Comment on "Dynamical systems and Poisson structures" [M. Gürses, G. Sh. Guseinov and K. Zheltukhin, J. Math. Phys. 50, 112703 (2009)]

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We show that theorems 8, 9 and 11 in the work cited in the title are incorrect in general. The existence of globally well-defined first integrals or flow invariant functions for dynamical systems in R^n can not be taken for granted.

Our concern are theorems 8, 9 and 11 in Gürses, Guseinov and Zheltukhin work, hereafter referred as paper GGZ[1]. For completeness, we reproduce the cited theorems from GGZ, in literal form:

"**Theorem 8**: All dynamical systems in R^3 are Hamiltonian. This means that any vector field in R^3 is Hamiltonian vector field. Furthermore, all dynamical systems in R^3 are bi-Hamiltonian.

Theorem 9: All dynamical systems in R^n are Hamiltonian. Furthermore, all dynamical systems in R^n are $n-1$-Hamiltonian.

Theorem 11: All autonomous dynamical systems in R^n are superintegrable."

It is well known[2] that, due to the theorem on straightening of trajectories, in a small neighborhood of any point $\mathbf{x}_0 \in R^n$ which is not an equilibrium position there exist coordinates $x_1, ..., x_n$ in which the differential equations of any dynamical system on R^n can be expressed as

$$\dot{x}_1 = 1\,, \quad \dot{x}_2 = ... = \dot{x}_n = 0\,. \tag{1}$$

Therefore, the coordinates $x_2, ..., x_n$ form a "complete" set of first integrals. Any other first integral is a function $x_2, ..., x_n$.

However, it is also well known that the existence of a complete set of *global* (as opposed to local) first integrals is a major question for any dynamical system. Sometimes, one can even have explicit expressions for the local first integrals, although the dynamical system admit no globally well-defined first integral at all. In each particular case there is the need of a separate analysis of the integrable or non-integrable character of the dynamics. Integrable systems are the exception rather than the rule, contrarily to the statements in the cited theorems of GGZ. The concept of integrability suppose the existence of invariant, regular foliations whose leaves are embedded flow-invariant submanifolds, of the smallest dimension possible. This by no means can be taken for granted.

In the same context, superintegrable systems are even more scarce. For instance, we have few superintegrable potentials on the two- and three-dimensional Euclidean spaces with invariants that are quadratic polynomials in the canonical momenta[3,4]. A similar analysis was provided for the 2D and 3D spheres[5], for the 2D hyperbolic plane[6,7] and for the 3D hyperbolic space[8]. In addition, the existence of superintegrable systems with two degrees of freedom possessing three independent globally defined constants of motion which are quadratic in the velocities was studied in an unified way on the 2D sphere and on the

2D hyperbolic plane[9].

Clearly the method for the construction of generalized Hamiltonian structures presented in GGZ and theorems 8, 9 and 11 in it neglect the need of explicitly finding globally well-defined first integrals (or at least invariant functions) of the flow, a highly non-trivial task. In contrast, there are alternative approaches[10,11] which can be successful in generating Poisson structures for 3D dynamical systems with only one (not necessarily two) available invariant function (see for instance the case of certain three-dimensional Lotka-Volterra systems[11]). Obviously any maximally superintegrable system can be easily cast in a generalized Hamiltonian form, following e.g. the Nambu mechanics scheme[12].

In conclusion, in practice the existence of at least one globally well-posed invariant function is a necessary condition for constructing any generalized Hamiltonians formalism.

The fact that in GGZ the authors consider complete sets of invariant functions which are not necessarily first integrals is immaterial for the present discussion.

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