On the Generalized Hamiltonian Structure of 3D Dynamical Systems

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Abstract

The Poisson structures for 3D systems possessing one constant of motion can always be constructed from the solution of a linear PDE. When two constants of the motion are available the problem reduces to a quadrature and the structure functions include an arbitrary function of them.

1 Introduction

Three dimensional dynamical systems have deserved much attention both in view of their intrinsic mathematical relevance and of their wide interest in domains such as Mechanics [1], optics [2]-[3], dynamic of interacting populations [4]-[7], modeling of fluid turbulence [8]-[9], wave interaction models [10]-[12], dynamo theory [13], and several other areas of physical, chemical or biological importance. A more recent related issue concerns the Poisson structures of these systems. This question has been contemplated both from the point of view of their existence [14] and of their explicit determination [15]-[18]. As a rule, 3D systems possess a Poisson structure whenever a sufficient number (e. g. *two*) of independent constants of motion exist [19]. Their explicit construction however is a partly open problem which, as we show in this letter, can always be solved when two constants of the motion are known.

In a recent paper Gumral and Nutku [16] reduced the problem of the determination of the structure functions associated to the Poisson structure of 3D dynamical systems to the solution of a quasi-linear partial differential equation (eq. (70) of their paper) in three independent variables. When two independent constants of the motion are known, the problem therefore reduces to a Ricatti equation. In this letter we show that in fact the Poisson structures of 3D systems possessing one constant of the motion can always be obtained from the solutions of a *linear* partial differential equation and that when two independent constants of the motion are known the problem reduces to a quadrature. In such cases the resulting structure functions may involve an arbitrary function of the constants of the motion. When only one constant of the motion is available, the problem can frequently be handled in the sense that a generalized Hamiltonian formalism may still be constructed in terms of some particular solution of the pertaining equations. To illustrate the procedure we shall consider examples of both types. An interesting example with only one constant of motion is a five parameter version of the 3D Lotka-Volterra system for which a Hamiltonian formalism can be constructed. A four parameter version of the same system was already known to possess a bi-Hamiltonian structure [6].

2 Generalized Hamiltonian Structures

In this section we consider a generic dynamical system in N dimensions

$$\dot{x}^\mu = v^\mu(\mathbf{x}, t)\,, \qquad \mu = 1, \ldots, N, \tag{1}$$

where v^μ is a sufficiently smooth vector field (in general $v^\mu \in C^\infty$), $\mathbf{x} = (x^1, x^2, \ldots, x^N)$, and the over dot denotes derivative with respect to t. In addition we consider a function $H(\mathbf{x}, t)$ satisfying

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}\,, \tag{2}$$

along any phase trajectory, that is

$$v^\mu\, \partial_\mu H = 0\,, \tag{3}$$

where ∂_μ indicates the partial derivative with respect to x^μ and repeated indices represent the Einstein's summation convention. In specific applications the function H is typically a time-independent first integral of (1) valid over a region in phase space. Finally we consider an *anti-symmetric* matrix $\mathcal{J}$ which satisfies the Jacobi identities [20]

$$J^{\mu[i}\,\partial_\mu J^{jk]} = 0\,, \tag{4}$$

and therefore provides a generalized definition for the Poisson bracket

$$[F,\,G] \equiv \partial_\mu F\, J^{\mu\nu}\,\partial_\nu G\,, \tag{5}$$

of functions F and G in phase space.
Definition: System (1) is said to be Hamiltonian iff there exists a function H satisfying (2) and an anti-symmetric matrix $\mathcal{J}$ satisfying the Jacobi identities such that

$$v^\mu \equiv J^{\mu\nu}\,\partial_\nu H\,. \tag{6}$$

In such case, H is called the Hamiltonian of (1) and $\mathcal{J}$ the associated Lie tensor or matrix of structure functions.

For given $H(\mathbf{x},t)$ let us label the variables such that $\partial_N H \neq 0$.
Lemma: if for some anti-symmetric matrix $J^{\mu\nu}$ and $H(\mathbf{x},t)$ satisfying (3),

$$v^s = J^{s\nu}\,\partial_\nu H\,, \qquad \text{for} \qquad s = 1\,,\dots\,,(N-1) \tag{7}$$

then

$$v^N = J^{N\nu}\,\partial_\nu H\,, \qquad \text{for} \qquad s = N\,. \tag{8}$$

The implication of the lemma is clear: if a dynamical system possess a constant of motion and $N-1$ of its equations are in a Hamiltonian-like form then its last equation is necessarily of the same form.

Proof: Use (7) to expand (3) in terms of $J^{s\nu}$

$$\sum_{s=1}^{N-1}\partial_s H\,J^{s\nu}\,\partial_\nu H + v^N\,\partial_N H = [H,\,H] + \partial_N H\left(v^N - J^{N\nu}\partial_\nu H\right) = 0. \tag{9}$$

The proposition now follows from the last equality, the property of the Poisson bracket, and the fact that $\partial_N H \neq 0$.

Corollary: When (7) holds, $(N-1)$ components of the Lie tensor can be represented in terms of the remaining $(N-1)(N-2)/2$ ones:

$$J^{N\mu} = -J^{\mu N} = \left(v^\mu - \sum_{\nu=1}^{N-1} J^{\mu\nu}\, \partial_\nu H \right) / \left(\partial_N H \right) \, . \tag{10}$$

The proof is constructed by solving (7) for $J^{N\nu}$.

Thus, any function satisfying (3) recast system (1) in the "pre-Hamiltonian" form (6). The algorithm consists of taking $(N-1)(N-2)/2$ arbitrary functions $J^{\mu\nu}$ $(\mu < \nu = 1, \ldots, (N-1))$ and filling up the Lie matrix with them, their anti- symmetric counterpart and the $J^{N\mu}$ given by (10). To complete the hamiltonization process we finally demand that $J^{\mu\nu}$ obey the Jacobi identities which then become their determining equations.

In 3D the Jacobi identities are compatible with a generic conformal rescaling of the structure functions [15]-[16]. Such invariance is, however, partially restricted for $J^{\mu\nu}$ constrained by relation (10). In fact, when a constant of the motion exists, the generic conformal invariance admitted by the Jacobi identities in 3D become a scale invariance by functions of the constants of motion and the Casimir only. We now prove this statement:

Suppose that $\bar{J}^{\mu\nu} \equiv \gamma(x) J^{\mu\nu}$ are structure functions for some Hamiltonian $\bar{H}$ when $J^{\mu\nu}$ are the corresponding structures functions for H. The equivalence of the two representations for the same system implies

$$J^{\mu\nu} (\gamma \partial_\nu \bar{H} - \partial_\nu H) = 0 \, . \tag{11}$$

As a Hamiltonian for a 3D autonomous system, $\bar{H}$ is necessarily a function of the two independent constants of the motion. Locally [21], these constants may be chosen as the Hamiltonian H and the Casimir of $J^{\mu\nu}$. Therefore, without loss of generallity,

$$\bar{H} = F(H, \bar{C}) \, . \tag{12}$$

Replacing $\bar{H} = F(H, \bar{C})$ in (11) and considering that the Casimir commutes with all functions of the dynamical variables yields

$$\gamma = 1 / \frac{\partial F}{\partial H} = \gamma(H, \bar{C}) \, . \tag{13}$$

To close the section we remark that in general the Jacobi identities form an overdetermined system of non linear equations in the unknown $J^{\mu\nu}$. In fact, when a constant of the motion exists they form a set of $N!/\,(3!(N-3)!)$ equations in $(N-1)(N-2)/2$ unknowns $\{J^{\mu\nu} : \mu < \nu = 1,\ldots,(N-1)\}$. For $N < 3$ the Jacobi identities are automatic and the hamiltonization process is trivial. For $N > 3$ the resulting system of equations is usually overdetermined. For $N = 3$ (which is the object of this letter) they become a *linear* partial differential equation. When two constants of motion are available, the problem is reduced to a first order ODE with solutions in terms of quadratures. The general solution for the structure functions contains an arbitrary function of the constants of the motion. The prove of this statement is presented in the following section.

3 Three dimensional systems

For $N = 3$, equations (7) read

$$v^1 = J^{12}\,\partial_2 H + J^{13}\,\partial_3 H\,, \tag{14}$$

$$v^2 = -J^{12}\,\partial_1 H + J^{23}\,\partial_3 H. \tag{15}$$

We can solve these equations in terms of some function J to be determined latter

$$\begin{aligned} J^{12} &= J\,, \\ J^{13} &= \Big(v^1 - J\,\partial_2 H\Big)/(\partial_3 H)\,, \\ J^{23} &= \Big(v^2 + J\,\partial_1 H\Big)/(\partial_3 H)\,. \end{aligned} \tag{16}$$

Recall that a different labeling of the variables must be adopted when $\partial_3 H \equiv 0$. This implies that equations (16) undergo a cyclic permutation of the indices $(1 \to 2 \to 3 \to 1)$ that exchanges ∂_3 with either ∂_2 or ∂_1. This permutation must be applied simultaneously to equations (18-19) bellow.

Equations (16) express all the components of the Lie tensor in 3D in terms of the symbol J determined by the Jacobi identity. To see this we substitute (16) into (4) and multiply the result by $\partial_3 H$. The cross derivative and the quadratic terms cancel out, and, after using (3) and some simple algebra, we obtain

$$v^\mu \partial_\mu J = A\,J + B, \tag{17}$$

where

$$A = \partial_\mu v^\mu - \frac{(\partial_3 v^\mu)(\partial_\mu H)}{\partial_3 H} \tag{18}$$

and

$$B = \frac{v^1\,\partial_3 v^2 - v^2\,\partial_3 v^1}{\partial_3 H}. \tag{19}$$

In this equations the indices 1, 2 and 3 are linked to the indices in equation (16) and, as already mentioned, should undergo a cyclic permutation whenever $\partial_3 H = 0$.

Equation (17), which is the Jacobi identity in terms of J, is the key equation in the solution of the hamiltonization problem of 3D systems. As mentioned before, this last condition is a linear first order PDE. When two constants of the motion are known, one of the characteristic equations for (17) becomes a linear first order ODE (two out of the three dynamical variables are solved in terms of a third one and the constants of motion). In this case J can be solved by quadratures and the structures functions may always involve an arbitrary function of the two constants of the motion.

It is important to remark here that the scale invariance referred to before is preserved by equations (17-19): if H is a Hamiltonian with associated Lie tensor (16) (expressed in terms of J) then $F(H, C)$, F arbitrary, is another valid Hamiltonian but with associated Lie tensor $\bar{J} \equiv J/(\partial F/\partial H)$. In fact, it can be easily checked that if J satisfies equation (17) for A and B calculated from H, then $\bar{J}$ satisfies the same equation with A and B calculated from $F(H, C)$. The multiplicity of solutions for the hamiltonization process is still greater if we consider that equation (17), being a first order linear PDE, in general admits an infinity of solutions. Sometimes, as we shall show in section 4, solutions in terms of arbitrary functions can also be found.

As final remark of practical interest we point that any particular solution to (17) provides a non trivial Poisson structure for the system under consideration. Interesting particular cases are obtained when the system has $B = 0$, in which case $J = 0$ is the simplest possible solution or when $B(\mathbf{x}, t) = -f(H, C)\,A(\mathbf{x}, t)$ in which case $J = f(H, C)$ is an equally simple particular solution.

4 Sample applications

To illustrate some of the various possibilities offered by the procedure proposed above we consider now three sample problems in detail. Several other systems can be equally treated by repeating a similar sequence of steps. A more exhaustive and detailed list of examples is being prepared and will soon be submitted for publication elsewhere.

4.1 The ice skate problem

As a first example we consider the ice skate problem studied by Lucey [17]. In the original treatment one of the equations was $\dot{x}^2 = 0$ which can be solved for $x^2 = -a =$ constant. This transforms the 4D system into a 3D one. We next relabel the variables using the replacements $x^4 \to x^3 \to x^2$. In this notation the ice skate system reads

$$\dot{x}^1 = -a\,, \qquad \dot{x}^2 = x^3\,, \qquad \dot{x}^3 = ax^3 \tan x^1. \tag{20}$$

We now verify that $H_1 = x^3 sec(x^1)$ is a constant of the motion and therefore a candidate as the Hamiltonian for the system. Use of this Hamiltonian as a source in (18-19) gives $A = 0$ and $B = -a\cos x^1$. We next verify that one of the characteristic equations of (17),

$$\frac{dJ}{dx^1} = \cos x^1, \tag{21}$$

is separate from the others and can be integrated in the form

$$J = \sin x^1 + F(H_1, H_2)\,, \tag{22}$$

where F is an arbitrary function of constants of the motion. The following structures functions are now obtained by substitution of (22) in (16)

$$\begin{aligned} J^{12} &= -J^{21} = \sin x^1 + F(H_1, H_2)\,, \\ J^{13} &= -J^{31} = -a\cos x^1\,, \\ J^{23} &= -J^{32} = x^3\left(\sec x^1 + F(H_1, H_2)\tan x^1\right)\,. \end{aligned} \tag{23}$$

To find a Casimir of the algebra it is necessary to specify the function F. In particular for $F = 0$, $C_1 = ax^2 + x^3 \tan x^1$, is a Casimir of the system.

We now verify that the ice skate problem is in fact completely integrable since C_1 is another constant of motion, functionally independent of H_1. We may therefore use $H_2 = C_1$ as an alternative Hamiltonian in which case $A = B = -a \cot x^1$ and

$$\bar{J} = -1 + \bar{F}(H_1, H_2) \sin x^1 \,, \tag{24}$$

for any arbitrary function $\bar{F}$ of the constants of the motion. This generates the alternative Poisson structure

$$\begin{aligned} \bar{J}^{12} &= -\bar{J}^{21} = -1 + \bar{F}(H_1, H_2) \sin x^1 \,, \\ \bar{J}^{13} &= -\bar{J}^{31} = -a\, \bar{F}(H_1, H_2) \cos x^1 \,, \\ \bar{J}^{23} &= -\bar{J}^{32} = \quad x^3 \left(-\tan x^1 + \bar{F}(H_1, H_2) \sec x^1 \right) . \end{aligned} \tag{25}$$

For $\bar{F} = 0$ the new algebra has Casimir $C_2 = H_1 = x^3 \sec x^1$. It is worthwhile remarking that the algebras corresponding to J and $\bar{J}$ are independent and not connected by conformal transformations. The rescaling invariance by functions of the Hamiltonian and the Casimir can still be applied to generate families of equivalent Poisson structures.

We have presented a complete and novel solution to the hamiltonization of the ice skate system. This problem illustrates clearly the basic features of the routine proposed. In the next subsection we analyze another sample problem and show that a well known system may still exhibits some novel and perhaps unexpected features.

4.2 The Euler top

We now revisit the Euler top problem and show that also this system possesses families of alternating Hamiltonian and Casimir with the associated Poisson structure involving arbitrary functions of the two.

The classical Euler top system [1] can be written as

$$\begin{aligned} \dot{x}^1 &= (I_2 - I_3) x^2 x^3 / (I_2 I_3) \,, \\ \dot{x}^2 &= (I_3 - I_1) x^3 x^1 / (I_3 I_1) \,, \\ \dot{x}^3 &= (I_1 - I_2) x^1 x^2 / (I_1 I_2) \,, \end{aligned} \tag{26}$$

and admit the (kinetic) energy

$$H = \frac{1}{2}\left(\frac{(x^1)^2}{I_1} + \frac{(x^2)^2}{I_2} + \frac{(x^3)^2}{I_3}\right) , \tag{27}$$

and the angular momentum

$$L = (x^1)^2 + (x^2)^2 + (x^3)^2 , \tag{28}$$

as independent constants of the motion. Taking the energy as the Hamiltonian we find $B = 0$ and

$$A = (I_1 - I_2)x^1x^2/(I_1I_2x^3) \equiv \dot{x}^3/x^3 . \tag{29}$$

For these values of A and B the solutions J to the basic equation (17) are easily found and the corresponding structure functions become

$$\begin{aligned} J^{12} &= -J^{21} = -x^3\,(1 + F(H,L)) , \\ J^{13} &= -J^{31} = \quad x^2\,(1 + I_3F(H,L)/I_2) , \\ J^{23} &= -J^{32} = -x^1\,(1 + I_3F(H,L)/I_1) , \end{aligned} \tag{30}$$

where again F is an arbitrary function of its arguments. When $F = 0$ the usual results are recovered with the angular momentum L as the Casimir.

Alternatively, starting with the angular momentum L as the Hamiltonian, leads to the same forms for A and B but the new structure functions become (where $\bar{F}$ is also an arbitrary function of its arguments)

$$\begin{aligned} \bar{J}^{12} &= -\bar{J}^{21} = x^3\left(1/I_3 + \bar{F}(H,L)\right) , \\ \bar{J}^{13} &= -\bar{J}^{31} = -x^2\left(1/I_2 + \bar{F}(H,L)\right) , \\ \bar{J}^{23} &= -\bar{J}^{32} = x^1\left(1/I_1 + \bar{F}(H,L)\right) . \end{aligned} \tag{31}$$

For $\bar{F} = 0$ the kinetic energy H is the Casimir of the new algebra.

The analysis presented here shows that the Euler top system admits a continuous family of Poisson structures in which the kinetic energy and the angular momentum play alternatively the roles of Hamiltonian and Casimir. Each pair of Hamiltonian and Casimir have associated Lie algebra that are independent, that is, that are not connected by conformal rescaling.

4.3 The 3D Lotka-Volterra system

The 3D Lotka-Volterra system and some of its special subsystems play an important role in modeling many physical, chemical and biological processes. Their associated vector field is defined (in their most general form with Verhulst terms $b_{ii} \neq 0$) by

$$v^k \equiv x^k(a_k + b_{k\mu}x^\mu) \qquad\qquad k = 1, \dots N\,. \tag{32}$$

In equation (32) and throughout the rest of this letter sum over the Latin index k is not implied.

Cairó and Feix [4] found several invariants or first integrals for N-dimensional Lotka-Volterra systems. In particular for $N = 3$ and $\det(b_{ij}) = 0$, one such invariant is

$$\mathcal{H} = H(x^1, x^2, x^3)\, e^{-s\,t} \equiv (x^1)^\alpha (x^2)^\beta (x^3)^\gamma e^{-s\,t} \tag{33}$$

where α, β, γ and s are given by

$$\begin{aligned} \alpha &= b_{22}b_{31} - b_{21}b_{32}\,, \\ \beta &= b_{11}b_{32} - b_{12}b_{31}\,, \\ \gamma &= b_{12}b_{21} - b_{11}b_{22}\,, \\ s &= a_1\alpha + a_2\beta + a_3\gamma\,. \end{aligned}$$

Under the constraint $s = 0$, that is, for

$$a_1(b_{22}b_{31} - b_{21}b_{32}) + a_2(b_{11}b_{32} - b_{12}b_{31}) + a_3(b_{12}b_{21} - b_{11}b_{22}) = 0\,, \tag{34}$$

the function H becomes a time-independent first integral. Bellow we present the derivation of a Poisson structure for this system under the less restrictive conditions that we can find. This will ultimately imply that among the initially free coefficients a_k and b_{ij} only five remain arbitrary. Unfortunately no other constant of the motion is known for the same number of free parameters and complete integrability cannot be sought. It is however interesting to verify that under appropriate conditions one can construct a Poisson structure without being actually able to integrate the equations completely.

In order to tackle the basic equation (17) we first calculate A and B in terms of the vector field v^μ and of the constant of the motion H. For this, equations (18-19) and (33) yield

$$A = a_1 + a_2 + b_{11}x^1 + b_{22}x^2 + b_{33}x^3 + (b_{1\mu} + b_{2\mu})x^\mu\,, \tag{35}$$

$$B = \frac{U}{\gamma H}[(a_1 b_{23} - a_2 b_{13}) + (b_{23}b_{1\mu} - b_{13}b_{2\mu})x^\mu]\,, \tag{36}$$

where we introduced the symbol $U = x^1x^2x^3$ to simplify the notation. To proceed we write the characteristic equation associated to (17)

$$\frac{d\,x^k}{x^k(a_k + b_{k\mu}x^\mu)} = \frac{dJ}{A\,J + B}\,, \tag{37}$$

which after some element but tedious algebra implies

$$\frac{dU}{U\,[A + a_3 + (b_{31} - b_{11})x^1 + (b_{32} - b_{22})x^2]} = \frac{dJ}{AJ + B}\,. \tag{38}$$

We now multiply the numerator and the denominator of the first term in equation (38) by $\epsilon/\gamma H$ where ϵ is an arbitrary constant to be determined later, and re-write the characteristic equations in the form

$$\frac{d\,x^k}{x^k(a_k + b_{k\mu}x^\mu)} = \frac{d\,[J - \epsilon U/(\gamma H)]}{A\,[J - \epsilon U/(\gamma H)] - B'\,(U/(\gamma H))}\,, \tag{39}$$

with B' defined by

$$\begin{aligned} B' &= a_1 b_{23} - a_2 b_{13} - \epsilon a_3 + [\epsilon(b_{11} - b_{31}) + b_{23}b_{11} - b_{13}b_{21}]\,x^1 \\ &+ [\epsilon(b_{22} - b_{32}) + b_{23}b_{12} - b_{13}b_{22}]\,x^2\,. \end{aligned} \tag{40}$$

Equation (39) is difficult to treat in its general form. One particular solution however can be readily found by imposing the additional condition $B' = 0$. As can be easily checked, under this condition

$$J = \frac{\epsilon U}{\gamma H} = \frac{\epsilon}{\gamma}(x^1)^{1-\alpha}(x^2)^{1-\beta}(x^3)^{1-\gamma}. \tag{41}$$

satisfies equation (39) and therefore the fundamental equation (17). The condition on B' produces the value of the arbitrary constant ϵ and (for arbitrary x^1 and x^2) two additional constraints on the coefficients of the system.

This represents a total of *four* constraints ($\det(b_{ij}) = 0$, $s = 0$ and two out of conditions (42-44) below) on the *twelve* initially free parameters. This corresponds to a total of eight free parameters which in fact reduces to *five* since three parameters can always be eliminated by rescaling.

Condition (40) for arbitrary x^1 and x^2, implies three equations one of which can be solved for ϵ when either (i) $a_3 \neq 0$ or (ii) $b_{31} \neq b_{11}$ or (iii) $b_{32} \neq b_{22}$. These conditions imply the following relations

$$
\begin{aligned}
(a_1 b_{23} - a_2 b_{13})(b_{31} - b_{11}) &= a_3(b_{23}b_{11} - b_{13}b_{21})\,, && (42)\\
(a_1 b_{23} - a_2 b_{13})(b_{32} - b_{22}) &= a_3(b_{23}b_{12} - b_{13}b_{22})\,, && (43)\\
(b_{23}b_{11} - b_{13}b_{21})(b_{32} - b_{22}) &= (b_{23}b_{12} - b_{13}b_{22})(b_{31} - b_{11})\,, && (44)
\end{aligned}
$$

which we interpret as follow:
i) when $a_3 \neq 0$ apply (42-43) and use

$$\epsilon = \frac{1}{a_3}(a_1 b_{23} - a_2 b_{13})\,; \tag{45}$$

and/or
ii) when $b_{31} \neq b_{11}$ apply (42,44) and use

$$\epsilon = (b_{23}b_{11} - b_{13}b_{21}))/(b_{31} - b_{11})\,; \tag{46}$$

and finally
iii) when $b_{32} \neq b_{22}$ apply (43-44) and calculate ϵ from

$$\epsilon = (b_{23}b_{12} - b_{13}b_{22})/(b_{32} - b_{22})\,. \tag{47}$$

To complete the calculation insert ϵ in (41) and substitute J into (16). This provides a Poisson structure for the 3D Lotka-Volterra system, namely:

$$
\begin{aligned}
J^{12} &= -J^{21} = \frac{\epsilon}{\gamma H} x^1 x^2 x^3\,, && (48)\\
J^{13} &= -J^{31} = \frac{x^1 x^3}{\gamma H}\left(b_{1\mu}x^\mu - \frac{\epsilon\beta}{\gamma}x^3\right)\,, && (49)\\
J^{23} &= -J^{32} = \frac{x^2 x^3}{\gamma H}\left(b_{2\mu}x^\mu + \frac{\epsilon\alpha}{\gamma}x^3\right)\,. && (50)
\end{aligned}
$$

In Nutku's analysis [6] of the Lotka-Volterra system, one of the first integrals was the logarithm of the function used above and no Verhulst (diagonal) terms were included. Like in the present treatment the determinant of the coefficient of the quadratic terms was set to zero and an additional constraint was imposed similarly to condition (34). This more constrained system (four free parameters) has a known second constant of motion (or a constant of motion and a Casimir) and consequently admits a bi-Hamiltonian structure. The component J^{12} of the Lie tensor in Nutku's analysis satisfies condition (17) but is a different particular solution.

5 Conclusions

We have presented a procedure for constructing the Poisson structure of dynamical systems possessing a constant of motion. In three dimensions the problem is reduced to the solution of a linear PDE. When two time-independent first integrals of the system are known the procedure can be applied twice and yields bi-Hamiltonian structures involving *arbitrary* functions of the constants of motion (or a constant of the motion and a Casimir). The technique was applied to a few sample systems to show how it operates. In particular we showed that the ice skate system analyzed by Lucey [17] is completely integrable, that the Euler top systems admits an infinite family of bi-Hamiltonian structures and that a five free parameter version of the 3D Lotka-Volterra system can be cast in a generalized Hamiltonian form. Other 3D systems for which one or more time-independent constants of motion are known can be equally treated. This is being done for several rescaled version of such systems found in the literature. The results are still in preparation (some preliminary results can be found in ref. [18]) and will be soon submitted for publication elsewhere.

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