Lie Symmetries of Generalized Ermakov Systems

Fernando Haas and Joao Goedert

Instituto de Física-UFRGS, Caixa Postal 15051, 91500-970 Porto Alegre, RS - Brazil

Abstract. The Lie point symmetry analysis of Ermakov systems is extended to the case where the frequency function in the equations of motion depends also on the dynamical variables and their derivatives. From this new standpoint, a quite general class of two dimensional oscillators with nonlinear coupling can be viewed as a Hamiltonian Ermakov system possessing a Lie point symmetry and is shown to be completely integrable.

1 Introduction

Ermakov systems, first studied around the end of the nineteenth century (Ermakov 1880), were rediscovered in the late sixties (Lewis 1967) and have since then deserved much attention as a paradigm for nonlinear integrable dynamical system. The scope of application of Ermakov systems covers several interesting fields, such as the exact quantisation (Lewis and Riesenfeld 1969; Ray and Hartley 1982) of the time-dependent oscillator, the search of coherent states for nonlinear, nonautonomous systems (Ray 1982), nonlinear elasticity (Shahinpoor and Nowinski 1971), shallow water waves theory (Rogers and Rangulam 1989), cosmological particle creation (Ray 1979), the optics of elliptic gaussian beams (Goncharenko et al. 1991) and the stability of orbits in accelerators (Courant and Snyder 1958). The interested reader may consult (Rogers et al. 1993; Schief et al. 1996) and references therein for a more detailed account of the literature.

The Ermakov equations (Schief et al. 1996) are currently written as,

$$\ddot{x} + \omega^2 x = \frac{1}{yx^2} f(y/x)\,, \qquad \ddot{y} + \omega^2 y = \frac{1}{xy^2} g(x/y)\,, \tag{1}$$

where ω is an arbitrary frequency that may depend on time, the coordinates and their derivatives. The functions f and g depend on the dynamical variables as indicated by their arguments. As is easy to verify (Ray and Reid 1979a), equations (1) admit the Ermakov invariant

$$I = \frac{1}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} f(\tau)\mathrm{d}\tau + \int^{x/y} g(\tau)\mathrm{d}\tau \tag{2}$$

whereby a nonlinear superposition law (Reid and Ray 1980) can be constructed explicitly whenever one of the equations decouples, a situation that

is more likely to occur when ω is a function of time only. The usual assumption that ω is a function of time only can be relaxed and, in general, ω may depend also on the dynamical variables and their derivatives (Reid and Ray 1980; Goedert 1989) with no impact on the existence of the Ermakov invariant and the corresponding superposition law. This extra freedom has not been exploited in any detail except for very few exceptions (Goedert 1989; Haas and Goedert 1996). For instance, the Lie point symmetry analysis of Ermakov systems for $\omega = \omega(t)$ has been thoroughly investigated (Leach 1991; Govinder et al. 1993; Govinder and Leach 1994a; Govinder and Leach 1994b) but not for ω dependent on the dynamical variables. The purpose of this work is to fill this gap in the literature and perform the Lie symmetry analysis of Ermakov systems in the general case. Theoretically ω could depend on the dynamical variables and their derivatives up to any arbitrary order, besides time. For physical reasons, however, we restrict ourselves to the cases where $\omega = \omega(t, x, y, \dot{x}, \dot{y})$ only.

The paper is organized as follows. In Sect. 2 it is shown how the Ermakov system may be recast in a more compact form by redefining appropriately the functions ω, f and g. This compact form is employed in the calculations of Sect. 3, where the most general Ermakov systems that admit Lie point symmetries are determined. Section 4 presents a coupled nonlinear oscillator which is an interesting example of generalized Ermakov systems endowed with geometric symmetries. The conclusions are presented in Sect. 5.

2 Compact Representation

Ermakov systems have been presented traditionally in the form (2) which involve three arbitrary functions ω, f and g. However, and in spite of the appearance of the equations (1), only two and not three independent arbitrary functions are necessary to specify the most general Ermakov system. As explained in Haas and Goedert (1996), let the relations

$$\Omega^2 = \omega^2 - \frac{1}{xy^3} g(x/y) \tag{3}$$

$$F(y/x) = f(y/x) - \frac{x^2}{y^2} g(x/y) \tag{4}$$

define new functions Ω and $F(y/x)$ in terms of the original ω, f and g. In terms of these functions, the pair of equations (1) is

$$\ddot{x} + \Omega^2 x = \frac{1}{yx^2} F(y/x), \tag{5}$$

$$\ddot{y} + \Omega^2 y = 0. \tag{6}$$

and the Ermakov invariant becomes

$$I = \frac{1}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} F(\tau)\mathrm{d}\tau \,. \tag{7}$$

Except for very special cases, Ω will now depend on the dynamical variables (x, y) and possibly on their derivatives. Nevertheless, the term "frequency" will still be used to refer to Ω. The representation (5–6) does not clog the usual calculations with unnecessary functions, but still covers appropriately all previously discussed Ermakov systems.

The use of generalized frequencies considerably expands the reach of the Ermakov equations. In particular several dynamical systems can be cast in the Ermakov form (5–6), although not so in their first appearance. Two examples will be mentioned here, with no intend of exhausting the full span of such possibilities. The first example is the Kepler-Ermakov system (Athorne 1991a)

$$\ddot{x} = -x\, h(y/x)/r^3 + f(y/x)/(yx^2) \tag{8}$$
$$\ddot{y} = -y\, h(y/x)/r^3 + g(x/y)/(xy^2), \tag{9}$$

where $h(y/x)$ is arbitrary and $r^2 = x^2 + y^2$. The reduction to the form (5–6) is obtained by choosing

$$\Omega^2 = h(y/x)/r^3 - g(x/y)/(xy^3) \tag{10}$$
$$F(y/x) = f(y/x) - (x^2/y^2)g(x/y)\,. \tag{11}$$

Another interesting example of a nonstandard Ermakov system is Lutzky's integrable system (Lutzky 1980),

$$\ddot{x} + \omega^2(t)x = F_1(x, y) \tag{12}$$
$$\ddot{y} + \omega^2(t)y = F_2(x, y), \tag{13}$$

where F_1 and F_2 satisfy

$$xF_2 - yF_1 = \psi(x/y)/(x\,y) \tag{14}$$

for some given ψ. To cast these equations in Ermakov form, it is only necessary to define

$$\Omega^2 = \omega^2 - F_2/y \tag{15}$$
$$F(y/x) = -(x/y)\psi(x/y)\,. \tag{16}$$

In fact F_1 and F_2 could depend on velocities and time while still preserving the Ermakov character of the dynamical equations. The first integral presented in Lutzky's paper is nothing but the Ermakov invariant (7).

3 Lie Point Symmetries

The outstanding examples presented in Sect. II attest the importance of specifying the Ermakov systems in terms of only two rather than three arbitrary functions, as our starting point for its Lie symmetry study. Following Govinder and Leach (Leach 1991; Govinder and Leach 1994b), we first consider the symmetries of the system equivalent to (5–6) given by

$$x\ddot{y} - y\ddot{x} + \frac{1}{x^2}F(y/x) = 0 \tag{17}$$
$$\ddot{y} + \Omega^2(\mathbf{x}, \dot{\mathbf{x}}, t)y = 0, \tag{18}$$

where we introduced the vectorial notation $\mathbf{x} = (x, y)$ to shorten the representation. The advantage in dealing with equation (17) relies in its independence on Ω, a feature that allows the straightforward determination of its symmetry generator. This has already been done when performing the symmetry analysis of usual Ermakov systems (Govinder and Leach 1994b). The symmetry is

$$G = \rho^2\frac{\partial}{\partial t} + \rho\dot{\rho}\,\mathbf{x}\cdot\frac{\partial}{\partial \mathbf{x}}\,, \tag{19}$$

where $\rho(t)$ is an arbitrary differentiable function. The first integral I of the Ermakov system remains invariant under the first extension of G as pointed out in Govinder and Leach (1994b). When $\rho = \{1, \sqrt{t}, t\}$, we obtain time translation, self-similarity and conformal transformations respectively, that is, the full $SL(2, R)$ group is recovered.

For the complete Ermakov system (equations (17–18)) to present the same symmetry obeyed by (17), that is, in order for equation (18) to also possess the symmetry generated by G as given by (19), some restriction must be imposed on the functional form of Ω. Applying the twice extended generator G on (18) and standard Lie techniques, one conclude that the most general admissible frequencies are of the form

$$\Omega^2 = -\frac{\ddot{\rho}}{\rho} + \frac{1}{\rho^4}\sigma(\mathbf{x}/\rho, \rho\dot{\mathbf{x}} - \dot{\rho}\mathbf{x})\,, \tag{20}$$

where σ is an arbitrary function of the indicated arguments. Note that velocity dependence is also possible, a fact that considerably expands the class of Ermakov systems that admit symmetry transformations.

Let us summarize the results so far: without any *a priori* assumption on the functional form of the frequency, we obtained the most general Ermakov system (5–6) invariant under a Lie point symmetry

$$\ddot{x} + \left(-\frac{\ddot{\rho}}{\rho} + \frac{\sigma}{\rho^4}\right)x = \frac{1}{yx^2}F(y/x), \tag{21}$$
$$\ddot{y} + \left(-\frac{\ddot{\rho}}{\rho} + \frac{\sigma}{\rho^4}\right)y = 0\,. \tag{22}$$

The symmetry group is generated by G as given by (19) and σ has the form

$$\sigma = \sigma(\mathrm{x}/\rho, \rho\dot{\mathrm{x}} - \dot{\rho}\mathrm{x})\,. \tag{23}$$

It is apparent that not all generalised Ermakov systems are invariant under the Lie point symmetry G. The Kepler-Ermakov problems, for instance, do not belong to the class of invariant systems for any conceivable function ρ, σ or F.

Let us show now that all previous results on the group theoretical structure of Ermakov systems are included as special cases of our present treatment. For the particular choice

$$\ddot{\rho} + \omega^2(t)\rho = \frac{k}{\rho^3} \tag{24}$$

$$\sigma = k - \frac{\rho^4}{xy^3}g(x/y), \tag{25}$$

where k is a constant, we arrive at the Ermakov system (1) with time-dependent frequency $\omega(t)$. At least for such simple frequencies there exists a quasi-invariance transformation (Burgan et al. 1979; Athorne et al. 1990)

$$\bar{x} = x/C \qquad \bar{y} = y/C \qquad \bar{t} = \int \mathrm{d}t/C^2, \tag{26}$$

where $C(t)$ satisfies the time dependent harmonic oscillator equation

$$\ddot{C} + \omega^2(t)C = 0, \tag{27}$$

which removes ω from the equations of motion

$$\bar{x}'' = \frac{1}{\bar{y}\bar{x}^2}f(\bar{y}/\bar{x}), \qquad \bar{y}'' = \frac{1}{\bar{x}\bar{y}^2}g(\bar{x}/\bar{y})\,. \tag{28}$$

In this case the symmetry group of the transformed Ermakov system is indeed $SL(2, R)$.

The symmetry analysis performed here does not specify any particular form for the functions σ and ρ, nor depend on any previous coordinate transformation like (26). Despite its apparent simplicity, the transformation (26) would not be well defined when ω depends on the dynamical variables, because in such a case the equation (27) is meaningless. Even for $\omega = \omega(t)$, the time dependent harmonic oscillator (27) generally does not possess a closed form solution, a fact that can prevent the application of (26) in practice.

4 Two Dimensional Oscillator with Nonlinear Coupling

Unlike the simple $SL(2, R)$ invariant Ermakov systems in two spatial dimensions, the invariant Ermakov systems (21–22) do not always admit complete integrability. This is the price to pay for the extra generality. Some additional structure, perhaps a Hamiltonian formalism, must exist in order to solve exactly the equations of motion. To gain insight in this respect, let us consider the equations for a two dimensional oscillator with nonlinear coupling introduced by Ray and Reid (Ray and Reid 1979b)

$$\ddot{x} + \omega^2(t)x = x\rho^{-4}G(xy/\rho^2) \tag{29}$$

$$\ddot{y} + \omega^2(t)y = y\rho^{-4}G(xy/\rho^2), \tag{30}$$

where $\rho(t)$ is an auxiliary function constrained by

$$\ddot{\rho} + \omega^2(t)\rho = 1/\rho^3\,. \tag{31}$$

Ray and Reid found the first integral of (29–30)

$$J = (\rho\dot{x} - \dot{\rho}x)(\rho\dot{y} - \dot{\rho}y) + \frac{xy}{\rho^2} - \int^{xy/\rho^2} G(\tau)\mathrm{d}\tau. \tag{32}$$

This system is in fact an Ermakov system with $F = 0$ and coordinate dependent frequency Ω given by

$$\Omega^2 = \omega^2(t) - \frac{1}{\rho^4}G\left(\frac{xy}{\rho^2}\right). \tag{33}$$

The corresponding Ermakov invariant is

$$I = \frac{1}{2}(x\dot{y} - y\dot{x})^2. \tag{34}$$

Moreover, it is a Hamiltonian Ermakov system (Cerveró and Lejarreta 1991; Athorne 1991b; Haas and Goedert 1996) with Hamiltonian

$$H = p_x p_y + \omega^2 xy - \frac{1}{\rho^2}\int^{xy/\rho^2} G(\tau)\mathrm{d}\tau. \tag{35}$$

H is not a constant of motion since the system is not conservative. However, J and the Ermakov invariant (34) are sufficient to assure complete integrability.

We now show that (33) is indeed of the form (20) which guaranties the invariance of (29–30). Using the auxiliary equation (31) to substitute $\omega(t)$ for ρ we find that the frequency becomes

$$\Omega^2 = -\frac{\ddot{\rho}}{\rho} + \frac{1}{\rho^4}\left(1 - G\left(\frac{xy}{\rho^2}\right)\right), \tag{36}$$

which in fact is of the general form (20).

The Hamiltonian structure and the symmetry can be now further exploited to reach complete integrability, that is to reduce the problem to quadrature. For this purpose canonical group coordinates (u, v) are the adequate transformations

$$u = x/\rho\,, \qquad v = y/\rho\,, \qquad T = \int \rho^{-2}\mathrm{d}t, \tag{37}$$

under which the symmetry can be viewed as a translation, $G = \partial/\partial T$. In the new variables the equations of motion reduce to the autonomous system

$$u'' + u = uG(uv) \tag{38}$$

$$v'' + v = vG(uv) \tag{39}$$

possessing the Hamiltonian

$$K = p_u p_v + uv - \int^{uv} G(\tau)\mathrm{d}\tau\,, \tag{40}$$

which is exactly the transformed first integral J. Since there are two constants of motion in involution for a Hamiltonian system, there is complete integrability. The Hamiltonian K of the Ermakov system in canonical group coordinates is quadratic in momentum. As explained in (Haas and Goedert 1996), the appropriated variables for quadrature of this Ermakov system are given by

$$q = uv\,, \quad s = v/u\,. \tag{41}$$

In these coordinates the constants of motion become

$$\sqrt{2I} = q\,s'/s, \tag{42}$$

$$J = (q'^2 - 2I)/4q + q - \int^q G(\tau)\mathrm{d}\tau\,. \tag{43}$$

where the prime denotes the derivative with respect to the new time. The quadrature of the last two equations gives successively $q(T)$ and $s(T)$. The map to the original variables is a purely algebric task performed by the equations

$$x^2 = \rho^2 q/s\,, \qquad y^2 = \rho^2 q\,s\,. \tag{44}$$

One final step is the determination of $T(t)$ which is obtained by integrating the last equation in (37). In the quadrature sense, this completes the integration process. Recall that all the required structure and the correct choice of variables for the problem was dictated by the Hamiltonian Ermakov character and by the presence of symmetry. In particular, the analytical form of the solution can be found explicitly in terms of elliptic functions for $G(\tau) = c_1/\tau^3 + c_2 + c_3\tau + c_4\tau^2$, where the c_i are arbitrary constants.

5 Conclusion

The generalised Ermakov systems (21–22) are very general in nature and the width of their application scope is still not completely determined. In this paper we determined their Lie point symmetry group and show that a wide set of nonlinearly coupled harmonic oscillators belonging to this class is in fact completely integrable. This gives a clear indication of the variety of possible applications into which generalised Ermakov systems may enter. In particular it can be shown that Lie symmetries of the type generated by (19) and some modifications of it occur in several problems in Physics. Subclasses of two and three dimensional charged particle motion, some perturbed magnetic monopole and time-dependent Kepler problems fall into the category of systems presenting such symmetries (Haas and Goedert 1997). Extensions of this work to include contact or nonlocal symmetries can be valuable. It is also expected that the results of this paper may serve to generalise the Ermakov system to more dimensions in the spirit of Leach's approach (Leach 1991).

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Hamilton's Principal Function and Integration by Quadratures for an N-Degree-of-Freedom Non - autonomous System Given N Invertible Invariants in Involution

H. Ralph Lewis[1] and Serge Bouquet[2]

[1] Dartmouth College, Department of Physics and Astronomy, Hanover, New Hampshire 03755-3528, USA

[2] Commissariat à l'Energie Atomique, Centre de Bruyères-le-Châtel, DPTA, BP 12, 91680 Bruyères-le-Châtel, France

Abstract. Hamilton's principal function for an N-degree-of-freedom non autonomous Hamiltonian system is expressed in terms of quadratures involving N, possibly time-dependent, invariants in involution. This determines a set of $2N$ canonical coordinates and momenta, each of which is an invariant.

Keywords. Hamilton's principal function, integrability, invariant

1 Introduction

One of the long-standing interests of Prof. Marc R. Feix has been the search for exact invariants of time-dependent Hamiltonian systems (Feix and Lewis 1985; Feix et al. 1987; Lewis et al. 1992). It is appropriate on the occasion of Prof. Feix' retirement to present an account of our recent contribution to integration by quadratures and solution of the Hamilton-Jacobi equation. The collaboration that has led to our results began with an extended visit by one of us (HRL) to the group of Prof. Feix at the Université d'Orléans. A more complete account will be presented elsewhere (Lewis and Bouquet 1998).

Integrability of classical Hamiltonian systems is much studied and has a variety of meanings (Arnold et al. 1988; Tabor 1989; Gutzwiller 1990; Lichtenberg and Lieberman 1992; Ott 1993). In discussions about integrability, attention is frequently restricted to autonomous Hamiltonians and invariants, and it is common to consider the properties of solutions of the equations of motion as functions of a complex time variable. This has led to much information about the chaotic or non-chaotic nature of the solutions of the equations of motion of Hamiltonian systems. The present paper concerns the concept known as integration by quadratures (Arnold et al. 1988), whose applicability depends only weakly on the nature of the solutions of the equations of motion. The discussion applies equally to non autonomous and autonomous Hamiltonians, and solutions of the equations of motion are considered as functions of a real time variable.

A system with N degrees of freedom is said to have been integrated by quadratures if either the solutions of the equations of motion or $2N$ invariants have been expressed in terms of quadratures (*i.e.*, integrals of definite functions). The possibility of integration by quadratures for an autonomous system was first discussed by E. Bour