ON THE HAMILTONIAN STRUCTURE OF ERMAKOV SYSTEMS

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Ermakov systems have been intensively studied since the late sixties both in view of their nice mathematical properties and the application potential of their celebrated invariants which ranges from the stability analysis of charged particle orbits in cyclotron accelerators [1] to the exact quantum description of the time-dependent harmonic oscillator [2], quantum optics [3], and two-layer, shallow water waves [4] . However, one of the original motivation for considering Ermakov systems was the desire to investigate the nature of the magnetic-moment series for a charged particle moving non-relativistically in certain electromagnetic field configurations [5] (see also [6] and [7] and references therein for an updated appraisal of the Ermakov system subject). More recently the identification of additional structures in the original Ermakov-Pinney system and in its generalized version, the Ermakov-Lewis-Ray-Reid (ELRR) system, has called for extra attention. After an early inventory of their symmetry properties [6] and the analysis of various generalization schemes [4], the attention has been, more recently, centered on the joint existence of a second independent constant of motion or a Hamiltonian for subclasses of Ermakov systems [8]–[9]. This particular topic is important *per se* because, for Ermakov systems, the existence of a second constant of motion usually implies complete integration. Most important, however, the Hamiltonian structure is fundamental in various contexts like quantization and perturbation theory.

The most general system in two configuration variables that qualifies as an Ermakov or ELRR system is usually written as

$$\ddot{x} + \omega^2 x = \frac{1}{yx^2} f(y/x), \tag{1}$$

$$\ddot{y} + \omega^2 y = \frac{1}{xy^2} g(x/y), \tag{2}$$

where the overdot stands for time derivative, f and g are arbitrary functions of their argument and ω is an arbitrary function of time t, and of x, y and the time derivatives of x and y of first and higher order. For practical reasons, we shell consider $\omega = \omega(t, x, y)$ only.

The system of equations (1–2) is known to possess the Lewis–Ray–Reid invariant

$$I = \frac{1}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} f(r)\,dr + \int^{x/y} g(r)\,dr\,. \tag{3}$$

This invariant persists for arbitrary dependence of ω on x and y. We can therefore redefine f and combine ω^2 and g in a single function

$$\Omega^2 \equiv \omega^2(t, x, y) - \frac{1}{xy^3}\, g(x/y) \tag{4}$$

$$F(s) = f(s) - s^{-2} g(1/s). \tag{5}$$

These redefinitions imply no loss of generality and simplify future considerations by casting the ELRR system in the more compact form

$$\ddot{x} + \Omega^2 x = \frac{1}{yx^2} F(y/x) \tag{6}$$

$$\ddot{y} + \Omega^2 y = 0. \tag{7}$$

The transformation (4–5) also indicates that the conventional ELRR system involves only *two* arbitrary functions and not three as implied by the traditional notation. So we shell consider the canonical Hamiltonian structure of ELRR systems in the representation (6–7).

As already mentioned, ω in (1– 2) can be a function of time and any combinations of the dynamical variables x, y and their time derivatives of arbitrary order. In this work we shell consider the Hamiltonian property of the ELRR system for which the frequency function is allowed to depend not only on time (as usually done) but also on x and y. In this case, the resulting constraint becomes less restrictive and we are able to find a much wider class of systems satisfying the Hamiltonian property.

We consider for the Hamiltonian of the Ermakov system (6–7) the following function

$$H = \frac{A}{2}p_x^2 + Bp_xp_y + \frac{C}{2}p_y^2 + V(x,y,t), \tag{8}$$

where A, B and C are numbers, not all zero, and $V(x,y,t)$ is a potential function depending on time and the spatial variables.

The *ansatz* (8) is justified by Douglas theory for two dimensional Lagrangian systems [10], which shows that the coefficients for the quadratic terms in the velocities in a Lagrangian are constants of motion, at least for velocity free force fields. In particular, these coefficients can be taken as numerical constants. As one can show, the addition of a term linear in the momenta does not alter the generality of the description. Finally, both cases of Hamiltonian Ermakov systems known in the literature [8]–[9] are of the proposed form.

We now impose that the canonical Hamilton equations generate the ELRR system (6–7) and obtain that the admissible frequencies satisfy

$$\Omega^2 = \frac{1}{y}(B\frac{\partial V}{\partial x} + C\frac{\partial V}{\partial y}), \tag{9}$$

and that the potential obey the linear first order partial differential equation

$$(Bx - Ay)\frac{\partial V}{\partial x} + (Cx - By)\frac{\partial V}{\partial y} = \frac{1}{x^2}F(y/x). \tag{10}$$

Equation (10) can be easily solved by the method of characteristics yielding

$$V = \frac{1}{2}\Lambda(q,t) + \frac{1}{q}\int^s F(r)dr. \tag{11}$$

Here and in the remaining text, $\Lambda(q,t)$ is an arbitrary function of its arguments, and the variables q and s are defined by

$$q = Ay^2 - 2Bxy + Cx^2, \tag{12}$$

$$s = y/x. \tag{13}$$

It is also convenient to define the function

$$\xi(s) = As^2 - 2Bs + C, \tag{14}$$

so that $q = x^2\xi(s)$.

The resulting Hamiltonian ELRR system implied by the admissible frequencies and potentials, in the conventional notation, reads

$$\ddot{x} + \Delta\frac{\partial\Lambda}{\partial q}x = \frac{1}{yx^2}\bar{f}(y/x), \tag{15}$$

$$\ddot{y} + \Delta\frac{\partial\Lambda}{\partial q}y = \frac{1}{xy^2}\bar{g}(x/y), \tag{16}$$

where

$$\bar{f}(s) = 2\Delta\frac{s}{\xi^2}\int^s F(r)dr + (As - B)\frac{s}{\xi}F(s), \tag{17}$$

$$\bar{g}(1/s) = 2\Delta\frac{s^3}{\xi^2}\int^s F(r)dr + (Bs - C)\frac{s^2}{\xi}F(s), \tag{18}$$

$$\Delta = AC - B^2. \tag{19}$$

It is now appropriate to compare our result with those in the literature. Cerveró and Lejarreta's subclass of Hamiltonian Ermakov systems [9] is obtained from the present formalism by setting $A = C = 1$, $B = 0$, and $\Lambda = \omega^2(t)q$ in (15–16). The resulting functions $\bar{f}$ and $\bar{g}$ do satisfy the Hamiltonian constraint stated in [9]. This Hamiltonian has already been used in the study of the propagation of elliptic gaussian beams in nonlinear, dispersive media [3]. The completely integrable class of Ermakov systems determined by Goedert [8], is derivable from the present formalism by taking $A = C = 0$, $B = 1$, and $\Lambda = 2\int^{-q/2}\omega^2(Q)dQ$. Needless to say, the functions $\bar{f}$ and $\bar{g}$ resulting from this prescription do satisfy the integrability condition stated in [8].

For time-independent Λ, H is a second constant of motion. In this case the canonical formalism can be used to solve elegantly the corresponding equations of motion. For this we make use of both constants of motion, the Hamiltonian and the Lewis- Ray-Reid invariant, to reduce the problem to two separable ordinary differential equations,

$$(dq/dt)^2 = 4\Delta(2qH - q\Lambda(q) - 2I), \tag{20}$$

$$(ds/dt)^2 = 2q^{-2}(t)\xi^2(s)\left(I - \int^s F(r)dr\right), \tag{21}$$

wich can be successively solved in terms of quadratures.

The generality of the dynamical system (15–16) (which contain two arbitrary functions, Ω and F) strongly supports the believe that it will play a central role in several physical applications. In particular, we have found the general solution for a problem arising in two-layer, shallow water theory. This and the propagation of elliptic gaussian beams in nonlinear, dispersive media is the subject of ongoing research and will be communicated in a forthcoming paper.

This work has been partially supported by the brazilian agencies Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Financiadora de Estudos e Projetos (FINEP).

References

[1] E. R. Courant and W. D. Snyder, Ann. Phys., **3**(1958)1.

[2] H. R. Lewis and W. Riesenfeld, J. Math. Phys., **10**(1969)1458.

[3] A. M. Goncharencko, Yu. A. Logvin, A. M. Samson, P. S. Shapovalov and S. I. Turovets, Phys. Lett. A, **160**(1991)138.

[4] C. Athorne, Phys. Lett. A, **159**(1991)375.

[5] H. R. Lewis, Phys. Rev. Lett., **18**(1967)510.

[6] K. S. Govinder and P. G. L. Leach, Phys. Lett. A, **186**(1994)391.

[7] C. Rogers, C. Hoenselaers and J. R. Ray, J. Phys. A: Math. Gen., **26**(1993)2625.

[8] J. Goedert, Phys. Lett. A, **136**(1989)391.

[9] J. M. Cerveró and J. D. Lejarreta, Phys. Lett. A, **156**(1991)201.

[10] J. Douglas, Trans. Am. Math. Soc., **50**(1941)71.