COMPLETELY INTEGRABLE ERMAKOV SYSTEMS

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1 Introduction

The question of the existence of invariants (constants of motion, first integrals) is one of central importance in the study of any dynamical system, be it classical or quantum. One basic problem of great interest in Physics, concerns the time-dependent harmonic oscilator. This class of non-autonomous systems has lead, in different occasions, to the study of the so called Ermakov[1]-[11] systems. One of the first relevant application of the Ermakov invariant was implemented by Lewis and Riesenfeld[2] who developed a quantization procedure for the time dependent harmonic oscilator. However, the original motivation for considering such systems was the desire to investigate the nature of the magnetic-moment series for a charged particle moving nonrelativistically in relatively simple electromagnetic field configurations[3]. Presently, the application scope of the subject extends from particle accelerators[4], to cosmological particle creation[5], to the construction of Feynman propagators for time dependent oscilators [6] down to two-layer long-wave theory[10].

Recently it has been shown[9] that a subclass of Ermakov systems admit a second constant of motion, a fact that might have an impact on the application of the related "non linear superposition law". The hability of constructing explicit second constant of motions implies, in such cases, in the possibility of a complete integration of the system. More recently it has been shown that there exists a large subclass of Ermakov systems that besides being twice integrable are also hamiltonian[11]. However the result was shown to apply only to the restricted Ermakov system in which the associated frequency ω if a function only of the time and not of any other dependent variable. In the present study, the class of completely integrable Ermakov systems that are also hamiltonian is extended in order to generalize two recent results[9],[11] and a much wider class of space and time-dependent harmonic oscilators that possess a second *exact* constant of motion is determined.

The proposed goal is achieved throught a global transformation that reduces the basic pair of time dependent equations to a set of autonomous equations. This autonomous system is further restricted following a strategy similar to either that adopted in reference [9] or that adopted in reference [11] which reduces the system to a hamiltonian system. In both cases it is trivial to write a second invariant which, toghether with the already known Ermakov invariant renders the system completly integrable. This complete integrability allows one to directly write down a nonlinear superposition law without having to perform any extra integration.

2 Basic theory and transformation

The Ermakov system was first studied in the last century[1], being independently rediscovered more recently[3] in connection with the study of charged particle motion in magnetic fields. This system is composed of a set of two nonlinear coupled ordinary differential equations

$$\ddot{x} + \omega^2 x = \frac{1}{yx^2} f(y/x)\,, \tag{1}$$

$$\ddot{y} + \omega^2 y = \frac{1}{xy^2} g(x/y)\,, \tag{2}$$

where f and g are arbitrary function of their argument, ω is an arbitrary function of time and of x and y and their time derivatives of any order. From the physical point of view it is interesting to restrict the consideration to the cases where ω depends only on x, y and t and on the first time derivatives of the dependent variables. Equations (1) and (2) admit the invariant[8]

$$I = \frac{1}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} f(\tau)d\tau + \int^{x/y} g(\tau)d\tau\,, \tag{3}$$

which shell be refered as the Ermakov invariant. In equations (1-3) and in what follows, an overdot represents the derivative with respect to the time.

We now introduce the global transformation $(x, y, t \to \hat{x}, \hat{y}, \hat{t})$ defined by

$$\hat{x} = x\, exp\left(\int^t \alpha(\tau)d\tau\right)\,, \tag{4}$$

$$\hat{y} = y\, exp\left(\int^t \alpha(\tau)d\tau\right)\,, \tag{5}$$

$$d\hat{t} = dt\, exp\left(2 \int^t \alpha(\tau)d\tau\right)\,, \tag{6}$$

where α is any solution of the Ricatti type equation,

$$\dot{\alpha} - \alpha^2 = \omega^2. \tag{7}$$

We notice that in the usual case when ω depends on time only, the transformation (4-7) reduces to the transformation used in a recent study[10] of the system. The transformed autonomous Ermakov system reads

$$\hat{x}'' = \frac{1}{\hat{y}\hat{x}^2} f(\hat{y}/\hat{x})\,, \tag{8}$$

$$\hat{y}'' = \frac{1}{\hat{x}\hat{y}^2} g(\hat{x}/\hat{y})\,, \tag{9}$$

where the prime stands for the derivative with respect to the new time $\hat{t}$.

The transformed Ermakov invariant $\hat{I}$ possesses the interesting property of invariance with respect to the transformation (4-7). In other words, the transformed invariant can be shown to have the form

$$\hat{I} = I(\hat{x}, \hat{y}, \hat{x}', \hat{y}')\,, \tag{10}$$

where I is given by equation (3).

3 New completely integrable Ermakov systems

The presence of two arbitrary functions in the transformed Ermakov pair (8-9) allows us to enforce additional constrains in order to obtain twice integrable systems. This can be done either by using Goedert's or Cerveró & Lejarreta's formulation. Each procedure leads, as we shell see bellow, to a different subclass of completely integrable systems.

In the first case we simply write Goedert's invariant (with $\omega \equiv 0$) for the transformed system (8-9)

$$\hat{J}_1(\hat{x}, \hat{y}, \hat{x}', \hat{y}') = \hat{x}'\hat{y}' + \frac{K}{\hat{x}^2} - \frac{1}{\hat{x}^2}\int^{\hat{y}/\hat{x}} \frac{1}{\tau} f(\tau)d\tau\,. \tag{11}$$

An application of the inverse of the transformation (8-9) to (11) yields the generalized second invariant for the Ermakov system

$$J_1(x, y, \dot{x}, \dot{y}, t) = \left((\dot{x} + \alpha x)(\dot{y} + \alpha y) + \frac{K}{x^2} - \frac{1}{x^2} \int^{y/x} \frac{1}{\tau} f(\tau) d\tau \right) exp\left(-2 \int^t \alpha(\tau) d\tau \right), \quad (12)$$

as long as g be derivable from f throught the relation

$$g(z) = -\frac{1}{z^2} f(\frac{1}{z}) - \frac{2}{z^2} \int^{1/z} \frac{1}{\tau} f(\tau)\, d\tau + \frac{2K}{z^2}, \quad (13)$$

where f remains arbitrary and K is an arbitrary numerical constant. Notice that in the result previously derived by Goedert, ω was restricted to depend on only the product xy whereas in the present case ω may depend freely on time and on the dependent variables, as long as one is able to solve the related equation of Ricatti type (7).

In the particular case where g is given by (13), the Ermakov invariant (3) becomes

$$\hat{I}_1 = \frac{1}{2}(\hat{x}\hat{y}' - \hat{x}'\hat{y})^2 + 2 \int^{\hat{y}/\hat{x}} f(\tau)\, d\tau + 2 \int^{\hat{y}/\hat{x}} \left(\int^u \frac{1}{\tau} f(\tau)\, d\tau \right) du - 2\, K \frac{\hat{y}}{\hat{x}}. \quad (14)$$

The non linear superposition law is now easely derived from (12) and (14). in the transformed coordinates. The necessary steps are accomplished by the elimination of $\hat{x}'$ between both invariants. The final implicit solution will read

$$S(\hat{x}, \hat{y}, \hat{y}', \hat{J}_1, \hat{I}_1) = 0, \quad (15)$$

for some well determined function S. In such cases it is correct to state that the solution $\hat{x}$ for the first of the Ermakov equation is expressed in terms of the solution of the second equation in the system, at least for the transformed variables. To transform back to the original coordinates one need to solve the Ricatti type equation (7) which is also expressed in terms of the invariants and y and $\dot{y}$. This will generally transform the differential Ricatti equation into an integro-differential equation.

We now turn into the second class of completely integrable Ermakov systems, namely the class of Ermakov systems that are at the same time hamiltonian, as recently analysed by Cerveró and Lejarreta. According to their derivation, an Ermakov system is also hamiltonian whenever g and f are related by

$$g(1/u) = \frac{2}{u} \int^u f(\tau)\, d\tau - f(u) + \frac{K}{u}. \quad (16)$$

In the specific case of the transformed Ermakov system(8-9), their hamiltonian (with $\hat{p}_1 \equiv \hat{x}'$ $\hat{p}_2 \equiv \hat{y}'$) reduces to

$$\hat{J}_2 \equiv \hat{H} = \frac{1}{2}(\hat{x}'^2 + \hat{y}'^2) + \frac{1}{\hat{y}^2} \int^{\hat{y}/\hat{x}} f(\tau)\, d\tau + \frac{K}{2\hat{y}^2}. \quad (17)$$

Because the system under consideration is autonomous $\hat{H}$ is the conserved energy which, when transformed back to the original variables, reads

$$J_2 = \left(\frac{1}{2}(\dot{x} + \alpha x)^2 + \frac{1}{2}(\dot{y} + \alpha y)^2 + \frac{1}{y^2} \int^{y/x} f(\tau)\, d\tau + \frac{K}{2y^2} \right) exp\left(-2 \int^t \alpha(\eta) d\eta \right). \quad (18)$$

In this second case, the Ermakov invariant becomes,

$$\hat{I}_2 = \frac{1}{2}(\hat{x}\hat{y}' - \hat{x}'\hat{y})^2 + \left(1 + (\frac{\hat{x}}{\hat{y}})^2 \right) \int^{\hat{y}/\hat{x}} f(\tau)\, d\tau + \frac{K}{2} \frac{x^2}{y^2}. \quad (19)$$

Again, in the transformed coordinates, we can construct a nonlinear superposition law by elimination of $\hat{x}'$ among the two invariants. The procedure parallels the previous one and will generate a similar relation (nonlinear superposition law) with $\hat{J}_2$ substituted for $\hat{J}_1$ and $\hat{I}_2$ substituted for $\hat{I}_1$. Again in this formalism ω may depend on the dependent variables and their derivative with respect to time, a feature that was not considered in previous works. We finally notice that all system derivable from (17) were already included in the class of once integrable systems derived by Leach et al.[12] and by Dhara and Lawande[13], but apparently in both studies the authors did not recognize this important connection (the fact that they were also of type Ermakov for appropriated subclasses) This point was missed in subsequent works as well.

To summarize we stress that the known class of twice integrable Ermakov system was hereby extended in two different directions. This completely integrable systems yield explicit nonlinear superposition laws that allow the direct solutions of a wide class of nonlinear second order ordinary differential equations. Specific applications can be found in the literature (see e. g. [10]) and some other have been announced in [11]. This results, in our believe, might bring an important contribution to the study of charged particle motion in the presence of electromagnetic fields, an area were the time dependent harmonic oscilator plays a fundamental role.

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