HAMILTONIAN STRUCTURE FOR RESCALED INTEGRABLE LORENZ SYSTEMS

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Abstract

It is shown that three among the known invariants for the Lorenz system recast the original equations into a hamiltonian form. This is made possible by an appropriate time-dependent rescaling and the use of a generalized formalism with non-trivial structure functions.

Introduction

In recent papers Nutku [1] unveiled the hamiltonian structure (in a generalized form [2]) a pair of two-dimensional dynamical systems of interest mainly in modeling biological systems and having at least one invariant. Yet more recently, Cairó and Feix [3] showed that this structures exists (and can actually be determined) for any second order system that possesses one time- independent invariant. Their analysis considered in detail the cases of the Lotka-Volterra equations with one such invariant. They also analyzed, via coordinate rescaling, a case of this system for which the invariant is time-dependent.

In this communication similar results are presented for the *three dimensional* Lorenz equations [4], one of the fundamental systems for modeling fluid mechanics. This system has recently attracted wide attention mainly in view of its ability to model the onset of chaotic behavior and the existence of strange attractors [5] in Bénard convection [6] cells. This kind of system and correspondingly peculiar behavior has been found, among others, in the description of convection in a circular fluid loop associated with the modified homopolar dynamo in geomagnetism [7], and in a generalized form, in connection with the pressure-gradient-driven interchange instability that occurs in laboratory plasmas [8] in a two-fluid description model.

The Lorenz equations support both periodic and non periodic regimes depending on the parameter values. Most frequently the non periodic regimes are the most interesting in view of their wide occurrence in nature. In laboratory conditions however, one is frequently more interested in stationary or relaxed regimes which correspond to parameter values that might imply the existence of invariant or integral of the motion. A few such invariants have been identified and collected in the literature [9]. All these known invariants are explicitly time-dependent and three among them admit coordinate rescaling that removes the time dependence. For these special cases, it is possible to recast the original system into a generalized hamiltonian structure, a result similar to that obtained for generic two dimensional systems or the Lotka-Volterra equations in particular with (at least) one time- independent first integral.

Basic equations, rescaling, and results

The Lorenz system

$$\dot{x} = \sigma y - \sigma x, \qquad \dot{y} = -y + rx - xz, \qquad \dot{z} = -bz + xy, \tag{1}$$

is known to possess first integral for several ranges of parameter values. Apart from the linear case ($\sigma = 0$), exact invariants were determined [9] for *six* different combinations of the parameters (σ, b, r). Among these, *three* cases have invariants that, as shown in the sequel, are rescalable and generate a hamiltonian structure for the corresponding rescaled systems.

CASE I: For arbitrary r and $b = 2\sigma$, the function $I_1 = (x^2 - 2\sigma z)\exp(2\sigma t)$ is an exact invariant of the system (1).

Let the independent variables be rescaled according to

$$x_1 = x\exp(\sigma t), \qquad x_2 = y\exp(t), \qquad x_3 = z\exp(2\sigma t). \tag{2}$$

This transformation recasts $H = x_1^2 - 2\sigma x_3$ and the Lorenz system into the time-dependent form

$$\dot{x}_1 = \sigma x_2 \exp{(\sigma - 1)t}, \dot{x}_2 = r x_1 \exp{(1-\sigma)t} - x_1 x_3 \exp{(1 - 3\sigma)t}, \dot{x}_3 = x_1 x_2 \exp{(\sigma - 1)t}. \tag{3}$$

As can be easily checked by inspection, system (3) can be written in the *hamiltonian* form

$$\dot{x} = J^{ik}\nabla_k H, \qquad i = 1, 2, 3, \tag{4}$$

where the structure functions J^{ik} are the elements of the antisymetric matrix

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} 0 & x_3 \exp{(1-3\sigma)t} & -x_2 \exp{(\sigma - 1)t} \\ -x_3 \exp{(1 - 3\sigma)t} & 0 & -x_1 r/\sigma \exp{(1-\sigma)t} \\ x_2 \exp{(\sigma - 1)t} & x_1 r/\sigma \exp{(1-\sigma)t} & 0 \end{pmatrix}. \tag{5}$$

Finally, to complete the generalized hamiltonian character of the rescaled Lorenz system (3) it can be directly checked that the elements of $\mathcal{J}$ do also satisfy the Jacobi identity

$$J^{i[k}\nabla_i J^{lm]} = 0, \tag{6}$$

where the square brackets indicate a sum over the cyclic permutation of the enclosed super indices and sum over i is also implied.

CASE II: For $b = 1, r = 0$ and arbitrary σ, $I_2 = (y^2 + z^2)\exp(2t)$ is a second time-dependent invariant of (1). For this parameter choice, the transformations

$$x_1 = x\exp(\sigma t), \qquad x_2 = y\exp(t), \qquad x_3 = z\exp(t), \tag{7}$$

recast I_2 into the time-independent invariant form $H = x_2^2 + x_3^2$ for the time-dependent rescaled system

$$\dot{x}_1 = \sigma x_2 \exp(\sigma - 1)t, \qquad \dot{x}_2 = -x_1 x_3 \exp(-\sigma t), \qquad \dot{x}_3 = x_1 x_2 \exp(-\sigma t) \tag{8}$$

It is now easy to check that in this case equations (8) take the hamiltonian form (4) as long as $\mathcal{J}$ be given by

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} 0 & \sigma\exp{(\sigma-1)t} & 0 \\ -\sigma\exp{(\sigma - 1)t} & 0 & -x_1\exp{(-\sigma t)} \\ 0 & x_1 \exp{(-\sigma t)} & 0 \end{pmatrix}. \tag{9}$$

which also satisfies the Jacobi identity.

CASE III: Finally the case characterized by $b = \sigma = 1$ and r arbitrary, admits the invariant $I_3 = (y^2 - rx^2 + z^2)\exp(2t)$ and can be similarly rescaled by the transformation

$$x_1 = x\exp(t), \qquad x_2 = y\exp(t), \qquad x_3 = z\exp(t). \tag{10}$$

This transformation recast I_3 into the time-independent invariant $H = x_2^2 - rx_1^2 + x_3^2$ and the dynamical system (1) into

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = rx_1 - x_1x_3\exp(-t), \qquad \dot{x}_3 = x_1x_2\exp(-t), \tag{11}$$

which can also be written into the hamiltonian form (4), this time through the following structure matrix

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -x_1\exp(-t) \\ 0 & x_1\exp(-t) & 0 \end{pmatrix}. \tag{12}$$

Again the structure function in (12) satisfy the Jacobi identity ensuring, therefore, the complete hamiltonian character of the rescaled Lorenz system for this third set of parameter values.

There are indications that the results briefly presented in this communication can be extended to all third order systems that either possess a time-independent invariant or possess a time-dependent invariant that can be rescaled. The rescaling has the role of transferring the time dependence into the structure functions and converting the invariant into the Hamiltonian for system. The proofs for these conjectures are still under analysis and the results should be submitted for publication soon.

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