Section III, the dispersion relation of the obliquely propagating electrostatic linear waves in magnetized quantum plasma with arbitrary degeneracy of electrons is obtained. The limiting cases of waves parallel or perpendicular to the the magnetic field are discussed, as well as the strongly magnetized ions limit. Section IV describes the modifications of the linear dispersion relation due to the inclusion of electron inertia and magnetization effects. Section V shows that the fluid theory is the limit case of quantum kinetic theory in the long wavelength limit, as it should be. In Section VI, using reductive perturbation methods, the ZK equation for two dimensional propagation of nonlinear ion-acoustic waves is derived for a magnetized degenerate electrons plasma with arbitrary temperature. The soliton solution is presented. Section VII illustrates the results, using suitable plasma parameters for observations, within the applicability range of the model. Section VIII contains the summary of the conclusions. Finally, Appendix A has a more detailed derivation of the static electronic response from quantum kinetic theory, necessary in Section VI.

II. DYNAMIC EQUATIONS FOR MAGNETIZED QUANTUM FLUIDS

Consider a quantum electron-ion plasma with arbitrary degeneracy of electrons, embedded in an external magnetic field $\mathbf{B}_0 = B_0\hat{x}$ directed along the x-axis. In principle, the electrostatic wave is assumed to propagate obliquely to the external magnetic field in the xy-plane i.e., $\nabla = (\partial_x, \partial_y, 0)$. In order to study the quantum MIAWs the ions are taken to be inertial, while electrons are assumed to be inertialess. The set of dynamic equations for MIAWs in a quantum plasma with arbitrary degeneracy of electrons is described as follows.

The ion continuity equation is given by

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_{ix}) + \frac{\partial}{\partial y}(n_i u_{iy}) = 0\,, \tag{1}$$

while the ion momentum equations in component form are

$$\frac{\partial u_{ix}}{\partial t} + \left(u_{ix}\frac{\partial}{\partial x} + u_{iy}\frac{\partial}{\partial y}\right)u_{ix} = -\frac{e}{m_i}\frac{\partial\phi}{\partial x}\,, \tag{2}$$

$$\frac{\partial u_{iy}}{\partial t} + \left(u_{ix}\frac{\partial}{\partial x} + u_{iy}\frac{\partial}{\partial y}\right)u_{iy} = -\frac{e}{m_i}\frac{\partial\phi}{\partial y} + \omega_{ci}u_{iz}\,, \tag{3}$$

$$\frac{\partial u_{iz}}{\partial t} + \left(u_{ix}\frac{\partial}{\partial x} + u_{iy}\frac{\partial}{\partial y}\right)u_{iz} = -\omega_{ci}u_{iy}\,. \tag{4}$$

The momentum equation for the inertialess quantum electron fluid is

$$0 = -\frac{\nabla p}{n_e} + e\nabla\phi + \frac{\alpha\hbar^2}{6m_e}\nabla\left[\frac{1}{\sqrt{n_e}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sqrt{n_e}\right]\,. \tag{5}$$

The Poisson equation is written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \frac{e}{\varepsilon_0}(n_e - n_i), \tag{6}$$

where ϕ is the electrostatic potential. The ion fluid density and velocity are represented by n_i and $\mathbf{u}_i = (u_{ix}, u_{iy}, u_{iz})$ respectively, while n_e is the electron fluid density. Also, m_e and m_i are the electron and ion masses, $-e$ is the electronic charge, ε_0 is the vacuum permittivity, $\hbar$ is the reduced Planck's constant and $\omega_{ci} = eB_0/m_i$ is the ion cyclotron frequency. In equilibrium, we have $n_{e0} = n_{i0} \equiv n_0$. The electron's fluid pressure $p = p(n_e)$ is specified by a barotropic equation of state which is given below. The last term on the right hand side of the momentum equation (5) for electrons is the quantum force, which arises from the Bohm potential, giving rise to quantum diffraction or tunneling effects due to the wave like nature of the charged particles. The dimensionless quantity α is selected in order to fit the kinetic linear dispersion relation in the long wavelength limit, in a Fermi-Dirac equilibrium, as shown in the continuation. Quantum effects on ions are ignored in view of their large mass in comparison to electrons. In addition, temperature effects on ions are disregarded. Finally, to avoid too much complexity and to focus on the interplay between degeneracy and quantum recoil, exchange effects are also ignored.

The equation of state can be obtained from the moments of a local Fermi-Dirac distribution function [29, 31] of an ideal Fermi gas and reads

$$p = \frac{n_e}{\beta} \frac{\mathrm{Li}_{5/2}(-e^{\beta\mu})}{\mathrm{Li}_{3/2}(-e^{\beta\mu})}\,, \tag{7}$$

where $\beta = (\kappa_B T)^{-1}$, κ_B is the Boltzmann constant, T is the temperature and μ is the chemical potential, satisfying

$$n_e = n_0 \frac{\mathrm{Li}_{3/2}(-e^{\beta\mu})}{\mathrm{Li}_{3/2}(-e^{\beta\mu_0})}\,. \tag{8}$$

The equilibrium chemical potential μ_0 is related to the equilibrium density n_0 through

$$-\frac{n_0}{\mathrm{Li}_{3/2}(-e^{\beta\mu_0})}\left(\frac{\beta m_e}{2\pi}\right)^{3/2} = 2\left(\frac{m_e}{2\pi\hbar}\right)^3 = \mathcal{A}\,, \tag{9}$$

where the quantity $\mathcal{A}$ was defined for later convenience.

Equations (7) and (8) contain the polylogarithm function $\mathrm{Li}_\nu(-z)$ with index ν, which for $\nu > 0$ can be defined [32] as

$$\mathrm{Li}_\nu(-z) = -\frac{1}{\Gamma(\nu)}\int_0^\infty \frac{s^{\nu-1}}{1+e^s/z}ds\,, \quad \nu > 0 \tag{10}$$

where $\Gamma(\nu)$ is the gamma function. For $\nu < 0$ one applies

$$\mathrm{Li}_\nu(-z) = \left(z\frac{\partial}{\partial z}\right)\mathrm{Li}_{\nu+1}(-z) \tag{11}$$

as many times as necessary, where $\nu + 1 > 0$.

The numerical coefficient α appearing in the Bohm potential term in Eq. (5) has been derived from finite-temperature quantum kinetic theory for low-frequency electrostatic excitations in [29],

$$\alpha = \frac{\mathrm{Li}_{3/2}(-e^{\beta\mu_0})\,\mathrm{Li}_{-1/2}(-e^{\beta\mu_0})}{[\mathrm{Li}_{1/2}(-e^{\beta\mu_0})]^2}, \tag{12}$$

expressed as a function of the equilibrium fugacity $z = \exp(\beta\mu_0)$. The treatment of [29] considered non-magnetized plasmas, but in Section V it is proved that Eq. (12) applies to MIAWs too. As discussed in [29], in the classical limit ($z \ll 1$) one has $\alpha \approx 1$, while in the full degenerate limit ($z \gg 1$) one has $\alpha \approx 1/3$. The same behavior is seen for α as a function of the chemical potential, as depicted in Fig. 1, showing a transition zone from classical to dense regimes.

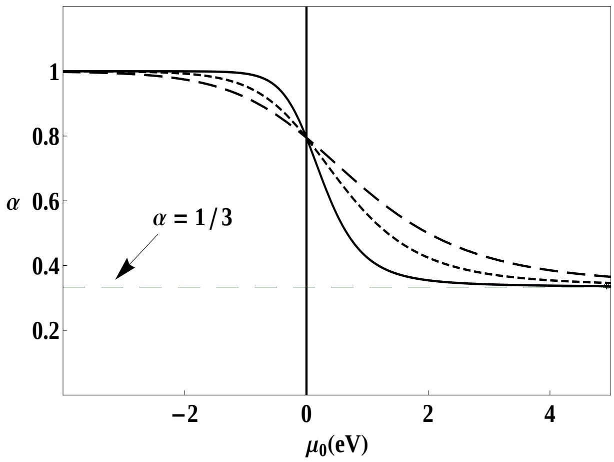

FIG. 1: Coefficient α from Eq. (12), as a function of the chemical potential μ_0. Solid line: $T = 4000\,\mathrm{K}$. Doted line: $T = 6000\,\mathrm{K}$. Dashed line: $T = 8000\,\mathrm{K}$.

It happens [33] that the finite-temperature quantum hydrodynamic equations using α from Eq. (12) are consistent with the results from orbital free density functional theory [34, 35].

III. LINEAR WAVES

In order to find the dispersion relation for electrostatic wave in a magnetized quantum plasma with arbitrary degeneracy of electrons, we linearize the system of equations (1)-(6) by considering

$$
\begin{aligned}
n_i &= n_0 + n_{i1}\,, \quad n_e = n_0 + n_{e1}\,, \quad u_{ix} = u_{ix1}\,, \\
u_{iy} &= u_{iy1}\,, \quad u_{iz} = u_{iz1}\,, \quad \phi = \phi_1\,,
\end{aligned} \tag{13}
$$

inducing a correction $\mu = \mu_0 + \mu_1$, where the subscript 1 denotes the first order quantities. In particular, using the expansion of the polylogarithm function to first order, i.e.,

$$\mathrm{Li}_\nu(-e^{\beta(\mu_0+\mu_1)}) = \mathrm{Li}_\nu(-e^{\beta\mu_0}) + \beta\mu_1 \mathrm{Li}_{\nu-1}(-e^{\beta\mu_0})\,, \tag{14}$$

and considering plane wave perturbations $\sim \exp[i(k_x x + k_y y - \omega t)]$, the result is

$$1 + \chi_i(\omega, \mathbf{k}) + \chi_e(0, \mathbf{k}) = 0\,, \tag{15}$$

where the ionic and electronic susceptibilities are respectively given by

$$\chi_i(\omega, \mathbf{k}) = -\frac{\omega_{pi}^2(\omega^2 - \omega_{ci}^2\cos^2\theta)}{\omega^2(\omega^2 - \omega_{ci}^2)}\,, \tag{16}$$

$$\chi_e(0, \mathbf{k}) = \omega_{pe}^2\left[\frac{1}{m_e}\left(\frac{dp}{dn_e}\right)_0 k^2 + \frac{\alpha\hbar^2k^4}{12m_e^2}\right]^{-1}\,, \tag{17}$$

where $\omega_{pj}^2 = n_0e^2/(m_j\varepsilon_0)$ for $j = i, e$ and $\mathbf{k} = k\,(\cos\theta, \sin\theta, 0)$. Due to the neglect of electrons inertia, only the static electronic susceptibility $\chi_e(0, \mathbf{k})$ is necessary. There is no loss of generality in assuming waves in the xy plane, due to the cylindrical geometry around the $x-$axis.

The dispersion relation (15) develops as a quadratic equation for ω^2 whose solution is

$$\omega^2 = \frac{1}{2}\Big[\omega_0^2 + \omega_{ci}^2 \pm \left((\omega_0^2 + \omega_{ci}^2)^2 - 4\omega_0^2\omega_{ci}^2\cos^2\theta\right)^{1/2}\Big]\,, \tag{18}$$

where ω_0 was already obtained [29] in the case of unmagnetized quantum ion-acoustic waves,

$$\omega_0^2 = \frac{c_s^2k^2\,[1 + H^2(k\lambda_D)^2/4]}{1 + (k\lambda_D)^2 + H^2(k\lambda_D)^4/4}\,. \tag{19}$$

In Eq. (19) one has the ion-acoustic speed c_s which follows from

$$c_s^2 = \frac{1}{m_i}\left(\frac{dp}{dn_e}\right)_0 = \frac{\kappa_B T}{m_i}\,\frac{\mathrm{Li}_{3/2}(-e^{\beta\mu_0})}{\mathrm{Li}_{1/2}(-e^{\beta\mu_0})}\,, \tag{20}$$

the generalized electronic screening length λ_D from

$$\lambda_D^2 = \frac{c_s^2}{\omega_{pi}^2} = \frac{\kappa_B T}{m_e\omega_{pe}^2}\,\frac{\mathrm{Li}_{3/2}(-e^{\beta\mu_0})}{\mathrm{Li}_{1/2}(-e^{\beta\mu_0})}\,, \tag{21}$$

as well as the quantum diffraction parameter H specified by

$$H = \frac{\beta\hbar\omega_{pe}}{\sqrt{3}}\left(\frac{\mathrm{Li}_{-1/2}(-e^{\beta\mu_0})}{\mathrm{Li}_{3/2}(-e^{\beta\mu_0})}\right)^{1/2}\,. \tag{22}$$

In the dilute plasma limit $e^{\beta\mu_0} \ll 1$, implying $\mathrm{Li}_\nu(-e^{\beta\mu_0}) \approx -e^{\beta\mu_0}$, one has $c_s \approx \sqrt{\kappa_B T/m_i}$, $\lambda_D = \sqrt{\kappa_B T/(m_e\omega_{pe}^2)}$, which respectively are the more traditional ion-acoustic speed and Debye length, and $H \approx \beta\hbar\omega_{pe}/\sqrt{3}$. On the other hand, in the fully degenerate case $e^{\beta\mu_0} \gg 1$, using $\mathrm{Li}_\nu(-e^{\beta\mu_0}) \approx -(\beta\mu_0)^\nu/\Gamma(\nu+1)$ one has $\mu_0 \approx E_F = \hbar^2(3\pi^2 n_0)^{2/3}/(2m_e)$, which is the Fermi energy, and $c_s \approx \sqrt{(2/3)E_F/m_i}$, $\lambda_D = \sqrt{2E_F/(3m_e\omega_{pe}^2)}$, which are respectively the quantum ion-acoustic speed and the Thomas-Fermi screening length, and $H \approx (1/2)\hbar\omega_{pe}/E_F$. The dispersion relation (18) is formally the same as for classical magnetized plasma [36, 37], provided the fully quantum ion-acoustic frequency ω_0 is replaced by its purely classical counterpart.

As apparent from the dispersion relation (15), ions are responsible for providing inertia effects, while electrons are responsible for kinetic energy (arising from the standard, thermodynamic temperature and/or Fermi pressure) and quantum diffraction is represented by the parameter H. It is convenient to rewrite Eq. (22) using Eq. (9), yielding

$$H^2 = -\frac{2\alpha_F}{3}\sqrt{\frac{2\beta m_e c^2}{\pi}}\,\mathrm{Li}_{-1/2}(-e^{\beta\mu_0})\,, \tag{23}$$

where $\alpha_F = e^2/(4\pi\varepsilon_0\hbar c) \approx 1/137$ is the fine structure constant. Obviously the theory is non-relativistic, in spite of the appearance of the rest energy $m_e c^2$ in Eq. (23).

For a fixed temperature, H is a simple function of the fugacity $z = \exp(\beta\mu_0)$, as shown in Fig. 2 below. It is seen that the pure wave like quantum effects are enhanced for larger densities (and fugacities) up to $z \approx 3.03$, while for larger degeneracy the quantum statistical effects prevail showing that the quantum force becomes less effective in denser systems, in view of Pauli's exclusion principle. Therefore for dilute systems H increases with the density and decreases with temperature, while for fully degenerate systems the leading order behavior shows a decreasing of H for an increasing density.

The positive sign in Eq. (18) corresponds to fast electrostatic waves, and a negative sign corresponds to slow electrostatic waves in a magnetized plasma. The effect of degeneracy for arbitrary angle θ is not entirely straightforward to identify, due to the somehow involved expression (18). However, in practical applications it is likely to have $\omega_0 \gg \omega_{ci}$, so that the fast mode becomes $\omega^2 \approx \omega_0^2 + \omega_{ci}^2\sin^2\theta$ while the slow mode becomes $\omega^2 \approx \omega_{ci}^2\cos^2\theta$. Since quantum effects are present only on $\omega_0^2 \gg \omega_{ci}^2$, it happens that the fast wave has an angular dependence appearing as a correction, while the slow wave is strongly angle dependent, but not so influenced by quantum effects. As an example, consider hydrogen

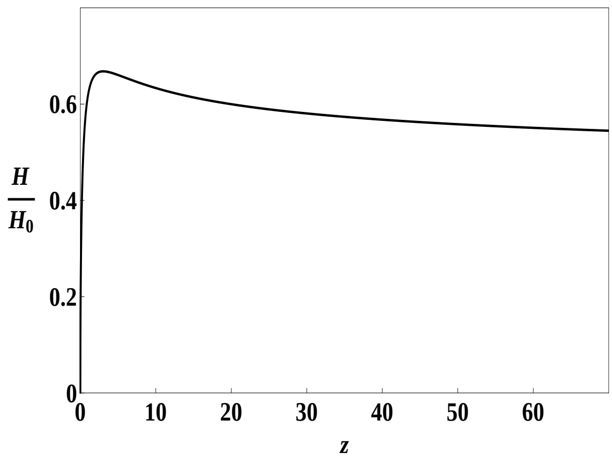

FIG. 2: Quantum diffraction parameter defined in Eq. (23), as a function of the equilibrium fugacity $z = e^{\beta\mu_0}$, and normalized to $H_0 = [(2\alpha_F/3)(2\beta m_e c^2/\pi)^{1/2}]^{1/2}$.

plasma with an ambient magnetic field $B_0 = 10^3\,\mathrm{T}$, yielding the ion cyclotron frequency $\omega_{ci} = 9.58 \times 10^{10}\,\mathrm{rad/s}$. For solid-density plasma with $T = 10^6\,\mathrm{K}$, $n_0 = 5 \times 10^{30}\,\mathrm{m}^{-3}$ and a wave-number $k = 2\pi \times 10^9\,\mathrm{m}^{-1}$, one has $\omega_0 = 6.50 \times 10^{14}\,\mathrm{rad/s} \gg \omega_{ci}$. These parameters are in accordance with the more detailed validity conditions discussed in Section VII.

In addition, some significant limiting cases are described below.

A. Wave propagation parallel to the magnetic field

Considering $\theta = 0$ in Eq. (18), one has either $\omega = \omega_{ci}$ (the ion cyclotron frequency) or $\omega = \omega_0$. The later has already been discussed in [29]. In the limit $k\lambda_D \ll 1$ one has $\omega_0 \approx c_s k$, where the ion-acoustic speed contains only quantum degeneracy effects as seen from Eq. (20).

B. Wave propagation perpendicular to the magnetic field

Considering $\theta = \pi/2$ in Eq. (18), one has a vanishing solution ($\omega^2 = 0$) as well as a quantum modified electrostatic ion cyclotron wave given by $\omega^2 = \omega_{ci}^2 + \omega_0^2$.

C. Strongly magnetized ions

For completeness we consider the strongly magnetized ions case. If $\omega_{ci} \gg \omega_0$ one has the fast mode

$$\omega^2 = \omega_{ci}^2 \left[1 + \frac{\omega_0^2}{\omega_{ci}^2} \sin^2\theta + \frac{\omega_0^4}{4\,\omega_{ci}^4} \sin^2(2\theta) + \mathcal{O}\left(\left(\frac{\omega_0}{\omega_{ci}} \right)^6 \right) \right] , \tag{24}$$

and the slow mode

$$\omega^2 = \omega_0^2 \cos^2\theta \left[1 - \frac{\omega_0^2}{\omega_{ci}^2} \sin^2\theta + \mathcal{O}\left(\left(\frac{\omega_0}{\omega_{ci}} \right)^4 \right) \right] . \tag{25}$$

IV. ELECTRON INERTIA AND MAGNETIZATION EFFECTS

In the purely classical case, the conditions of applicability of the model are well-known [18, 19]. In the quantum case, it is interesting to include electron inertia and magnetization effects, to measure the limitations of Eq. (15). In this context one adds to Eqs. (1)-(4) and (6) the electron continuity equation

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_{ex}) + \frac{\partial}{\partial y}(n_e u_{ey}) = 0\,, \tag{26}$$

and replace Eq. (5) by

$$\begin{aligned} m_e\Big(\frac{\partial \mathbf{u}_e}{\partial t} \; + \; \mathbf{u}_e \cdot \nabla \mathbf{u_e}\Big) &= -\frac{\nabla p}{n_e} - e\Big(-\nabla\phi + \mathbf{u}_e \times \mathbf{B}_0\Big) \\ &+ \; \frac{\alpha \hbar^2}{6 m_e} \nabla \Big[\frac{1}{\sqrt{n_e}} \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Big) \sqrt{n_e} \Big] \,, \end{aligned} \tag{27}$$

where $\mathbf{u}_e = (u_{ex}, u_{ey}, u_{ez})$ is the electron fluid velocity. Proceeding as in Section III and also supposing linear perturbations where $\mathbf{u}_e = \mathbf{u}_{e1}$, the result is

$$1 + \chi_i(\omega, \mathbf{k}) + \chi_e(\omega, \mathbf{k}) = 0\,, \tag{28}$$

where the ionic susceptibility is still given by Eq. (16) and

$$\chi_e(\omega, \mathbf{k}) = -\frac{\omega_{pe}^2(\omega^2 - \omega_{ce}^2 \cos^2\theta)}{\omega^4 - (k^2 v_T^2(k) + \omega_{ce}^2)\omega^2 + k^2 v_T^2(k) \omega_{ce}^2 \cos^2\theta} \,, \tag{29}$$

where $\omega_{ce} = eB_0/m_e$ is the electron cyclotron frequency and

$$v_T^2(k) = \frac{1}{m_e} \left(\frac{dp}{dn_e} \right)_0 + \frac{\alpha \hbar^2 k^2}{12 m_e^2} \,. \tag{30}$$

It is not the purpose of this work to develop the full consequences of the dispersion relation (28), but it is useful to observe that in the formal limit $\omega \to 0$ the electron response (29) regains the static electronic response $\chi_e(0, \mathbf{k})$ given by Eq. (17). Moreover, by inspection of Eq. (29) it is found that such a limit is attended for a warm electron fluid, where $k^2 v_T^2(k) \gg \omega_{ce}^2$ so that the electrons magnetization could be disregarded, and $k^2 v_T^2(k) \gg \omega^2$, which is attainable for low frequency excitations. In addition, notice that $v_T(k)$ from Eq. (30) depends not only on pressure but also on quantum diffraction effects. On the other hand, ions are assumed to be cold and non-quantum enough.

V. COMPARISON TO KINETIC THEORY

The results from hydrodynamics should agree with kinetic theory in the long wavelength limit. Therefore it is necessary to compare the ionic and electronic responses found from kinetic theory, to the susceptibilities shown in Eqs. (16) and (17). Since ions are safely assumed as classical in most cases, their particle distribution function $f_i = f_i(\mathbf{r}, \mathbf{v}, t)$ satisfy Vlasov's equation, which presently is

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{e}{m_i}(-\nabla\phi + \mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial}{\partial \mathbf{v}}\right] f_i = 0 \,. \tag{31}$$

On the other hand, the quantum nature of electrons deserves the use of the quantum Vlasov equation satisfied by the electronic Wigner quasi-distribution $f_e = f_e(\mathbf{r}, \mathbf{v}, t)$,

$$\begin{aligned} \frac{\partial f_e}{\partial t} &+ \mathbf{v} \cdot \nabla f_e - \frac{ie}{\hbar}\left(\frac{m_e}{2\pi\hbar}\right)^3 \times \\ &\times \int d\mathbf{s}\, d\mathbf{v}' \exp\left(\frac{i m_e (\mathbf{v}' - \mathbf{v}) \cdot \mathbf{s}}{\hbar}\right) \times \\ &\times \left[\phi\left(\mathbf{r} + \frac{\mathbf{s}}{2}, t\right) - \phi\left(\mathbf{r} - \frac{\mathbf{s}}{2}, t\right)\right] f_e(\mathbf{r}, \mathbf{v}', t) = 0 \,. \end{aligned} \tag{32}$$

All integrals run from $-\infty$ to ∞, unless otherwise stated. Moreover, under the same assumption as before, namely large electron thermal and quantum (statistical and diffraction) effects, the magnetic force on electrons was omitted in Eq. (32).

The scalar potential is self-consistently determined by Poisson's equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \frac{e}{\varepsilon_0}\left(\int f_e \, d\mathbf{v} - \int f_i \, d\mathbf{v}\right) \,, \tag{33}$$

where we are taking spatial variations in the xy-plane only.

Proceeding as in Section III, assuming plane wave perturbations $\sim \exp[i(k_x x + k_y y - \omega t)]$ around isotropic in velocities equilibria, the dispersion relation $1 + \chi_i(\omega, \mathbf{k}) + \chi_e(\omega, \mathbf{k}) = 0$ is easily derived. Disregarding the negligibly small Landau damping of MIAWs in the case of cold ions, it is found [38, 39] that the ionic susceptibility from kinetic theory coincides with the fluid expression (16). On the other hand, for low frequency waves, the static limit $\chi_e(0, \mathbf{k})$ is sufficient for electrons, reading

$$\chi_e(0, \mathbf{k}) = \frac{e^2}{\varepsilon_0 \hbar k^2} \int \frac{d\mathbf{v}}{\mathbf{k} \cdot \mathbf{v}} \left[F\left(\mathbf{v} - \frac{\hbar \mathbf{k}}{2m_e}\right) - F\left(\mathbf{v} + \frac{\hbar \mathbf{k}}{2m_e}\right) \right], \tag{34}$$

where the principal value of the integral is understood if necessary and where the equilibrium electronic Wigner function is $f_e = F(\mathbf{v})$.

Consider a Fermi-Dirac equilibrium,

$$F(\mathbf{v}) = \frac{\mathcal{A}}{1 + e^{\beta(m_e v^2/2 - \mu_0)}}, \quad v = |\mathbf{v}|, \tag{35}$$

where the normalization constant $\mathcal{A}$ is given in Eq. (9), assuring that $\int F(\mathbf{v})\, d\mathbf{v} = n_0$.

It turns out that the right-hand side of Eq. (34) can be evaluated as a power series of the quantum recoil $q = \sqrt{\beta/(2m_e)}\, \hbar k/2$, supposed to be a small quantity for long wavelengths and/or sufficiently large electronic temperature:

$$\begin{aligned}
\chi_e(0, \mathbf{k}) &= \frac{\beta m_e \omega_{pe}^2}{\sqrt{\pi}\, \mathrm{Li}_{3/2}(-z) k^2} \Big[\Gamma\left(\frac{1}{2}\right) \mathrm{Li}_{1/2}(-z) + \\
&+ \Gamma\left(-\frac{1}{2}\right) \mathrm{Li}_{-1/2}(-z) \frac{q^2}{3} \\
&+ \Gamma\left(-\frac{3}{2}\right) \mathrm{Li}_{-3/2}(-z) \frac{q^4}{5} + \ldots \Big] \\
&= \frac{\beta m_e \omega_{pe}^2}{\sqrt{\pi}\, \mathrm{Li}_{3/2}(-z) k^2} \sum_{j=0}^{\infty} \Gamma\left(\frac{1}{2} - j\right) \times \\
&\times \mathrm{Li}_{1/2-j}(-z) \frac{q^{2j}}{2j+1},
\end{aligned} \tag{36}$$

where $z = e^{\beta \mu_0}$. The derivation is detailed in the Appendix A. The expression (36) is exact, as long as the series converges. Moreover, it coincides with the static limit of Eq. (29) of [40], where only the leading $\mathcal{O}(q^2)$ quantum recoil correction was calculated.

For the sake of comparison, the hydrodynamic result from Eq. (17) can be rewritten as

$$\begin{aligned}\chi_e(0,\mathbf{k}) &= \frac{\beta m_e \omega_{pe}^2 \mathrm{Li}_{1/2}(-z)}{\mathrm{Li}_{3/2}(-z)k^2}\left(1+\frac{2q^2}{3}\frac{\mathrm{Li}_{-1/2}(-z)}{\mathrm{Li}_{1/2}(-z)}\right)^{-1} \\ &= \frac{\beta m_e \omega_{pe}^2}{\mathrm{Li}_{3/2}(-z)k^2}\Big(\mathrm{Li}_{1/2}(-z) - \frac{2q^2}{3}\mathrm{Li}_{-1/2}(-z) \\ &+ \mathcal{O}(q^4)\Big)\,, \end{aligned} \tag{37}$$

which coincides with Eq. (36) in the long wavelength limit, in view of $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(-1/2) = -2\sqrt{\pi}$. This completes the justification of α in Eq. (12) in the magnetized case.

VI. ZAKHAROV-KUZNETSOV EQUATION FOR ARBITRARY DEGENERACY

In order to derive the ZK equation for obliquely propagating MIAWs in arbitrary degenerate plasma, it is convenient to make use of normalized quantities. The dispersion relation (18) suggests the use of the dimensionless variables $(\tilde{x},\tilde{y}) = (x,y)/\lambda_D$, $\tilde{t} = \omega_{pi} t$, $(\tilde{u}_{ix},\tilde{u}_{iy},\tilde{u}_{iz}) = (u_{ix},u_{iy},u_{iz})/c_s$ as well as $\tilde{\phi} = e\phi/(m_i c_s^2)$, $\tilde{n}_j = n_j/n_0$, where $j = e,i$. Equations (1)-(6) are then written as

$$\frac{\partial \tilde{n}_i}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}(\tilde{n}_i \tilde{u}_{ix}) + \frac{\partial}{\partial \tilde{y}}(\tilde{n}_i \tilde{u}_{iy}) = 0\,, \tag{38}$$

$$\frac{\partial \tilde{u}_{ix}}{\partial \tilde{t}} + \left(\tilde{u}_{ix}\frac{\partial}{\partial \tilde{x}} + \tilde{u}_{iy}\frac{\partial}{\partial \tilde{y}}\right)\tilde{u}_{ix} = -\frac{\partial \tilde{\phi}}{\partial \tilde{x}}\,, \tag{39}$$

$$\frac{\partial \tilde{u}_{iy}}{\partial \tilde{t}} + \left(\tilde{u}_{ix}\frac{\partial}{\partial \tilde{x}} + \tilde{u}_{iy}\frac{\partial}{\partial \tilde{y}}\right)\tilde{u}_{iy} = -\frac{\partial \tilde{\phi}}{\partial \tilde{y}} + \Omega \tilde{u}_{iz}\,, \tag{40}$$

$$\frac{\partial \tilde{u}_{iz}}{\partial \tilde{t}} + \left(\tilde{u}_{ix}\frac{\partial}{\partial \tilde{x}} + \tilde{u}_{iy}\frac{\partial}{\partial \tilde{y}}\right)\tilde{u}_{iz} = -\Omega \tilde{u}_{iy}\,, \tag{41}$$

$$\begin{aligned} 0 &= \tilde{\nabla}\tilde{\phi} - \frac{\mathrm{Li}_{1/2}(-e^{\beta\mu_0})}{\mathrm{Li}_{1/2}(-e^{\beta\mu})}\tilde{\nabla}\tilde{n}_e \\ &+ \frac{H^2}{2}\tilde{\nabla}\left[\frac{1}{\sqrt{\tilde{n}_e}}\left(\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2}\right)\sqrt{\tilde{n}_e}\right]\,, \end{aligned} \tag{42}$$

$$\left(\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2}\right)\tilde{\phi} = \tilde{n}_e - \tilde{n}_i\,, \tag{43}$$

where $\Omega = \omega_{ci}/\omega_{pi}$ has been defined and where $\tilde{\nabla} = (\partial/\partial\tilde{x}, \partial/\partial\tilde{y}, 0)$, while Eq. (8) becomes

$$\tilde{n}_e = \frac{\mathrm{Li}_{3/2}(-e^{\beta\mu})}{\mathrm{Li}_{3/2}(-e^{\beta\mu_0})}\,. \tag{44}$$

In the following calculations, for brevity the tilde sign used for defining normalized quantities will be omitted.

In order to find a nonlinear evolution equation describing the magnetized plasma, the stretching of the independent variables x, y and t is defined under the assumption of strong magnetization as follows [41–44],

$$X = \varepsilon^{1/2}(x - V_0 t)\,, \quad Y = \varepsilon^{1/2} y\,, \quad \tau = \varepsilon^{3/2} t\,, \tag{45}$$

where ε is a formal small expansion parameter and V_0 is the phase velocity of the wave, to be determined later on. The perturbed quantities can be expanded in powers of ε as follows,

$$n_j = 1 + \varepsilon n_{j1} + \varepsilon^2 n_{j2} + \ldots\,, \quad j = e, i\,, \tag{46}$$

$$u_{ix} = \varepsilon u_{x1} + \varepsilon^2 u_{x2} + \varepsilon^3 u_{x3} \ldots\,, \tag{47}$$

$$u_{i\perp} = \varepsilon^{3/2} u_{\perp 1} + \varepsilon^2 u_{\perp 2} + \varepsilon^{5/2} u_{\perp 3} \ldots\,, \quad \perp = y, z\,, \tag{48}$$

$$\phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \ldots\,, \tag{49}$$

$$\mu = \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \ldots \tag{50}$$

In the present model, the ion velocity components (u_{iy}, u_{iz}) in the perpendicular to the magnetic field directions are taken as higher order perturbations compared to the parallel component u_{ix} since in the presence of a strong magnetic field, the plasma is anisotropic so that the ion gyro-motion becomes a higher order effect.

The lowest ε order terms ($\sim \varepsilon^{3/2}$) from the set of equations (38)-(42) give

$$-V_0 \frac{\partial n_{i1}}{\partial X} + \frac{\partial u_{x1}}{\partial X} = 0\,, \tag{51}$$

$$V_0 \frac{\partial u_{x1}}{\partial X} = \frac{\partial \phi_1}{\partial X}\,, \tag{52}$$

$$u_{z1} = \frac{1}{\Omega} \frac{\partial \phi_1}{\partial Y}\,, \tag{53}$$

$$-\Omega u_{y1} = 0\,, \tag{54}$$

$$\frac{\partial \phi_1}{\partial X} = \frac{\partial n_{e1}}{\partial X}\,. \tag{55}$$

The velocity u_{z1} appears in Eq. (53) due to the $E \times B$ drift.

The lowest ε order terms ($\sim \varepsilon$) from equations (43) and (44) give

$$n_{i1} = n_{e1} = \frac{\mathrm{Li}_{1/2}(-e^{\beta\mu_0})}{\mathrm{Li}_{3/2}(-e^{\beta\mu_0})}\,\beta\mu_1\,. \tag{56}$$

Solving the system (51)-(56), we get $V_0 = \pm 1$. We set $V_0 = 1$ (the normalized phase velocity of the MIAW) without loss of generality.

Collecting the next higher order terms of the ion continuity ($\sim \varepsilon^{5/2}$) and of the X , Y and Z components of the ion momentum equations ($\sim \varepsilon^{5/2}, \varepsilon^2, \varepsilon^2$), and after a rearrangement, we find

$$\frac{\partial n_{i1}}{\partial \tau} - \frac{\partial n_{i2}}{\partial X} + \frac{\partial u_{x2}}{\partial X} + \frac{\partial}{\partial X}(n_{i1}u_{x1}) + \frac{\partial u_{y2}}{\partial Y} = 0\,, \tag{57}$$

$$\frac{\partial u_{x1}}{\partial \tau} - \frac{\partial u_{x2}}{\partial X} + u_{x1}\frac{\partial}{\partial X}u_{x1} = -\frac{\partial \phi_2}{\partial X}\,, \tag{58}$$

$$-\frac{\partial u_{y1}}{\partial X} = \Omega u_{z2}\,, \tag{59}$$

$$\frac{\partial u_{z1}}{\partial X} = \Omega u_{y2}\,. \tag{60}$$

Using Eq. (56) and the next higher order terms $\sim \varepsilon^{5/2}$ from the equations of motion of the inertialess degenerate electrons in the X and Y directions, we get

$$\frac{\partial n_{e2}}{\partial X} = \frac{\partial \phi_2}{\partial X} + \alpha n_{e1}\frac{\partial n_{e1}}{\partial X} + \frac{H^2}{4}\left(\frac{\partial^3}{\partial X^3} + \frac{\partial}{\partial X}\frac{\partial^2}{\partial Y^2}\right)n_{e1} \tag{61}$$

and

$$\frac{\partial n_{e2}}{\partial Y} = \frac{\partial \phi_2}{\partial Y} + \alpha n_{e1}\frac{\partial n_{e1}}{\partial Y} + \frac{H^2}{4}\left(\frac{\partial}{\partial Y}\frac{\partial^2}{\partial X^2} + \frac{\partial^3}{\partial Y^3}\right)n_{e1}, \tag{62}$$

where α has been defined in equation (12).

Now collecting the ε^2 order terms from Poisson's equation, we have

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)\phi_1 = n_{e2} - n_{i2}\,, \tag{63}$$

while the next higher terms from Eqs. (57), (58) and (60) give

$$\begin{aligned}\frac{\partial n_{i2}}{\partial X} &= \frac{\partial n_{i1}}{\partial \tau} + \frac{\partial}{\partial X}(n_{i1}u_{x1}) + \frac{1}{\Omega}\frac{\partial}{\partial Y}\left(\frac{\partial u_{z1}}{\partial X}\right) \\ &\quad + \frac{\partial u_{x1}}{\partial \tau} + u_{x1}\frac{\partial u_{x1}}{\partial X} + \frac{\partial \phi_2}{\partial X}\,.\end{aligned} \tag{64}$$

Differentiating Eq. (63) with respect to X and using Eqs. (61) and (64) together with $n_{i1} = n_{e1} = u_{x1} = \phi_1$, $u_{z1} = (1/\Omega)\,\partial\phi_1/\partial Y$, it is finally possible to write the ZK equation for obliquely propagating quantum MIAWs in terms of $\phi_1 \equiv \varphi$,

$$\frac{\partial \varphi}{\partial \tau} + A\varphi\frac{\partial \varphi}{\partial X} + \frac{\partial}{\partial X}\left(B\frac{\partial^2 \varphi}{\partial X^2} + C\frac{\partial^2 \varphi}{\partial Y^2}\right) = 0\,. \tag{65}$$

The nonlinearity coefficient A and the dispersion coefficients B and C in the parallel and perpendicular directions of the magnetic field, respectively, are defined as

$$A = \frac{1}{2}\,(3-\alpha)\;, \tag{66}$$

$$B = \frac{1}{2}\left(1 - \frac{H^2}{4}\right)\;, \tag{67}$$

$$C = \frac{1}{2}\left(1 + \frac{1}{\Omega^2} - \frac{H^2}{4}\right)\;. \tag{68}$$

In the purely classical limit the nonlinearity and dispersion coefficients become $A = 1$, $B = 1/2$ and $C = (1/2)\,(1 + 1/\Omega^2)$ in agreement with Refs. [17, 43, 44] treating MIAWs in a classical electron-ion plasma. In addition, the ZK equation for fully degenerate plasma will have $A = 4/3$ and $H = (1/2)\,\hbar\omega_{pe}/E_F$ in the coefficients B, C. The associated fully degenerate ZK equation does not matches the results from Refs. [20–22], after comparison using physical (dimensional and non-stretched) coordinates, restricted to the case of electron-ion plasmas. Note that the ZK equations from the previous works do not match the purely classical result.

Provided $l_x^2 B + l_y^2 C \neq 0$, the soliton solution of the ZK equation (65) for obliquely propagating MIAWs is given by

$$\varphi = \varphi_0\,\mathrm{sech}^2(\eta/W)\,, \tag{69}$$

where $\varphi_0 = 3u_0/(Al_x)$ is the height and where $W = \sqrt{4l_x(l_x^2 B + l_y^2 C)/u_0}$ is the width of the soliton in terms of the stretched coordinates. The polarity of the soliton depends on the sign of φ_0. The transformed coordinate η in the co-moving frame is defined as $\eta = l_x X + l_y Y - u_0\tau$, where $u_0 \neq 0$ is the speed of the nonlinear pulse and where $l_x > 0$ and l_y are direction cosines, so that $l_x^2 + l_y^2 = 1$. To obtain localized structures, decaying boundary conditions as $\eta \to \pm\infty$ were applied. Following the habitual usage the terminology "soliton" is applied to the solitary wave (69), although the ZK equation does not belong to the class of completely integrable evolution equations. The dispersion effects arising from the combination of charge separation and finite ion Larmor radius balances the nonlinearity in the system to form the soliton.

Defining $\delta V = \varepsilon u_0/l_x$, in the laboratory frame the solution reads

$$\begin{aligned}\varphi = \frac{3\,\delta V}{A}\,\mathrm{sech}^2\Bigg\{&\frac{1}{2}\left(\frac{\delta V}{l_x^2 B + l_y^2 C}\right)^{1/2} \times \\ &\times\Big[l_x\Big(x - (V_0 + \delta V)\,t\Big) + l_y y\Big]\Bigg\}\,.\end{aligned} \tag{70}$$

It is apparent that $V_0 + \delta V$ corresponds to the velocity at which travels the intersection between a plane of constant phase and a field line, down the same field line [42].

From Eq. (70) the width L of the soliton in the laboratory frame is

$$L = 2\left(\frac{l_x^2 B + l_y^2 C}{\delta V}\right)^{1/2} = \sqrt{\frac{2}{\delta V}}\left(1 - \frac{H^2}{4} + \frac{l_y^2}{\Omega^2}\right)^{1/2} \tag{71}$$

From this expression one concludes that while in the non-quantum $H = 0$ limit necessarily $\delta V > 0$ (bright soliton propagating at supersonic speed), in the quantum case one has the theoretical possibility of $\delta V < 0$ (dark soliton propagating at subsonic speed), provided $H^2/4 > 1 + l_y^2/\Omega^2$.

It is interesting to single out the different quantum effects in the soliton (70). The amplitude/dip of bright/dark solitons is inversely proportional to A, which depends only on quantum degeneracy and not on quantum diffraction. More degenerate systems produce a smaller α - as seen e.g. from Fig. 1 - and hence a bigger A. Therefore the soliton amplitude/dip decreases with the degeneracy. On the other hand, from the width (71) one find the dependence on the quantum diffraction parameter H^2 (which is also dependent on the degeneracy degree) shown in Fig. 3 below. For semiclassical bright solitons ($\delta V > 0$) one has L^2 decreasing with quantum effects, while for quantum, dark solitons ($\delta V < 0$) one has L^2 increasing with quantum effects.

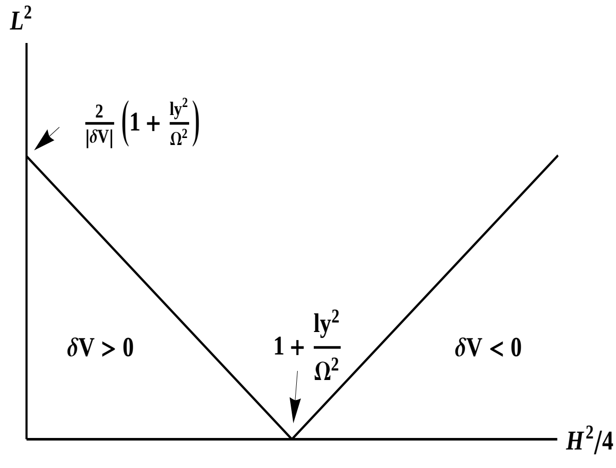

FIG. 3: Squared width of the localized structure as shown in Eq. (71), as a function of the quantum diffraction parameter H^2.

In unmagnetized quantum plasmas, by construction the term $\sim 1/\Omega^2$ does not appears. In this case when $H = 2$ the corresponding Korteweg-de Vries (KdV) equation collapses to the Burger's equation, producing an ion-acoustic shock wave structure instead of a soliton [8].

In the magnetized case, a further possibility happens when $C = 0$, or, equivalently, $1 + 1/\Omega^2 = H^2/4$, which is not allowed in the classical limit ($H \equiv 0$). When $C = 0$, from Eq. (65) one has

$$\frac{\partial\varphi}{\partial\tau} + A\varphi\frac{\partial\varphi}{\partial X} - \frac{1}{2\Omega^2}\frac{\partial^3\varphi}{\partial X^3} = 0\,, \tag{72}$$

which transforms to the KdV equation in its standard form by means of $\varphi \to -\varphi, X \to -X, \tau \to \tau$. Therefore, in this particular situation the problem becomes completely integrable.

Finally, if $B = 0$, it means that $1 - H^2/4 = 0$ due to which $C = 1/2\Omega^2$, so that Eq. (65) becomes

$$\frac{\partial\varphi}{\partial\tau} + A\varphi\frac{\partial\varphi}{\partial X} + \frac{1}{2\Omega^2}\frac{\partial}{\partial X}\frac{\partial^2\varphi}{\partial Y^2} = 0, \tag{73}$$

This is a KdV-like equation having perpendicular to the magnetic field dispersion effects, due to the obliquely propagating MIAW.

Regarding the ranges of validity of the parameters A, B and C in Eqs. (66–68), first we observe that from Eq. (12) one has $1/3 < \alpha < 1$, so that $1 < A < 4/3$. In addition, since H^2 from Eq. (23) can in principle attain any non-negative value, B and C are not positive definite. However, strictly speaking, very large values of H^2 are associated with strongly coupling effects which can have a large impact on soliton propagation, to be addressed in a separate extended theory. Only in such generalized framework one could be able to make more precise statements on pure quantum soliton existence or non-existence. Nevertheless, significant values of H^2 are certainly physically acceptable, as found from the present treatment. These issues are best discussed in the following Section.

VII. APPLICATIONS

It is important to discuss the validity domain of the general theory devised in the last Sections. Moreover, it is highly desirable to offer precise physical parameters where the predicted linear and nonlinear waves could be searched in practice. Obviously, the theory is more relevant in the intermediate regimes, where the thermal and Fermi temperatures are not significantly different. Otherwise, the fully degenerate or dilute limits could be sufficiently accurate. Therefore, in this Section frequently we assume

$$T = T_F \,, \tag{74}$$

where $T_F = E_F/\kappa_B$ is the electrons Fermi temperature.

To start, consider the normalization condition (9), which can be expressed [26] as

$$\mathrm{Li}_{3/2}(-z) = -\frac{4}{3\sqrt{\pi}}\,(\beta E_F)^{3/2} \,, \tag{75}$$

where $z = \exp(\beta\mu_0)$. For equal thermal and Fermi temperatures, $\beta E_F = 1$, which from Eq. (75) gives the equilibrium fugacity $z = 0.98$ and from Eq. (12) a parameter $\alpha = 0.80$, definitely in the intermediate dilute-degenerate situation as explicitly seen e.g. in Fig. 1.

Besides quantum degeneracy, quantum diffraction effects can also provide qualitatively new aspects as found e.g. in the extra dispersion of linear waves in Section III and the modified width of the solitons in Section VI. Therefore it would be interesting to investigate systems with a large parameter H. However, realistically speaking it is not possible to increase H without limits, which would enter the strongly coupled plasma regime, not included in the present formalism. For instance, the ideal Fermi gas equation of state for electrons would be unappropriated. Therefore, it is necessary to analyze the coupling parameter g for electrons, which can be defined [45] as $g = l/a$, where

$$l = -\frac{e^2}{12\pi\varepsilon_0\kappa_B T}\,\frac{n_0\Lambda_T^3}{\mathrm{Li}_{5/2}(-z)} \tag{76}$$

is a generalized Landau length involving the thermal de Broglie wavelength $\Lambda_T = [2\pi\hbar^2/(m_e\kappa_B T)]^{1/2}$, and $a = (4\pi n_0/3)^{-1/3}$ is the Wigner-Seitz radius. In the dilute case, one has $e^2/(4\pi\varepsilon_0\, l) = (3/2)\kappa_B T$, so that l would be the classical distance of closest approach in a binary collision, for average kinetic energy. The general expression (76) accounts for the degeneracy effects on the mean kinetic energy. A few calculations yield

$$g = -\frac{2\alpha_F\,\sqrt{2\beta m_e c^2}}{3\,(3\sqrt{\pi})^{1/3}}\,\frac{[\mathrm{Li}_{3/2}^2(-z)]^{2/3}}{\mathrm{Li}_{5/2}(-z)} \,, \tag{77}$$

an expression similar to the one for H^2 in Eq. (23). Hence, it is legitimate to suspect that the indiscriminate increase of quantum diffraction gives rise to nonideality effects such as dynamical screening and bound states [45]. Incidentally Eq. (77) agrees with Eq. (16) of [29], found from related but different methods.

The resemblance between the coupling and quantum diffraction parameters is confirmed in Fig. 4 below, which can be compared to Fig. 2 for H^2. Moreover, it is apparent in Fig. 5, that we have $H^2 < 0.5$ for the whole span of degeneracy regimes, as far as $g < 1$. Strictly speaking, the dark soliton, shock wave and the completely integrable case associated to the KdV equation (72) are therefore outside the validity domain of the model, since they need large values of quantum diffraction parameter H. Nevertheless, the influence of the wave nature of the electrons can still provide important corrections by its own, at least for reasonable values of H^2, as is obvious, for instance, in the width of the ZK soliton in Eq. (71).

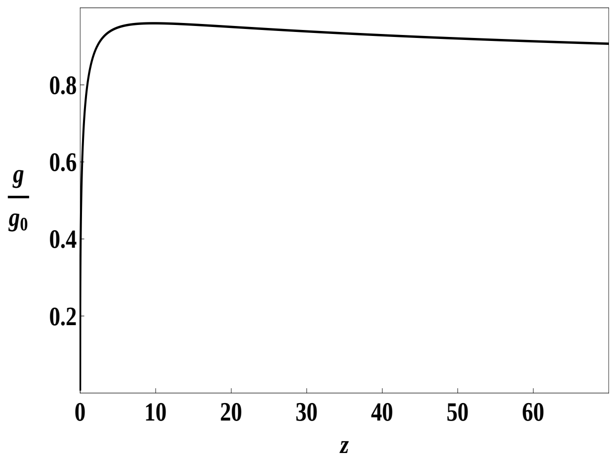

FIG. 4: Coupling parameter defined in Eq. (77), as a function of the equilibrium fugacity $z = e^{\beta\mu_0}$, and normalized to $g_0 = 2\alpha_F\sqrt{2\beta m_e c^2}/[3(3\sqrt{\pi})^{1/3}]$.

The behavior of the parameters g, H^2 can be summarized in Fig. 6, for $T = T_F$ and considering hydrogen plasma parameters. We observe that $g < 1$ for $n_0 > 5.23 \times 10^{28}\,\mathrm{m}^{-3}$, or $E_F > 5.11\,\mathrm{eV}$, which starts becoming realizable for typical densities in solid-density plasmas [46, 47].

It is necessary to discuss additional points about the validity conditions of the model. Both the static electronic response and long wavelength (and hence fluid) assumptions are collected in Eq. (27) of [29], reproduced here for convenience,

$$k_{\min} \equiv \frac{2\sqrt{3}m_e c_s}{\hbar} \ll k \ll \frac{\omega_{pi}}{c_s} \equiv k_{\max}\,. \tag{78}$$

For hydrogen plasma and $T = T_F$, from Eq. (78) one has $k_{\max} > k_{\min}$ for $n_0 < 9.81 \times 10^{35}\,\mathrm{m}^{-3}$. The later condition is safely satisfied for non-relativistic plasma. At such high

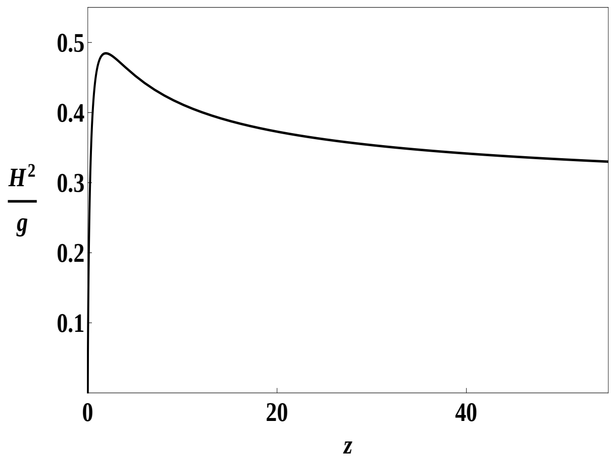

FIG. 5: Ratio between the diffraction parameter H^2 from Eq. (23) and the coupling parameter g from Eq. (77), as a function of the equilibrium fugacity $z = e^{\beta\mu_0}$.

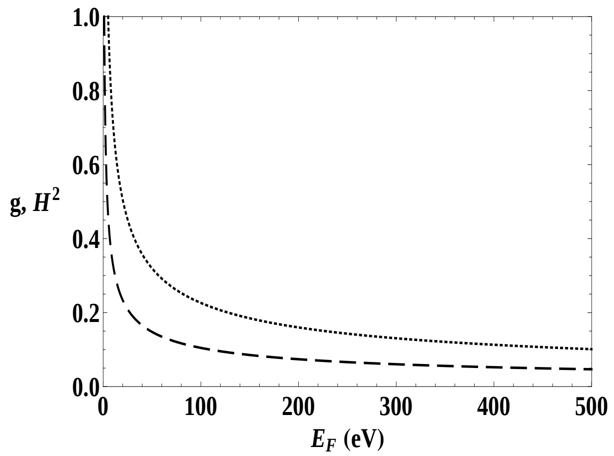

FIG. 6: Upper, dotted curve: coupling parameter g from Eq. (77); lower, dashed curve: quantum diffraction parameter H^2 from Eq. (23), as a function of the electronic Fermi energy E_F in eV, for hydrogen plasma and the intermediate dilute-degenerate regime where $T = T_F$.

densities available in compact astrophysical objects like white dwarfs and neutron stars, the Fermi momentum becomes comparable to $m_e c$, asking for a relativistic treatment. Strictly speaking, the first inequality in Eq. (78) could be removed, but in this case the quantum recoil would be less significant. In this case, only quantum degeneracy effects would be relevant.

In terms of the wavelength λ, Eq. (78) yields a suitable range $\lambda_{\rm min} = 2\pi/k_{\rm max} \ll \lambda \ll$

$\lambda_{\rm max} = 2\pi/k_{\rm min}$, shown in Fig. 7, in the nanometric scale from extreme ultraviolet to soft X-rays.

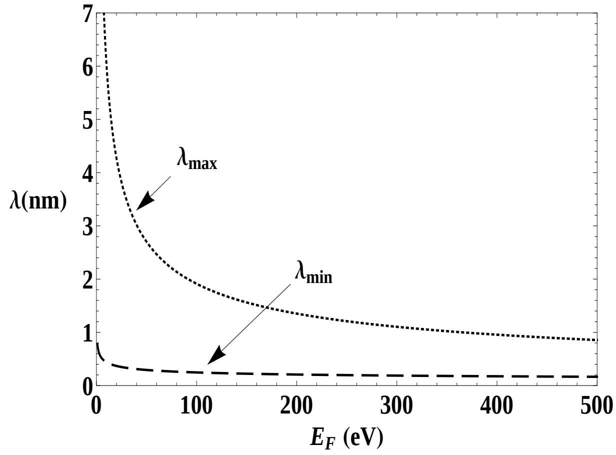

FIG. 7: Upper, dotted curve: maximum wavelength $\lambda_{\rm max}$; lower, dashed curve: minimal wavelength $\lambda_{\rm min}$, consistent with Eq. (78), as a function of the electronic Fermi energy E_F in eV, for hydrogen plasma and the intermediate dilute-degenerate regime where $T = T_F$.

As discussed in Section IV, the model also assumes that the pressure effects are significantly larger than magnetic field effects regarding electrons, or $k^2 v_T^2(k) \gg \omega_{ce}^2$. In the worse case where $k \approx k_{\rm min}$, the quantum diffraction is typically a correction in the expression (30) for $v_T(k)$. We then find

$$\frac{\hbar\omega_{ce}}{E_F} \ll 2\sqrt{3}\left(\frac{m_e}{m_i}\right)^{1/2} \frac{{\rm Li}_{3/2}(-z)}{{\rm Li}_{1/2}(-{\rm z})} \approx 0.10\,, \tag{79}$$

for $T = T_F$. The condition (79), which is easier to satisfy for larger densities, is safely satisfied except for very strong magnetic fields. For instance, for $E_F \approx 10\,{\rm eV}$, we just need $B_0 \ll 8.81\,{\rm kT}$.

Ions have been assumed to be cold, classical and ideal (disregarding strong ion coupling effects). Denoting T_i as the ionic temperature, when $T_F \gg T_i$ the MIAW phase speed becomes much larger than the ionic thermal speed, justifying the cold ions assumption. On the other hand, there is the need of a small ionic coupling parameter g_i. For a non-degenerate ionic fluid we then [45] have

$$g_i = \frac{e^2}{4\pi\varepsilon_0 a}\left(\frac{3\kappa_B T_i}{2}\right)^{-1} \ll 1\,. \tag{80}$$

Otherwise one would have a ionic liquid or even an ionic crystal, as is believed to happens for $g_i \approx 172$ in an one-component plasma [48]. The joint requirements of cold and non-strongly coupled ions is represented in Fig. 8 below, where the allowable region is between the straight lines $T_i = T_F, g_i = 1$. A minimal number density $n_0 = 3.63 \times 10^{28}\,\mathrm{m}^{-3}$ is found to be necessary, which again is accessible for typical solid-density plasmas.

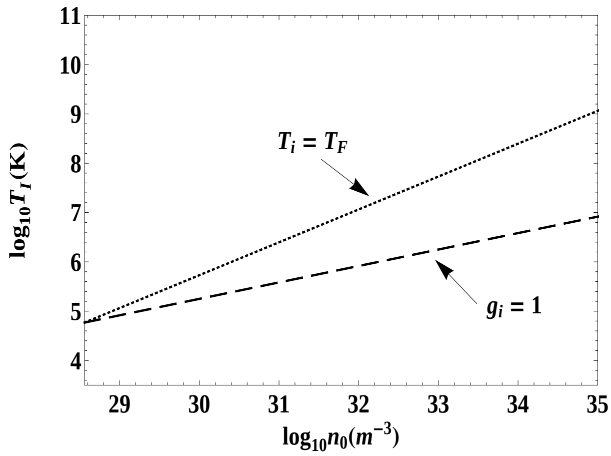

FIG. 8: Cold and weakly coupled ions are below the upper, dotted straight line (where the ionic temperature T_i equals the electronic Fermi temperature T_F) and above the lower, dashed straight line (where the ionic coupling parameter $g_i = 1$).

For the sake of illustration, consider the bright soliton solution from Eq. (70), for parameters representative of solid-density plasmas, namely, $n_0 = 5 \times 10^{30}\,\mathrm{m}^{-3}, H^2 = 0.10, g = 0.22$ and $T = T_F = 1.24 \times 10^6\,\mathrm{K}$. In this case the allowable wavenumbers satisfy $0.24\,\mathrm{nm} < \lambda < 1.85\,\mathrm{nm}$, the ionic temperatures are in the range $3.04 \times 10^5\,\mathrm{K} < T_i < 1.57 \times 10^6\,\mathrm{K}$ and the magnetic field should have a strength $B_0 < 9.23 \times 10^4\,\mathrm{T}$. The soliton is shown in Fig. 9, where $B_0 = 10^3\,\mathrm{T}, l_x = l_y = \sqrt{2}/2$ and $\delta V = 0.10$.

VIII. SUMMARY

The main results of the work are the dispersion relation (18) and the ZK equation (65), both of which describe quantum MIAWs in the linear and nonlinear regimes, respectively, allowing for arbitrary electrons degeneracy degree. The results significantly generalize the previous literature, restricted to either dilute (Maxwell-Boltzmann) or fully degenerate plas-

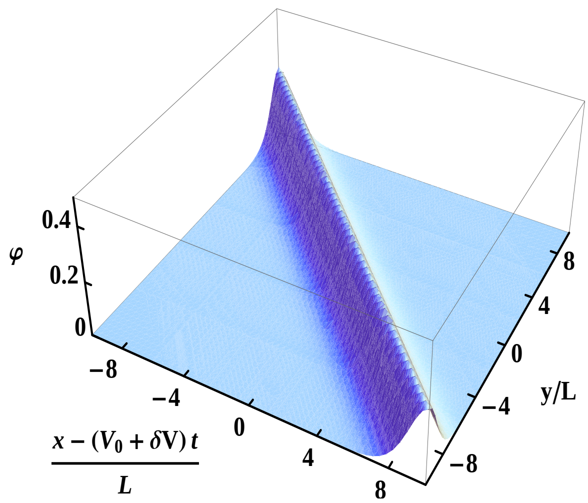

FIG. 9: The two dimensional profile of the MIAW hump soliton structure (70) moving with supersonic speed in the laboratory frame, using normalized coordinates. Parameters: $n_0 = 5 \times 10^{30}\,\mathrm{m}^{-3}, H^2 = 0.10, g = 0.22, T = T_F = 1.24 \times 10^6\,\mathrm{K}, B_0 = 10^3\,\mathrm{T}, l_x = l_y = \sqrt{2}/2$ and $\delta V = 0.10$

mas. The conditions for applications are investigated in depth, pointing the importance of the findings, for e.g. magnetized solid-density plasma. While the physical parameters were more focused on hydrogen plasma in the intermediate dilute-degenerate limit where $T = T_F$, adaptation to fully ionized electron-ion cases with atomic number $Z \neq 1$ and arbitrary temperatures is not difficult at all. With minor remarks, the whole parametric analysis of Section VII also applies to the unmagnetized case. It is hoped that the detailed assessment of physical parameters thus developed, will incentive experimental and observational verifications of linear and nonlinear quantum ion-acoustic waves, both in laboratory and nature, considering a large span of degeneracy regimes. A possibly important next step is the rigorous (non ad hoc) incorporation of exchange-correlation effects, which are beyond the reach of the present communication.

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APPENDIX A: DERIVATION OF EQ. (36)

From Eqs. (34) and (35) one has

$$
\begin{aligned}
&\chi_e(0,\mathbf{k})\\
&= \frac{\mathcal{A}e^2}{\varepsilon_0\hbar k^2}\int d\mathbf{v}\left[\frac{1}{\mathbf{k}\cdot\mathbf{v}+\hbar k^2/(2m_e)}-\frac{1}{\mathbf{k}\cdot\mathbf{v}-\hbar k^2/(2m_e)}\right]\times\\
&\times\ \frac{1}{1+\exp(\beta m_e v^2/2)/z}\\
&= \frac{2\pi\mathcal{A}e^2}{\varepsilon_0\hbar k^2}\int_{-\infty}^{\infty}dv_\parallel\left[\frac{1}{kv_\parallel+\hbar k^2/(2m_e)}-\frac{1}{kv_\parallel-\hbar k^2/(2m_e)}\right]\times\\
&\times\ \int_0^\infty dv_\perp\,\frac{v_\perp}{1+\exp[\beta m_e(v_\perp^2+v_\parallel^2)/2]/z}\,,
\end{aligned}
\tag{A1}
$$

where $\mathbf{v} = v_\parallel\mathbf{k}/k+\mathbf{v}_\perp\,,\mathbf{k}\cdot\mathbf{v}_\perp = 0$, $z = \exp(\beta\mu_0)$ and all integrals consider the principal value sense.

Performing the $v_\perp$ integral, considering the expression of $\mathcal{A}$ in Eq. (9) and applying a simple change of variables we get

$$
\begin{aligned}
\chi_e(0,\mathbf{k}) \;&=\; \frac{\beta m_e\omega_{pe}^2}{4\sqrt{\pi}\,\mathrm{Li}_{3/2}(-z)qk^2}\int_{-\infty}^{\infty}\frac{ds}{s}\times\\
&\times\;\left[\ln\left(1+ze^{-(s+q)^2}\right)-\ln\left(1+ze^{-(s-q)^2}\right)\right]\,,
\end{aligned}
\tag{A2}
$$

where $q = \sqrt{\beta/(2m_e)}\,\hbar k/2$.

Expanding in powers of the quantum recoil one has

$$
\begin{aligned}
&\ln\Big(1\;+\;ze^{-(s+q)^2}\Big)-\ln\Big(1+ze^{-(s-q)^2}\Big)\\
&\quad=\;-4q\Big(g(s)+\frac{q^2}{3!}g''(s)+\frac{q^4}{5!}g^{(iv)}(s)+\mathcal{O}(q^6)\Big)\,,
\end{aligned}
\tag{A3}
$$

where $g(s)\equiv s/[1+\exp(s^2)/z]$. At this point notice that the possible divergence at $s=0$ in the integral (A2) was explicitly removed.

After integrating by parts, it is found that

$$
\begin{aligned}
\chi_e(0,\mathbf{k}) \;=\; -&\frac{2\beta m_e\omega_{pe}^2 z}{\sqrt{\pi}\,\mathrm{Li}_{3/2}(-z)k^2}\Big(\int_0^\infty\frac{ds}{z+e^{s^2}}\\
-\;&\frac{2q^2}{3}\int_0^\infty\frac{ds\,e^{s^2}}{(z+e^{s^2})^2}\\
+\;&\frac{4q^4}{15}\int_0^\infty\frac{ds\,e^{s^2}(e^{s^2}-z)}{(z+e^{s^2})^3}+\mathcal{O}(q^6)\Big)\,.
\end{aligned}
\tag{A4}
$$

From Eqs. (10) and (11), each term on the right hand side of (A4) can be evaluated in terms of polylogarithms. For instance,

$$
\begin{aligned}
\mathrm{Li}_{1/2}(-z) &= -\frac{2z}{\sqrt{\pi}}\int_0^\infty \frac{ds}{z+e^{s^2}}\,, \\
\mathrm{Li}_{-1/2}(-z) &= -\frac{2z}{\sqrt{\pi}}\int_0^\infty \frac{ds\,e^{s^2}}{(z+e^{s^2})^2}\,, \\
\mathrm{Li}_{-3/2}(-z) &= \frac{2z}{\sqrt{\pi}}\int_0^\infty \frac{ds\,e^{s^2}(z-e^{s^2})}{(z+e^{s^2})^3}\,,
\end{aligned}
\tag{A5}
$$

clearly related to the integrals in expression (A4). In this way Eq. (36), which is also independently confirmed by numerical evaluation, is proved.

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