Lie symmetries and conservation laws for the one-dimensional Schrödinger-Maxwell system

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Abstract

This work considers the Schrödinger-Maxwell system which is frequently used to describe the one-dimensional expansion of a quantum electron gas. The Lie symmetry group of the system is determined. A variational formulation for the dynamical equations is also proposed and Noether's theorem is applied to generate the conservation laws associated with the symmetries that preserve the action functional.

1 Introduction

Charged particle expansion into vacuum is a basic phenomenon that has attracted attention since the early days of plasma theory. Plasma expansion occurs in several different natural and laboratory situations, notably in astrophysics, in fusion research and in the polar wind. On the theoretical ground, analytical, numerical and experimental efforts have been exercised constantly. Expanding plasma have been described in a classical framework, both by kinetic and hydrodynamic theories. The nonlinear character of the process is manifest in a variety of ways, such as shock waves, wave-breaking and anomalous dissipative effects. One of the main features of these phenomena is the self-similar behavior of the assymptotic motion. For an extended review of the subject we refer to Ch. Sack and H. Schamel [1].

This work considers the expansion into vacuum of a *quantum* electron gas, for which some results are already available. Although the Schrödinger-Maxwell or the Wigner-Maxwell systems have long been used in their stationary versions to describe semiconductor devices [2], time-dependent studies are much scarcer. Mola *et all.* [3] have used the Schrödinger-Poisson system to model the one-dimensional expansion of a quantum electron gas, and concluded that the classical regime acts as a universal attractor. Their analysis relied basically in rescaling thecniques and computer simulation.

A natural generalization of the rescaling methods is the Lie theory of extended groups, a powerful tool to search exact solutions for nonlinear problems [4]. In the classical context, Lie groups have been used to generate exact, time-dependent solutions for the Vlasov-Maxwell and the Vlasov-Poisson equations [6]–[8]. These solutions are much more general in structure than the uniformly translated BGK equilibrium solutions [9]. The purpose of this work is to investigate, by means of the Lie theory of extended groups, the quantum analogue of the classical Vlasov-Maxwell system, namely the Schrödinger-Maxwell system.

In section two, the Schrödinger-Maxwell system for the one-dimensional expansion of an electron gas is presented. The Lie symmetries of the problem are determined. In section three, these symmetries and the existence of a variational principle for the dynamical equations are used to find conservation laws via Noether's theorem. In the conclusions, a brief discussion of the results and a list of open questions on the subject are presented.

2 Symmetries of the one-dimensional Schrödinger-Maxwell system

The Schrödinger-Maxwell equations for the one-dimensional expansion of an electron gas are given by

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - eV\psi\,, \tag{1}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{Ne}{\varepsilon_0}|\psi|^2\,, \tag{2}$$

$$\frac{\partial^2 V}{\partial x\partial t} = -\frac{iNe\hbar}{2m\varepsilon_0}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)\,. \tag{3}$$

In what follows we will set the Planck's constant $\hbar$, the electron mass m , the density of particles per unit length N and the dielectric constant in vacuum ε_0 all equal to one by a convenient choice of rescaling.

Some general remarks are in order now. The Schrödinger-Maxwell model is a quite naïve approximation to the real problem, since it uses a one-particle wave function to describe a many-particle system. This is, however, analogous to the Vlasov approximation in classical plasma theory, where a one-particle distribution function represents the whole system. Spin and relativistic effects are also neglected. Nevertheless, the model can be used as an interesting starting point for more serious treatment. Another remark is that equation (3), a consequence of Ampère's Law, is usually not included in the system. This is a mistake that has already been pointed out in the classical case [5, 10].

In the search of Lie point symmetries we adopt the viewpoint of Bluman and Cole [4]. First he symmetries of the Schrödinger-Poisson system (1)–(2) are uncovered and then restricted by the additional imposition that they be also symmetries of the complete system (1)–(3). The algebraic procedure, though somewhat lengthy, is well known and we merely state the results here. For the Schrödinger-Poisson system the point symmetries are

$$
\begin{aligned}
\bar{t} &= t + \epsilon(c_1 t + c_2)\,, && (4)\\
\bar{x} &= x + \epsilon(c_1 x/2 + a_1(t))\,, && (5)\\
\bar{\psi} &= \psi + i\epsilon\,(ic_1 + \dot{a}_1 x + a(t))\,\psi\,, && (6)\\
\bar{V} &= V + \epsilon(-c_1 V + \ddot{a}_1 x + \dot{a})\,, && (7)
\end{aligned}
$$

where ϵ is an infinitesimal parameter characterizing the transformation, c_1 and c_2 are real numerical constants and $a_1(t)$ and $a(t)$ are real arbitrary functions. For the Schrödinger-Maxwell system the symmetries are given by

$$
\begin{aligned}
\bar{t} &= t + \epsilon(c_1 t + c_2)\,, && (8)\\
\bar{x} &= x + \epsilon(c_1 x/2 + c_3 t^2 + c_4 t + c_5)\,, && (9)\\
\bar{\psi} &= \psi + i\epsilon\,(ic_1 + (2c_3 t + c_4)x + a(t))\,\psi\,, && (10)\\
\bar{V} &= V + \epsilon(-c_1 V + 2c_3 x + \dot{a})\,, && (11)
\end{aligned}
$$

where all the numerical constants c_i and the arbitrary function $a(t)$ are real. As one can see, the symmetry transformations are specified by five numeric parameters and one arbitrary function. The parameter c_1 corresponds to scale transformations, c_2 to time translation, c_3 to a more subtle symmetry, c_4 to Galilean boosts, c_5 to space translation and $a(t)$ to local changes in the phase of the wave function.

3 Noether's theorem

A nice feature concerning the Schrödinger-Poisson system (1)–(2) is the existence of a variational principle, for the fields ψ and V with a Lagrangian density given by

$$
L = \frac{i}{2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) + \frac{1}{2}\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x} - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 - V|\psi|^2\,. \qquad (12)
$$

This Lagrangian density can be easily derived from the quantum electrodynamics (QED) action (see for instance [11]), but to our knowledge it has not been used in the present context.

In the Lagrangian formalism, Noether's theorem states a definite relationship between point symmetries that preserves the action functional and conservation laws in the form

$$
\partial\rho/\partial t + \partial J/\partial x = 0\,. \qquad (13)
$$

The charge ρ and current J are obtained directly from the Lagrangian density and the transformation group [12]. We refrain from listing their explicit form here for obvious reasons of space limitation but their existence is important for example in applications like checking numerically obtained solutions. This problem will be considered in the work in progress.

The only symmetry of the Schrödinger-Poisson system that do not preserve the action is the rescaling transformation (specified in equations (4)–(7) by the parameter

c_1). Here we stress that energy, probability and charge conservation are obtained via Noether's theorem using respectively the symmetries of time translation, change of wave function phase and the transformation specified by $a_1(t)$ in (4)–(7). Note also that these conservation laws do not use the more restricted Schrödinger-Maxwell symmetry group (8)–(11).

This work can be further developed in several directions. The determination of exact similarity solutions, for instance, is an interesting topic. We have found similarity solutions that are much more general than the well known self-similar solution. On a different direction, spin effects certainly are relevant for some ranges of temperature and density. Although the exact account of these effects can lead to formidable obstacles in the analytical or numerical grounds, phenomenological models (such as Fokker- Planck type models) can be valuable to include spin corrections. Another desirable improvement on the present methods concerns the extension of the theory to treat more dimensions and to include other plasma components.

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