# *NOETHER SYMMETRIES FOR CHARGED PARTICLE MOTION UNDER A MAGNETIC MONOPOLE AND GENERAL ELECTRIC FIELDS*

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**ABSTRACT:** *The search for Noether point symmetries for non-relativistic charged particle motion is reduced to the solution for a set of two coupled, linear partial differential equations for the electromagnetic field. These equations are completely solved when the magnetic field is produced by a fixed magnetic monopole. The result is applied to central electric field cases, in particular to the time-dependent monopole-oscillator problem. As an additional example of the theory, we found all Noether point symmetries and invariants for a constant magnetic field and a time-dependent harmonic electric field with a forcing term.*
**KEY WORDS:** *Noether symmetry, constant of motion, charged particle motion, magnetic monopole.*

## 1 Introduction

The charge-monopole problem is a classical subject in Physics. In the present work, in particular, we consider symmetries and conservation laws for the Lorentz equations of the form

$$\ddot{\mathbf{r}} = -\nabla V(\mathbf{r}, t) + g\frac{\dot{\mathbf{r}} \times \mathbf{r}}{r^3}\,, \tag{1}$$

applying Noether's theorem. Here, $\mathbf{r} = (x, y, z)$ is the position vector in $R^3$. The system (1) describes three-dimensional, non-relativistic charged particle motion under an electric field $\mathbf{E} = -\nabla V(\mathbf{r}, t)$ and a fixed magnetic monopole field with strength $g$. However, the scope of our work is more general, since our formalism apply to general electromagnetic fields. We make an effort to extend the results of [3, 4], where all the Noether and Lie points symmetries for two-dimensional, non-relativistic charged particle motions were found. Unlike the planar case, the fully three-dimensional case seems to be not accessible to a complete solution. Hence, we focus mainly on the magnetic monopole field case, which is amenable to complete calculations.

Let us review the state of the art on the search for constants of motion for the charge-monopole system. In the simpler case when the electric force is central and time-independent ($V = V(r)$, $r = (x^2 + y^2 + z^2)^{1/2}$), the Lorentz equations admit the vector first integral

$$\mathbf{D} = \mathbf{r} \times \dot{\mathbf{r}} - g\hat{r}\,, \tag{2}$$

where $\hat{r}$ is the unit vector in the radial direction. The vector $\mathbf{D}$, the so-called Poincaré vector, was used by Poincaré [9] to obtain the exact solution for the motion when only the magnetic monopole is present ($V = 0$). It should be mentioned that the Poincaré vector survives as a constant of motion even if there is an explicit time-dependence of the scalar potential.

More recently [6], it were considered the scalar potentials

$$V = \frac{\omega_0^2 r^2}{2} + \frac{g^2}{2r^2} \tag{3}$$

and

$$V = -\frac{\mu_0}{r} + \frac{g^2}{2r^2}\,, \tag{4}$$

where $\omega_0$ and $\mu_0$ are numerical constants. All bounded trajectories are periodic [6] when the scalar potential is given by (3) or (4). For the potential (3), linked to the isotropic harmonic oscillator, there is a tensor conserved quantity, whose components, using complex notation, are

$$T_{ij} = (\dot{u}_i + i\omega_0 u_i)(\dot{u}_j - i\omega_0 u_j)\,, \tag{5}$$

where $\mathbf{u} = \mathbf{D} \times \hat{r}$. In the case of the potential (4), related to the Coulomb or Kepler forces, we have the vector constant of motion

$$\mathbf{F} = \mathbf{D} \times \dot{\mathbf{r}} + \mu_0 \hat{r}\,, \tag{6}$$

a generalization of the Laplace-Runge-Lenz vector for the Kepler problem. Both (5) and (6) are constants of motion in the form of quadratic functions of the velocity. It can be demonstrated [13] that (3) and (4) are among the few time-independent central potentials for which (1) has quadratic integrals other than the energy. At the quantum level, there is degeneracy of the spectra for these potentials, in connection [5] with the invariance algebra $su(2) \oplus su(2)$.

In contradistinction to these earlier works, here we consider the effects of the superposition of a non-central, time-dependent electric force on the motion of charged particles under a fixed magnetic monopole field. An immediate result of the presence of non-central electric fields is the non-conservation of the Poincaré vector (2). However, at least for particular forms of $V(\mathbf{r}, t)$, it can be expected that some conservation law is present. To address the question, we pursue here the analysis of Noether point symmetries.

As an additional example of the theory, we found all Noether point symmetries and invariants for a constant magnetic field and a time-dependent harmonic electric field with a forcing term. In this case there is a 12-parameter or a 8-parameter group of symmetries, according to a condition on the parameters specifying the electromagnetic field. Apart from it's obvious physical significance, the extra example shows how to apply the general theory of this work to situations for which the electromagnetic field has a more particular form, known in advance.

The article is structured as follows. In Section II, we investigate the Noether point symmetries for non-relativistic charged particle motion under general electromagnetic fields. The whole problem is reduced to a set of two coupled, linear partial differential equations for the electromagnetic field. In Section III, we completely solve this system of equations when the magnetic field is in the form of a fixed magnetic monopole field. Two classes of electric field are determined, related to quadratic or linear constants of motion. These electric fields can be freely superimposed to the magnetic monopole field, with no harm on the existence of Noether point symmetries. In Section IV, the previous results are applied to the central electric field cases. In particular, we study the time-dependent monopole-oscillator system. In Section V, the case of a constant magnetic field is analyzed. Unlike the magnetic monopole problem, here we will not try to find the general class of admissible electric fields. Instead, we focus only on linear electric fields, which are amenable to exact calculations. Section VI is dedicated to the conclusion.

## 2 Noether point symmetries

The necessary and sufficient condition [11] for a vector field

$$G = \tau(\mathbf{r},t)\frac{\partial}{\partial t} + \mathbf{n}(\mathbf{r},t)\cdot\frac{\partial}{\partial \mathbf{r}} \tag{7}$$

to be a generator of Noether point symmetries for the action functional

$$S[\mathbf{r}(t)] = \int_{t_0}^{t_1} L(\mathbf{r},\dot{\mathbf{r}},t)\,dt\,, \tag{8}$$

where $L(\mathbf{r},\dot{\mathbf{r}},t)$ is the Lagrange function, is the existence of a function $F(\mathbf{r},t)$ such that

$$\tau\frac{\partial L}{\partial t} + \mathbf{n}\cdot\frac{\partial L}{\partial \mathbf{r}} + (\dot{\mathbf{n}} - \dot{\tau}\dot{\mathbf{r}})\cdot\frac{\partial L}{\partial \dot{\mathbf{r}}} + \dot{\tau}\,L = \dot{F}(\mathbf{r},t)\,. \tag{9}$$

Notice that the generator $G$ in (7) does not include derivatives of the coordinates, so that dynamical symmetries are not being considered here.

Associated to the symmetries satisfying (9), there is a first integral of the form

$$I = (\frac{\partial L}{\partial \dot{\mathbf{r}}}\cdot\dot{\mathbf{r}} - L)\,\tau - \mathbf{n}\cdot\frac{\partial L}{\partial \dot{\mathbf{r}}} + F\,, \tag{10}$$

conserved along the trajectories of the Euler-Lagrange equations.

Noether's theorem is applicable to any Lagrangian system, as is the case for non-relativistic motion of a charged particle under a general electromagnetic field. Introduce vector $\mathbf{A}(\mathbf{r},t)$ and scalar $V(\mathbf{r},t)$ potentials, so that the Lagrangian is given by

$$L = \frac{1}{2}\dot{\mathbf{r}}^2 + \mathbf{A}(\mathbf{r},t)\cdot\dot{\mathbf{r}} - V(\mathbf{r},t)\,. \tag{11}$$

The corresponding electromagnetic fields are

$$\mathbf{B} = \nabla\times\mathbf{A} \quad , \quad \mathbf{E} = -\nabla V - \partial\mathbf{A}/\partial t \quad . \tag{12}$$

Inserting $L$ in the symmetry condition (9), we find a polynomial form of the velocity components. The coefficient of each monomial should be zero. Such a prescription results in a system of linear partial differential equations determining both the symmetry generator and the vector and scalar potentials. Actually, we will show in the continuation that it is possible to reduce the discussion to the electromagnetic field only, without any mention to the electromagnetic potentials.

Putting forward the calculation of Noether symmetries, the monomial of third order on the velocity gives, using component notation,

$$\frac{\partial \tau}{\partial r_i} = 0\,, \tag{13}$$

thus implying

$$\tau = \rho^2(t)\,, \tag{14}$$

where $\rho(t)$ is an arbitrary function of time. Now, the monomial of second order on the velocity imposes

$$\frac{\partial n_i}{\partial r_j} + \frac{\partial n_j}{\partial r_i} - 2\,\delta_{ij}\rho\dot{\rho} = 0\,, \tag{15}$$

where the Kronecker delta was used. The solution is

$$\mathbf{n} = \rho\dot{\rho}\mathbf{r} + \mathbf{\Omega}(t) \times \mathbf{r} + \mathbf{a}(t)\,, \tag{16}$$

where $\mathbf{\Omega}(t)$ and $\mathbf{a}(t)$ are arbitrary vector functions of time.

Assembling (14-16), we conclude that the most general form of the Noether point symmetry generator is

$$G = \rho^2(t)\frac{\partial}{\partial t} + (\rho\dot{\rho}\mathbf{r} + \mathbf{\Omega}(t) \times \mathbf{r} + \mathbf{a}(t)) \cdot \frac{\partial}{\partial \mathbf{r}}\,, \tag{17}$$

for arbitrary $\rho(t)$, $\mathbf{\Omega}(t)$ and $\mathbf{a}(t)$. The resulting symmetries include a generalized rescaling, a time-dependent rotation and a time-dependent space translation. Up to this point, there is no restriction on the electromagnetic field. Notice that (17) is the proper extension of the Noether point symmetries generator for two-dimensional non-relativistic charged particle motion derived in [3].

The remaining equations implied by the symmetry condition (9) are

$$\nabla F = G\mathbf{A} + \rho\dot{\rho}\mathbf{A} + \mathbf{A} \times \mathbf{\Omega} + \partial \mathbf{n}/\partial t\,, \tag{18}$$
$$\partial F/\partial t = -G\,V - 2\rho\dot{\rho}V + \mathbf{A} \cdot \partial \mathbf{n}/\partial t\,, \tag{19}$$

in which the form (17) was taken into account. Also, we have used the definition

$$G\,W = \rho^2(t)\frac{\partial\,W}{\partial t} + (\rho\dot{\rho}\mathbf{r} + \mathbf{\Omega}(t) \times \mathbf{r} + \mathbf{a}(t)) \cdot \frac{\partial\,W}{\partial \mathbf{r}}\,, \tag{20}$$

valid for a generic function $W = W(\mathbf{r}, t)$.

Equations (18–19) have a solution $F$ if and only if

$$\frac{\partial^2 F}{\partial r_i \partial r_j} = \frac{\partial^2 F}{\partial r_j \partial r_i} \quad , \quad \frac{\partial^2 F}{\partial r_i \partial t} = \frac{\partial^2 F}{\partial t \partial r_i} \,. \tag{21}$$

Using (18–19) in (21), we obtain

$$G\mathbf{B} = -2\rho\dot{\rho}\mathbf{B} - 2\dot{\mathbf{\Omega}} + \mathbf{\Omega} \times \mathbf{B} \,, \tag{22}$$

$$G\mathbf{E} = -3\rho\dot{\rho}\mathbf{E} + \mathbf{\Omega} \times \mathbf{E} + \mathbf{B} \times \frac{\partial \mathbf{n}}{\partial t} + \frac{\partial^2 \mathbf{n}}{\partial t^2} \,, \tag{23}$$

involving only the electromagnetic fields, not the electromagnetic potentials. Hence, the choice of gauge does not have any influence in the search for Noether point symmetries.

Equations (22–23) are the fundamental equations for the determination of Noether point symmetries for non-relativistic charged particle motion. It is a system of coupled, linear partial differential equations for the fields $\mathbf{E}$ and $\mathbf{B}$, involving the functions $\rho$, $\mathbf{\Omega}$ and $\mathbf{a}$ that define the generator $G$. The system to be satisfied by the electromagnetic fields is the proper three-dimensional extension of the system (34-36) found in [3] in the planar charged particle motion case. As long as we know, this is the first time equations (22-23) are explicitly written.

Unlike the two-dimensional case, it seems that the complete solution of (22-23) is not available. Thus, a complete classification of the electromagnetic fields for which the three-dimensional non-relativistic charged particle motion possesses Noether point symmetries is not known. The technical difficulties arising in the fully three-dimensional case are related to the problem of finding the canonical group coordinates associated to the generator (17), for arbitrary functions $\rho(t)$, $\mathbf{\Omega}(t)$ and $\mathbf{a}(t)$. Nevertheless, equations (22-23) are useful in the investigation of specific electromagnetic fields. For instance, in what follows we show that when the magnetic field is produced by a fixed magnetic monopole it is possible to obtain the general solution for (22-23).

Of course, the electromagnetic fields satisfying (22-23) must also comply with Maxwell equations, a condition that is not immediately assured. The non-homogeneous Maxwell equations may always be satisfied by a convenient choice of charge and current densities. There remains the Gauss law for magnetism and Faraday's law. Gauss law is removed when magnetic monopoles are present. In these cases, Faraday's law is the only extra requirement.

Once the system (22–23) is solved for the electromagnetic field, the Noether invariant follows from (10), reading

$$I = (\dot{\mathbf{r}}^2/2 + V)\rho^2 - (\dot{\mathbf{r}} + \mathbf{A}) \cdot \mathbf{n} + F\,. \tag{24}$$

The invariant $I$ is a quadratic polynomial on the velocities when $\rho \neq 0$. Otherwise, for $\rho = 0$, $I$ is a linear polynomial on the velocities. It is apparent from (24) that the Noether invariant needs the electromagnetic potentials as well as the function $F(\mathbf{r}, t)$ obtained from (18–19). However, as we will see in concrete examples, the form of $I$ is in practice independent of gauge, as expected.

Equations (22-23), restricted to planar motions, were completely solved in [3]. In the next Section we pursue a less ambitious program, taking $\mathbf{B}$ as the field of a fixed magnetic monopole and studying the consequences of this choice on the generator $G$ and on the electric field.

# 3 Magnetic monopole

Here we apply the formalism of Section II to the case of a magnetic monopole fixed at origin and with strength $g$,

$$\mathbf{B} = \frac{g\,\mathbf{r}}{r^3}\,. \tag{25}$$

Inserting (25) in (22), the result is a condition on the functions $\rho$, $\mathbf{\Omega}$ and $\mathbf{a}$ composing the symmetry,

$$g\,(r^2\mathbf{a} - 3\mathbf{a} \cdot \mathbf{r}\mathbf{r})r^{-5} - \dot{\mathbf{\Omega}} = 0\,. \tag{26}$$

Notice that $\rho$ is not present, remaining arbitrary. Equation (26) is identically satisfied if and only if

$$\mathbf{a} = \dot{\mathbf{\Omega}} = 0\,. \tag{27}$$

Consequently, the generator of Noether point symmetries is specified by

$$G = \rho^2 \frac{\partial}{\partial t} + (\rho\dot{\rho}\mathbf{r} + \mathbf{\Omega} \times \mathbf{r}) \cdot \frac{\partial}{\partial \mathbf{r}}\,. \tag{28}$$

In other words,

$$G = G_\rho + \mathbf{\Omega} \cdot \mathbf{L}\,, \tag{29}$$

where

$$G_\rho = \rho^2 \frac{\partial}{\partial t} + \rho\dot{\rho}\mathbf{r} \cdot \frac{\partial}{\partial \mathbf{r}} \,, \tag{30}$$

is the generator of quasi-invariance transformations [1, 8], $\mathbf{\Omega}$ is from now on a constant vector and $\mathbf{L} = (L_1, L_2, L_3)$ is defined by

$$L_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \,, \quad L_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \,, \quad L_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \,. \tag{31}$$

We recognize $L_1$, $L_2$ and $L_3$ as the generators of the $so(3)$ algebra.

The electric fields compatible with Noether point symmetry satisfy (23). With generator given by (28), this equation for $\mathbf{E}$ reads

$$G\mathbf{E} = -3\rho\dot{\rho}\mathbf{E} + \mathbf{\Omega} \times \mathbf{E} + (\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho})\mathbf{r} \,. \tag{32}$$

Notice that the strength $g$ does not appear on (32).

Before considering (32) in the general case, it is instructive to examine first the case $\mathbf{E} = 0$, in which only the magnetic monopole is present. For no electric field, (32) reduces to

$$\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho} = 0 \,, \tag{33}$$

whose general solution is

$$\rho^2 = c_1 + c_2 t + c_3 t^2 \,, \tag{34}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants. We conclude by the existence of three symmetry generators,

$$G_1 = \frac{\partial}{\partial t} \,, \quad G_2 = t\frac{\partial}{\partial t} + \frac{\mathbf{r}}{2} \cdot \frac{\partial}{\partial \mathbf{r}} \,, \quad G_3 = t^2\frac{\partial}{\partial t} + t\mathbf{r} \cdot \frac{\partial}{\partial \mathbf{r}} \,. \tag{35}$$

These generators are associated, respectively, to time translation, self-similarity and conformal transformations, composing the $so(2,1)$ algebra, with commutation relations

$$[G_1, G_2] = G_1 \,, \quad [G_1, G_3] = 2G_2 \,, \quad [G_2, G_3] = G_3 \,. \tag{36}$$

Therefore, the problem where only the magnetic monopole is present is endowed with the $SO(2,1) \times SO(3)$ group of Noether point symmetries. Such a result was already obtained by Jackiw [2], using dynamical Noether transformations, and by Moreira et al. [7], using Lie point symmetries and no variational formulation. Lie's approach has the advantage of no necessity of electromagnetic potentials, which are always singular when magnetic monopoles are present. However, as seen in Section II, the basic equations (22–23) can be formulated in terms of the electromagnetic field only. Moreover, we shall see in practice the gauge invariance of the Noether invariant. Our procedure is simpler than, for instance, the use of fiber bundles to avoid the singularity of the vector potential [12].

The solution for (32) comprises two categories, one for $\rho \neq 0$ and the other for $\rho = 0$. Accordingly, (24) shows that each class of solution is associated to quadratic or linear constants of motion, respectively.

### 3.1 The case $\rho \neq 0$

Using the method of characteristics, we find that when $\rho \neq 0$ the solution for (32) is given by

$$\mathbf{E} = \frac{\ddot{\rho}}{\rho}\mathbf{r} + \frac{1}{\rho^4}\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) + \frac{1}{\rho^3}R(\Omega\bar{t})\cdot\bar{\mathbf{E}}(\bar{\mathbf{r}})\,, \tag{37}$$

where we have used the definitions

$$\bar{t} = \int^t d\tau/\rho^2(\tau)\,, \quad \bar{\mathbf{r}} = \frac{1}{\rho}R^T(\Omega\bar{t})\cdot\mathbf{r}\,, \tag{38}$$

for $R(\Omega\bar{t})$ the rotation matrix about $\mathbf{\Omega}$ by an angle $\Omega\bar{t}$. The symbol $T$ is for the transpose. Also, in (37) $\bar{\mathbf{E}}$ is an arbitrary vector function of the indicated argument. For $\mathbf{\Omega} = (0,0,\Omega)$, the explicit form of the rotation matrix is

$$R(\Omega\bar{t}) = \begin{pmatrix} \cos\Omega\bar{t} & -\sin\Omega\bar{t} & 0 \\ \sin\Omega\bar{t} & \cos\Omega\bar{t} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{39}$$

The reader can directly verify that (37) satisfy (32). For this check, the relations

$$\frac{\partial}{\partial\bar{t}} = \rho^2\frac{\partial}{\partial t} + (\rho\dot{\rho}\mathbf{r} + \mathbf{\Omega}\times\mathbf{r})\cdot\frac{\partial}{\partial\mathbf{r}}\,, \tag{40}$$

$$\frac{\partial}{\partial\bar{\mathbf{r}}} = \rho\,R^T(\Omega\bar{t}(t))\cdot\frac{\partial}{\partial\mathbf{r}} \tag{41}$$

are useful. Moreover, (28) and (40) shows that

$$G = \partial/\partial\bar{t} \tag{42}$$

that is, $\bar{\mathbf{r}}$ and $\bar{t}$ are canonical group coordinates for the Noether point symmetries with $\rho \neq 0$, so that $G$ is the generator of translations along $\bar{t}$.

We see, on the electric field, the presence of the arbitrary functions $\rho$ and $\bar{\mathbf{E}}$. However, for the vector field (37) to qualify as a true electric field, it must complain with Faraday's law,

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,, \tag{43}$$

which imposes, for $\mathbf{E}$ as in (37),

$$R(\Omega\bar{t}) \cdot (\bar{\nabla} \times \bar{\mathbf{E}}(\bar{\mathbf{r}})) = 0\,, \tag{44}$$

where $\bar{\nabla} = \partial/\partial\bar{\mathbf{r}}$. As the rotation matrix $R$ is non singular, the only way of satisfying (44) is

$$\bar{\mathbf{E}} = -\bar{\nabla}\, U(\bar{\mathbf{r}}) \tag{45}$$

for some function $U(\bar{\mathbf{r}})$. The remaining Maxwell equations (with exception of Gauss law for magnetism, which has been ruled out) can be satisfied by an appropriate choice of charge $n_q$ and current $\mathbf{J}_q$ densities,

$$n_q \equiv \nabla \cdot \mathbf{E} = 3\frac{\ddot{\rho}}{\rho} - 2\frac{\mathbf{\Omega}^2}{\rho^4} - \frac{1}{\rho^2}\nabla^2 U\,, \tag{46}$$

$$\begin{aligned} \mathbf{J}_q \equiv \nabla \times \mathbf{B} - \partial\mathbf{E}/\partial\, t &= (\frac{\dot{\rho}\ddot{\rho}}{\rho^2} - \frac{\dddot{\rho}}{\rho})\mathbf{r} + 4\frac{\dot{\rho}}{\rho^5}\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \\ &\quad -2\frac{\dot{\rho}}{\rho^3}\nabla U + \frac{1}{\rho^2}\nabla\frac{\partial U}{\partial t}\,. \end{aligned} \tag{47}$$

To resume, taking into account (45) and also (41), it follows that

$$\mathbf{E} = \frac{\ddot{\rho}}{\rho}\mathbf{r} + \frac{1}{\rho^4}\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) - \frac{1}{\rho^2}\nabla\, U(\bar{\mathbf{r}})\,, \tag{48}$$

is the general form of the admissible electric fields, compatible with Noether point symmetries (with $\rho \neq 0$) and a magnetic monopole field. The electric field depend on the arbitrary functions $\rho$ and $U$, as well as on the constant vector $\mathbf{\Omega}$. Central fields are obtained as particular cases taking $\mathbf{\Omega} = 0$ and $U = U(\bar{r})$, where $\bar{r}$ is the norm of $\bar{\mathbf{r}}$.

The Noether invariant (24) require the electromagnetic potentials, as well as the function $F$ solution of (18–19). In full generality, the electromagnetic potentials are given by

$$\mathbf{A} = \frac{g\,z}{r(x^2+y^2)}(y,-x,0) + \nabla\lambda(\mathbf{r},t)\,, \tag{49}$$

$$V = -\frac{\ddot{\rho}\,r^2}{2\rho} + \frac{1}{2\rho^4}(\mathbf{\Omega}\times\mathbf{r})^2 + \frac{1}{\rho^2}U(\bar{\mathbf{r}}) - \frac{\partial\lambda}{\partial\,t}(\mathbf{r},t)\,, \tag{50}$$

where $\lambda(\mathbf{r},t)$ is an arbitrary gauge function. Inserting the electromagnetic potentials into (18-19), there results a system whose solution is

$$F = \frac{1}{2}(\dot{\rho}^2 + \rho\ddot{\rho})\,r^2 + g\,r\frac{(\Omega_1 x + \Omega_2 y)}{(x^2+y^2)}\,G\lambda\,, \tag{51}$$

so that the Noether first integral, from (24), is

$$I = \frac{1}{2}(\rho\dot{\mathbf{r}} - \dot{\rho}\mathbf{r} - \mathbf{\Omega}\times\mathbf{r}/\rho)^2 + U(\bar{\mathbf{r}}) + g\mathbf{\Omega}\cdot\hat{r}\,. \tag{52}$$

As anticipated, $I$ is not dependent on the gauge function $\lambda$.

## 3.2 The case $\rho = 0$

For $\rho = 0$, (32) reduces to

$$\mathbf{\Omega}\times\mathbf{r}\cdot\frac{\partial\mathbf{E}}{\partial\mathbf{r}} = \mathbf{\Omega}\times\mathbf{E}\,. \tag{53}$$

Since $\mathbf{\Omega}$ is a constant vector, there is no loss of generality if we set $\mathbf{\Omega} = (0,0,\Omega)$ in (53), with $\Omega \neq 0$, so that $G$ becomes $L_3$, the generator of rotations about the $z$ axis. In this situation, the general solution for (53) is

$$\mathbf{E} = E_r(r,\theta,t)\hat{r} + E_\theta(r,\theta,t)\hat{\theta} + E_\phi(r,\theta,t)\hat{\phi}\,, \tag{54}$$

using spherical coordinates $(r,\theta,\phi)$ with unit vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ and such that $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, and $z = r\cos\theta$. The class (54) is the general class of electric fields compatible with azimuthal symmetry. With an appropriated scalar potential $V = V(r,\theta,t)$ as well as the vector potential (49), it can be proven that

$$F = G\lambda \tag{55}$$

is the solution for (18–19). The corresponding Noether invariant (24) is

$$I_3 = \Omega_3(y\dot{x} - x\dot{y} + g\,z/r)\,, \tag{56}$$

gauge independent. This first integral is proportional to the third component of the Poincaré vector (2).

If, in addition to the symmetry of rotation about one axis, we impose the existence of symmetry of rotation about a different axis, it can be easily proven that the solution for (53) is

$$\mathbf{E} = E(r,t)\hat{r}\,, \tag{57}$$

the general class of central, time-dependent fields. The result is explained by the $so(3)$ algebra. For instance, the presence of the extra symmetry of rotations about the $y$ axis imply rotational symmetry about the $x$ axis, since $[L_2, L_3] = -L_1$ and by definition the symmetry algebra is closed. In a similar way to (56), it can be then verified that the Noether invariants associated to the $SO(3)$ group are the three components of the Poincaré vector. Of course, this is not the first time that the Poincaré vector is shown to be associated to rotational symmetry (see, for instance, reference [2]).

# 4 Central electric forces

When the electric field is central, the Poincaré vector is immediately conserved, as a result of the $SO(3)$ symmetry. In this case, the discussion can be reduced to essentially one-dimensional, time-dependent motion. To see this, choose axis so that the conserved Poincaré vector may be written $\mathbf{D} = (0,0,D)$. Decomposing $\mathbf{D}$ in components and using spherical coordinates, we get

$$\cos\theta = -g/D\,, \quad \dot{\phi} = D/r^2\,, \tag{58}$$

while the radial component of the Lorentz equation reads, from (57) and (58),

$$\ddot{r} = E(r,t) + \frac{D^2 - g^2}{r^3}\,. \tag{59}$$

Eq. (58) shows that the motion is on a circular cone whose vertex contains the monopole, and that the angle $\phi$ can be obtained from a simple quadrature once the radial variable is found from the solution of (59). This latter equation involves only $r$ and time.

The presence of extra Noether invariants helps for the integration of (59). Since the electric field is central, the category of Noether symmetries described in subsection III.1 are admitted if and only if

$$\mathbf{\Omega} = 0\,, \quad U = U(\bar{r})\,, \quad \bar{r} = r/\rho\,, \tag{60}$$

In the following, we will consider in more detail a case in which the function $U$ can be conveniently chosen, so that there is an extra Noether symmetry.

Let us illustrate the initial results of the Section with the electric field

$$\mathbf{E} = -\omega^2(t)\mathbf{r} + \sigma^2\mathbf{r}/r^4\,. \tag{61}$$

where $\omega(t)$ is an arbitrary function of time and $\sigma$ is a numerical constant. As observed in the Introduction, for constant $\omega$ and $\sigma = g$, all bounded trajectories are periodic for this electric field. In the general, time-dependent case, (61) produces the time-dependent monopole-oscillator problem, with in addition a repulsive force.

As the electric force is central, $SO(3)$ is known to be admitted in advance, the Poincaré vector being conserved. Besides this obvious symmetry, there is also symmetry in the form of a quasi-invariance transformation. To see this, notice that the scalar potential

$$V = \omega^2(t)r^2/2 + \sigma^2/2r^2 \tag{62}$$

can be put in the form (50) with $\lambda = 0$ if and only if

$$U\left(\frac{r}{\rho}\right) = \frac{1}{2}(\ddot{\rho} + \omega^2(t)\rho)\rho\, r^2 + \frac{\sigma^2\rho^2}{2r^2}\,. \tag{63}$$

The right-hand side of (63) is properly a function of $r/\rho$ if and only if $\rho$ satisfies Pinney's [10] equation

$$\ddot{\rho} + \omega^2(t)\rho = k/\rho^3\,, \tag{64}$$

where $k$ is a constant. In this case, we have

$$U(\bar{r}) = \frac{1}{2}k\bar{r}^2 + \frac{\sigma^2}{2\bar{r}^2}\,. \tag{65}$$

Noether's invariant (52) is

$$I = \frac{1}{2}(\rho\dot{\mathbf{r}} - \dot{\rho}\mathbf{r})^2 + \frac{k}{2}\left(\frac{r}{\rho}\right)^2 + \frac{\sigma^2}{2}\left(\frac{\rho}{r}\right)^2 . \tag{66}$$

In conclusion, the time-dependent monopole oscillator system with an extra repulsive force do have, besides $SO(3)$ symmetry, quasi-invariance transformations as Noether symmetries, provided the function $\rho$ satisfies Pinney's equation.

A more convenient formulation of the invariance properties of the system is provided by the linearising transform

$$\psi = \rho^2 , \tag{67}$$

so that Pinney's equation becomes, upon differentiation,

$$\dddot{\psi} + 4\omega^2\dot{\psi} + 4\omega\dot{\omega}\psi = 0 . \tag{68}$$

The general solution for this last equation is any linear combination of three independent particular solutions $\psi_1$, $\psi_2$ and $\psi_3$,

$$\psi = c_1\psi_1 + c_2\psi_2 + c_3\psi_3 , \tag{69}$$

where $c_1$, $c_2$ and $c_3$ are numerical constants. To each solution $\psi_i$ correspond one associated Noether point symmetry, with generator of the form (30) with $\psi_i = \rho_i^2$,

$$G_i = \psi_i\frac{\partial}{\partial t} + \frac{\dot{\psi}_i\mathbf{r}}{2}\cdot\frac{\partial}{\partial\mathbf{r}} . \tag{70}$$

The associated Noether invariants follows from (66),

$$I_i = \frac{1}{2}\left(\psi_i\dot{\mathbf{r}}^2 - \dot{\psi}_i\, r\dot{r} + (\frac{\ddot{\psi}_i}{2} + \omega^2\psi_i)\, r^2 + \frac{\sigma^2\psi_i}{r^2}\right) , \tag{71}$$

with $i = 1, 2, 3$, and where we have eliminated the constant $k$ using Pinney's equation.

The Noether symmetries and invariants can be explicitly shown when the general solution for (68) is available. In particular, when

$$\omega = \omega_0 , \tag{72}$$

a constant, the general solution is

$$\psi = c_1 + c_2 \cos(2\omega_0 t) + c_3 \sin(2\omega_0 t)\,. \tag{73}$$

The corresponding generators (70), obtained for $c_1 = 1$, $c_2 = c_3 = 0$ and cyclic permutations, are

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial t}\,, \\
G_2 &= \cos(2\omega_0 t)\frac{\partial}{\partial t} - \omega_0 \sin(2\omega_0 t)\mathbf{r}\cdot\frac{\partial}{\partial \mathbf{r}}\,, \\
G_3 &= \sin(2\omega_0 t)\frac{\partial}{\partial t} + \omega_0 \cos(2\omega_0 t)\mathbf{r}\cdot\frac{\partial}{\partial \mathbf{r}}\,.
\end{aligned} \tag{74}$$

These generators, together with the generators of $SO(3)$, determine the algebra

$$\begin{aligned}
[G_1, G_2] &= -2\omega_0 G_3 \quad , \quad [G_2, G_3] = 2\omega_0 G_1 \quad , \\
[G_3, G_1] &= -2\omega_0 G_2 \quad , \quad [L_i, L_j] = -\epsilon_{ijk} L_k \quad , \\
[G_i, L_j] &= 0 \quad , \quad i, j = 1, 2, 3 \quad .
\end{aligned} \tag{75}$$

Therefore, the Noether point symmetry algebra for the time-independent monopole-oscillator with an extra repulsive force have a $so(2,1) \oplus so(3)$ structure, the same symmetry algebra as in the simple magnetic monopole case.

As already seen, $SO(3)$ invariance is associated to the Poincaré vector. On the other hand, invariance under $G_1$, $G_2$ and $G_3$ corresponds, respectively, to the constants of motion

$$\begin{aligned}
I_1 &= \frac{1}{2}(\dot{\mathbf{r}}^2 + \omega_0^2 r^2 + \sigma^2/r^2)\,, \qquad (76)\\
I_2 &= \frac{1}{2}\dot{\mathbf{r}}^2 \cos(2\omega_0 t) + \omega_0\, r\dot{r} \sin(2\omega_0 t) - \frac{1}{2}\omega_0^2 r^2 \cos(2\omega_0 t) + \frac{\sigma^2}{2r^2}\cos(2\omega_0 t)\,, \\
I_3 &= \frac{1}{2}\dot{\mathbf{r}}^2 \sin(2\omega_0 t) - \omega_0\, r\dot{r} \cos(2\omega_0 t) - \frac{1}{2}\omega_0^2 r^2 \sin(2\omega_0 t) + \frac{\sigma^2}{2r^2}\sin(2\omega_0 t)\,.
\end{aligned}$$

The six Noether invariants, namely the components of $\mathbf{D}$ and $I_i$ above, are not all independent, since

$$\omega_0^2 \mathbf{D}^2 = I_1^2 - I_2^2 - I_3^2\,. \tag{77}$$

As discussed in the beginning of the Section, the fact that the electric field is central allows to reduce the problem to the solution for the radial variable. Here, the existence of the Noether invariants allows the direct solution for $r(t)$ by elimination of $\dot{\mathbf{r}}$ between the invariants $I_1$, $I_2$ and $I_3$, with the result

$$r^2(t) = \frac{1}{\omega_0^2}(I_1 - I_2\cos(2\omega_0 t) - I_3\sin(2\omega_0 t))\,, \tag{78}$$

Inserting $r(t)$ into (58) and integrating, it follows that the azimuthal variable is

$$\phi(t) = \phi_0 + \arctan\left(\frac{-I_3 + (I_1 + I_2)\tan(2\omega_0 t)}{\omega_0 D}\right)\,, \tag{79}$$

where $\phi_0$ is a reference angle.

Formulae (58) and (78–79) are the exact solution for the time-independent monopole-oscillator problem with an repulsive force. The exact solution involves four independent integration constants, $I_1, I_2, I_3$ and $\phi_0$, while $D$ is functionally dependent on these constants through (77). The exact solution does not contain six integration constants since, from the very beginning, two components of the Poincaré vector were annulled. The remaining two constants can be easily incorporated, with the price of a less clear presentation. Finally, it should be stressed that any time-dependent frequency such that (68) can be exactly solved leads to exact solution in the same way as the time-independent case.

# 5 Constant magnetic field

In this Section, we consider the physically relevant case of a constant magnetic field,

$$\mathbf{B} = (0, 0, B_0)\,, \tag{80}$$

where $B_0$ is a numerical constant, and look for Noether point symmetries and invariants for appropriate electric fields. More precisely, unlike the magnetic monopole case, we do not search for the more general class of electric fields for which some Noether point symmetry is available. In fact, we restrict the treatment to time-dependent linear electric fields. Such a restriction is again physically meaningful.

Inserting (80) into (22) there results an equation for the vector $\mathbf{\Omega}$ responsible for time-dependent rotations,

$$\dot{\mathbf{\Omega}} = \frac{1}{2}\mathbf{\Omega} \times \mathbf{B} - \rho\dot{\rho}\mathbf{B}\,. \tag{81}$$

The above system is easily solved,

$$\begin{aligned} \Omega_1 &= c_1 \cos(B_0 t/2) + c_2 \sin(B_0 t/2)\,, \\ \Omega_2 &= -c_1 \sin(B_0 t/2) + c_2 \cos(B_0 t/2)\,, \\ \Omega_3 &= c_3 - B_0 \rho^2/2\,, \end{aligned} \tag{82}$$

where the $c_i$ are integration constants. With the result (83), the determining equation (23) for the electric field is expressed as

$$\begin{aligned} G\,\mathbf{E} &= -3\rho\dot{\rho}\mathbf{E} + \mathbf{\Omega} \times \mathbf{E} + (\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho} + B_0^2\rho\dot{\rho})\mathbf{r} + \\ &+ (\mathbf{B}\cdot\mathbf{r})(\frac{1}{4}\mathbf{\Omega} \times \mathbf{B} - \rho\dot{\rho}\mathbf{B}) + \frac{1}{4}(\mathbf{\Omega} \times \mathbf{B}) \cdot \mathbf{r}\,\mathbf{B} + \mathbf{B} \times \dot{\mathbf{a}} + \ddot{\mathbf{a}}\,. \end{aligned} \tag{83}$$

We have not found the general solution for (83), so that the general class of admissible electric fields remains to be determined. However, there is at least one case amenable to exact calculations, namely the particular case of linear electric fields of the form

$$\mathbf{E} = -\omega_\perp^2(t)\mathbf{r}_\perp - \omega_\parallel^2(t)\mathbf{r}_\parallel + \mathbf{f}(t)\,, \tag{84}$$

where $\mathbf{r}_\perp = (x, y, 0)$, $\mathbf{r}_\parallel = (0, 0, z)$, $\mathbf{f} = (f_1(t), f_2(t), f_3(t))$ and $\omega_\perp$, $\omega_\parallel$ and the $f_i$ are time-dependent functions. When the electric field is linear, both sides of (83) are linear functions of the coordinates. Equating to zero the coefficients of each coordinate and of the independent term, the result is a coupled system of ordinary differential equations for the functions $\rho$ and $\mathbf{a}$ composing the symmetry generator,

$$\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho} + (B_0^2 + 4\omega_\perp^2)\rho\dot{\rho} + 2\omega_\perp\dot{\omega}_\perp\rho^2 = 0\,, \tag{85}$$

$$\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho} + 4\omega_\parallel^2\rho\dot{\rho} + 2\omega_\parallel\dot{\omega}_\parallel\rho^2 = 0\,, \tag{86}$$

$$\ddot{a}_1 + \omega_\perp^2\,a_1 = B_0\dot{a}_2 + d_1(t)\,, \tag{87}$$

$$\ddot{a}_2 + \omega_\perp^2\,a_2 = -B_0\dot{a}_1 + d_2(t)\,, \tag{88}$$

$$\ddot{a}_3 + \omega_\parallel^2\,a_3 = d_3(t)\,. \tag{89}$$

Moreover, the following algebraic relations must be satisfied,

$$\Omega_1(\omega_\parallel^2 - \omega_\perp^2 - B_0^2/4) = \Omega_2(\omega_\parallel^2 - \omega_\perp^2 - B_0^2/4) = 0\,. \tag{90}$$

In (87-89), the vector $\mathbf{d} = (d_1, d_2, d_3)$ is defined according to

$$\mathbf{d} = \rho^2\dot{\mathbf{f}} + 3\rho\dot{\rho}\mathbf{f} + \mathbf{f}\times\mathbf{\Omega}\,. \tag{91}$$

Once the system (85-90) is solved, the associated Noether invariant is found from (24), which require both the electromagnetic potentials and the function $F$, solution for (18-19). The electromagnetic potentials are

$$\begin{aligned} \mathbf{A} &= \frac{B_0}{2}(-y, x, 0) + \nabla\lambda(\mathbf{r}, t)\,, & (92)\\ V &= \frac{1}{2}\omega_\perp^2(t)(x^2+y^2) + \frac{1}{2}\omega_\parallel^2(t)\,z^2 - \mathbf{f}(t)\cdot\mathbf{r} - \frac{\partial\lambda}{\partial t}(\mathbf{r}, t)\,, & (93) \end{aligned}$$

where $\lambda(\mathbf{r}, t)$ is an arbitrary gauge function. The function $F$, the last ingredient for the Noether invariant, follows from the use of these electromagnetic potentials in the system (18-19), and reads

$$F = \frac{1}{2}(\rho\ddot{\rho} + \dot{\rho}^2)r^2 + \dot{a}\cdot\mathbf{r} + \frac{1}{2}\mathbf{B}\cdot\mathbf{a}\times\mathbf{r} + \int^t d\mu\,\mathbf{f}(\mu)\cdot\mathbf{a}(\mu) + G\,\lambda\,. \tag{94}$$

The Noether invariant (24) is then expressed as

$$\begin{aligned} I &= \frac{1}{2}(\rho\dot{\mathbf{r}} - \dot{\rho}\mathbf{r})^2 + \mathbf{\Omega}\cdot\dot{\mathbf{r}}\times\mathbf{r} + \frac{\rho^2}{2}(\omega_\perp^2(x^2+y^2) + \omega_\parallel^2 z^2) + \frac{1}{2}\rho\ddot{\rho}\,r^2 + \\ &+ \frac{1}{2}\mathbf{\Omega}\cdot(\mathbf{B}\times\mathbf{r})\times\mathbf{r} - \mathbf{a}\cdot\dot{\mathbf{r}} + \dot{\mathbf{a}}\cdot\mathbf{r} - \mathbf{a}\cdot\mathbf{B}\times\mathbf{r} + \\ &+ \int^t d\mu\mathbf{f}(\mu)\cdot\mathbf{a}(\mu)\,, \end{aligned} \tag{95}$$

gauge independent as it must be. For the explicit form of the Noether invariant, we have to solve the system (85-90) giving the functions $\rho$ and $a_i$ which remain to be obtained.

After a detailed but straightforward analysis, we distinguish two classes of solutions for the system (85-90), according to the functions $\omega_\perp$ and $\omega_\parallel$ entering the electric field and the magnetic field strength. The two classes of solutions are treated separately.

### 5.1 The $\omega_{\parallel}^2(t) = \omega_{\perp}^2(t) + B_0^2/4$ case

In the situation where the relation $\omega_{\parallel}^2(t) = \omega_{\perp}^2(t) + B_0^2/4$ is valid, the condition (90) becomes an identity, so that the components $\Omega_1$ and $\Omega_2$ are left free. Referring to equation (83) defining the vector $\mathbf{\Omega}$ for a constant magnetic field, this means that the constants $c_1$ and $c_2$ are left free. Moreover, the equations (85) and (86) are identical, becoming, using the linearising transform $\psi = \rho^2$,

$$\dddot{\psi} + 4\omega_{\parallel}^2\dot{\psi} + 4\omega_{\parallel}\dot{\omega}_{\parallel}\psi = 0\,, \tag{96}$$

a third-order linear equation. Denote the general solution as

$$\psi = c_4\psi_1 + c_5\psi_2 + c_6\psi_3\,, \tag{97}$$

where $c_i$ are numerical constants and $\psi_i$ independent particular solutions. Taking into account the three arbitrary numerical constants entering $\mathbf{\Omega}$, plus $c_4$, $c_5$, $c_6$ and the six integration constants for the system (87-89), we arrive at a 12-parameter group of Noether point symmetries.

The construction of the symmetry group is best explained with a concrete example. In the last part of the Section, let us study in more detail the case

$$\mathbf{f} = 0\,, \quad \dot{\omega}_{\perp} = \dot{\omega}_{\parallel} = 0\,, \tag{98}$$

that is, the cases of time-independent harmonic fields with no forcing term. In this context, (96) is easily solved, giving

$$\psi = c_4 + c_5\cos(2\omega_{\parallel}t) + c_6\sin(2\omega_{\parallel}t)\,. \tag{99}$$

In addition, the system (87-89) has the general solution

$$\begin{aligned} a_1 &= c_7\cos(\omega_1 t) + c_8\sin(\omega_1 t) + c_9\cos(\omega_2 t) - c_{10}\sin(\omega_2 t)\,, \\ a_2 &= -c_7\sin(\omega_1 t) + c_8\cos(\omega_1 t) + c_9\sin(\omega_2 t) + c_{10}\cos(\omega_2 t)\,, \\ a_3 &= c_{11}\cos(\omega_{\parallel}t) + c_{12}\sin(\omega_{\parallel}t)\,, \end{aligned} \tag{100}$$

where $c_i$ are integration constants and

$$\omega_1 = \frac{1}{2}(B_0 + \sqrt{B_0^2 + 4\omega_{\perp}^2})\,, \quad \omega_2 = \frac{1}{2}(-B_0 + \sqrt{B_0^2 + 4\omega_{\perp}^2})\,. \tag{101}$$

Choosing $c_i = \delta_{ij}$, for $i = 1, ..., 12$ and $j = 1, ..., 12$, the twelve Noether point symmetry generators can be constructed from (17), (83) and (99-101).

They read

$$
\begin{aligned}
G_1 &= \cos(B_0t/2)(y\partial/\partial z - z\partial/\partial y) - \sin(B_0t/2)(z\partial/\partial x - x\partial/\partial z)\,, \\
G_2 &= \sin(B_0t/2)(y\partial/\partial z - z\partial/\partial y) + \cos(B_0t/2)(z\partial/\partial x - x\partial/\partial z)\,, \\
G_3 &= x\partial/\partial y - y\partial/\partial x\,, \quad G_4 = \partial/\partial t - (B_0/2)(x\partial/\partial y - y\partial/\partial x)\,, \\
G_5 &= \cos(2\omega_\parallel t)\,(\partial/\partial t - (B_0/2)(x\partial/\partial y - y\partial/\partial x)) - \omega_\parallel \sin(2\omega_\parallel t)\,\mathbf{r}\cdot\nabla\,, \\
G_6 &= \sin(2\omega_\parallel t)\,(\partial/\partial t - (B_0/2)(x\partial/\partial y - y\partial/\partial x)) + \omega_\parallel \cos(2\omega_\parallel t)\,\mathbf{r}\cdot\nabla\,, \\
G_7 &= \cos(\omega_1 t)\partial/\partial x - \sin(\omega_1 t)\partial/\partial y\,, \\
G_8 &= \sin(\omega_1 t)\partial/\partial x + \cos(\omega_1 t)\partial/\partial y\,, \\
G_9 &= \cos(\omega_2 t)\partial/\partial x + \sin(\omega_2 t)\partial/\partial y\,, \\
G_{10} &= -\sin(\omega_2 t)\partial/\partial x + \cos(\omega_2 t)\partial/\partial y\,, \\
G_{11} &= \cos(\omega_\parallel t)\partial/\partial z\,, \quad G_{12} = \sin(\omega_\parallel t)\partial/\partial z\,.
\end{aligned}
\tag{102}
$$

The conserved quantities associated to the above generators follows from (95),

$$
\begin{aligned}
I_1 &= \cos(B_0t/2)(z(\dot y + B_0x/2) - y\dot z) + \sin(B_0t/2)(z(\dot x - B_0y/2) - x\dot z)\,, \\
I_2 &= \sin(B_0t/2)(z(\dot y + B_0x/2) - y\dot z) - \cos(B_0t/2)(z(\dot x - B_0y/2) - x\dot z)\,, \\
I_3 &= x\dot y - y\dot x + (B_0/2)(x^2+y^2)\,, \\
I_4 &= \dot{\mathbf{r}}^2/2 + (B_0/2)(x\dot y - y\dot x) + (1/2)((\omega_\perp^2 + B_0^2/2)(x^2+y^2) + \omega_\parallel^2 z^2)\,, \\
I_5 &= (1/2)\dot{\mathbf{r}}^2\cos(2\omega_\parallel t) + \omega_\parallel r\dot r\sin(2\omega_\parallel t) + \\
&+ (B_0/2)(x\dot y - y\dot x)\cos(2\omega_\parallel t) - (1/2)(\omega_\perp^2(x^2+y^2) + \omega_\parallel^2 z^2)\cos(2\omega_\parallel t)\,, \\
I_6 &= (1/2)\dot{\mathbf{r}}^2\sin(2\omega_\parallel t) - \omega_\parallel r\dot r\cos(2\omega_\parallel t) + \\
&+ (B_0/2)(x\dot y - y\dot x)\sin(2\omega_\parallel t) - (1/2)(\omega_\perp^2(x^2+y^2) + \omega_\parallel^2 z^2)\sin(2\omega_\parallel t)\,, \\
I_7 &= (\dot x + \omega_2 y)\cos(\omega_1 t) - (\dot y - \omega_2 x)\sin(\omega_1 t)\,, \\
I_8 &= (\dot x + \omega_2 y)\sin(\omega_1 t) + (\dot y - \omega_2 x)\cos(\omega_1 t)\,, \\
I_9 &= (\dot x - \omega_1 y)\cos(\omega_2 t) + (\dot y + \omega_1 x)\sin(\omega_2 t)\,, \\
I_{10} &= -(\dot x - \omega_1 y)\sin(\omega_2 t) + (\dot y + \omega_1 x)\cos(\omega_2 t)\,, \\
I_{11} &= \dot z\cos(\omega_\parallel t) + \omega_\parallel z\sin(\omega_\parallel t)\,, \quad I_{12} = \dot z\sin(\omega_\parallel t) - \omega_\parallel z\cos(\omega_\parallel t)\,.
\end{aligned}
\tag{103}
$$

The invariants $I_i$ for $i = 7, ...12$ are sufficient, from elimination of the velocities, for the general solution of the equations of motion,

$$
x = \frac{1}{2\omega_\parallel}(I_7\sin(\omega_1 t) - I_8\cos(\omega_1 t) + I_9\sin(\omega_2 t) + I_{10}\cos(\omega_2 t))\,,
$$

$$
\begin{aligned}
y &= \frac{1}{2\omega_\parallel}\left(I_7\cos(\omega_1 t) + I_8\sin(\omega_1 t) - I_9\cos(\omega_2 t) + I_{10}\sin(\omega_2 t)\right), \quad (104)\\
z &= \frac{1}{\omega_\parallel}\left(I_{11}\sin(\omega_\parallel t) - I_{12}\cos(\omega_\parallel t)\right).
\end{aligned}
$$

The remaining invariants $I_i$ for $i = 1, ...6$ are functionally dependent on the invariants associated to time-dependent translations.

## 5.2 The $\omega_\parallel^2(t) \neq \omega_\perp^2(t) + B_0^2/4$ case

In this situation, condition (90) is satisfied only if $\Omega_1 = \Omega_2 = 0$. This rules out rotational symmetries about the $x$ and $y$ axis, and imply $c_1 = c_2 = 0$ in (83). In addition, equations (85) and (86) are equivalent only if $\rho = c_4$, a numerical constant. In fact, it is possible, in principle, to have non constant solutions for $\rho$ satisfying both (85-86). However, this possibility is allowed only if $\omega_\parallel$ and $\omega_\perp$ are related by a somewhat complicated relation which we refrain from writing here. In conclusion, when $\omega_\parallel^2(t) \neq \omega_\perp^2(t) + B_0^2/4$ generically there is a 8-parameter group of Noether point symmetries. This group comprises rotations about the $z$ axis, time-translation and the six time-dependent space translations determined by the solution of (87-89). Accordingly, in the time-independent case with no forcing term, specified by (98), the symmetries generated by $G_2$, $G_3$, $G_5$ and $G_6$ are lost, as well as the associated Noether invariants. Even if the symmetry structure is less rich, it is sufficient for the general solution of the equations of motion, since the six time-dependent translational symmetries are not broken.

# 6 Conclusion

We have obtained the system of partial differential equations to be satisfied by the electromagnetic field so that the action functional for non-relativistic motion is invariant under continuous point transformations. This system of equations has been completely solved when the magnetic field is produced by a fixed magnetic monopole. The associated constants of motion can have linear or quadratic dependencies on the velocity. These constants of motion can be used to integrate the Lorentz equations, as in the monopole-oscillator problem with an extra repulsive force field. Moreover, we have treated the case of a constant magnetic field plus a harmonic electric field with a forcing term.

The main open problem that still deserves attention is the complete solution of the basic system (22-23) for the electromagnetic field. The technical drawback here is the determination of the canonical group coordinates for the generator $G$ in (17) with arbitrary $\rho(t)$, $\mathbf{\Omega}(t)$ and $\mathbf{a}(t)$. It can be verified that this issue can be solved at least when $\mathbf{\Omega}(t)$ has a fixed direction. Other particular solutions may be valuable. In addition, the system (22-23) is worth to be considered in the case of other particular electromagnetic fields, for which the magnetic field is not in the form of a magnetic monopole or of a constant field.

Other direction of research is the search for Lie point symmetries for non-relativistic charged particle motion under generic electromagnetic fields. In the two-dimensional case, this problem can be completely solved [4]. In three dimensions, certainly the difficulties are greater than those we are faced in the case of Noether point symmetries, since the Noether group of point symmetries is a subgroup of the Lie group of point symmetries. Again, the stumbling block is the finding of canonical group coordinates for the generator of Lie symmetries. Finally, further extensions involve the use of transformations of more general character, such as dynamical or nonlocal transformations.

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# References

[1] L. CHETOUANI, L. GUECHI & T. F. HAMMANN, *Generalized canonical transformations and path integrals*, Phys. Rev. A **40** (1989) 1157-1164.

[2] R. JACKIW, *Dynamical symmetry of the magnetic monopole*, Ann. Phys. **129** (1980) 183-200.

[3] F. HAAS & J. GOEDERT, *Noether symmetries for two-dimensional charged particle motion*, J. Phys. A: Math. Gen. **32** (1999) 6837-6852.

[4] F. HAAS & J. GOEDERT, *Lie symmetries for two-dimensional charged particle motion*, J. Phys. A: Math. Gen. **33** (2000) 4661-4677.

[5] S. LABELLE, M. MAYRAND & L. VINET, *Symmetries and degeneracies of a charged oscillator in the field of a magnetic monopole*, J. Math. Phys. **32** (1991) 1516-1521.

[6] H. V. MCINTOSH & A. CISNEROS, *Degeneracy in the presence of a magnetic monopole*, J. Math. Phys. **11** (1970) 896-916.

[7] I. C. MOREIRA, O. M. RITTER & F. C. SANTOS, *Lie symmetries for the charge-monopole problem*, J. Phys. A **18** (1985) L427-L430.

[8] A. MUNIER, J. R. BURGAN, M. FEIX & E. FIJALLOW, *Asymptotic solutions for a variable mass two-body problem*, Astron. Astrophys. **94** (1981) 373-376.

[9] H. POINCARÉ, C. R. Acad. Sci. Paris **123** (1896) 530.

[10] E. PINNEY, *The nonlinear differential equation* $y'' + p(x)y + cy^{-3} = 0$, Proc. Am. Math. Soc. **1** (1950) 681-681.

[11] W. SARLET & F. CANTRIJN, *Generalizations of Noether's theorem in classical mechanics*, SIAM Rev. **23** (1981) 467-494.

[12] V. V. SOKOLOV, *Theory of magnetic monopoles and the Dirac-Schwinger quantization condition without strings*, Sov. J. Nucl. Phys. **23** (1976) 330-334.

[13] G. THOMPSON, *Second-order systems with Runge-Lenz-type vectors*, Lett. Math. Phys. **14** (1987) 69-75.